R-TORSION ON (2, N) TORUS KNOTS

By

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This thesis study the (2,n)-torus knots, uses the Khovanov Homology and Reidemeister Torsion. We show the study made to the Khovanov Homology Groups H^r, for $r = -n, -n+1, -1, 0$. These are isomorphics to either $\mathbb{Z}$, $\mathbb{Z}_2$ or some direct sum of copies of them. Finally, we applied the Redeimeister Torsion to the chain subcomplexes with polynomial degree of $-3n$, $-3n+2$ and $2-n$ using the same class of knots.

Resumen de Disertación Presentado a Escuela Graduada
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En esta tésis se presenta el estudio realizado sobre los nudos toroidales (2,n) haciendo uso de la Homología de Khovanov y la Torsión de Reidemeister. Se da a conocer el análisis hecho a los grupos homológicos H^r del Complejo de Khovanov, para $r = -n, -n+1, -1, 0$. Estos grupos homológicos son isomorfos a $\mathbb{Z}$, $\mathbb{Z}_2$ o una suma directa de copias de estos. Por último, se realizó el cálculo de la Torsión de Reidemeister a los subcomplejos de cadena con grado polinomial de $-3n$, $-3n+2$ y $2-n$ del Complejo de Khovanov para esta misma clase de nudos.

To

my mother Victoría Pérez-Ramírez

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LIST OF ABBREVIATIONS

R-Torsion	Reidemeister Torsion.
gcd	Greatest Common divisor.
snf	Smith Normal Form

LIST OF SYMBOLS

$\emptyset$	Empty set.
$\mathbb{R}$	Real numbers set.
$\mathbb{Z}$	Whole numbers set.
$\mathbb{Z}_2$	$\mathbb{Z}/2\mathbb{Z}$.
F	A field.
F^*	Group of units of F.
$M_{n\times m}(F)$	Set of all matrixes with n rows and m columns over F.
F	A vector space.
$V^{\otimes n}$	Tensor product of n copies of V.
S^n	n-sphere.
R	A ring with identity
R^t	Free group
$R/(a)$	Torsion group

CHAPTER 1
INTRODUCTION

1.1 Problem

Many researches have been presented as contributions in knot theory: some introduced new knot invariants, while others provided information about them and their relations.

The main purpose of this thesis is to analyze the (2, n)-torus knots using the Khovanov Homology and the Reidemeister torsion on the Khovanov complex. Also, this work expands previous research done about torus knots [11]. Rocha studied the Khovanov Homology for (3, n)-torus knots and found a formula for the 0^{st}-homology group and information about the r^{th}-homology groups when r is 1, $2k$ or $2k-1$.

Moreover, Ortiz (2007) introduced the concept of a volume form for the Khovanov Homology, which is also a knot invariant; [9]. It consists in apply the R-Torsion on the Khovanonv Chain Complex. Therefore, the focus in this thesis is to study the Reidemeister Torsion (R-Torsion) on the Khovanov Chain Complex for (2, n)-torus knots.

1.2 Background

Knots have been studied for many years: first organized studies were related to physics. Sir Thomson started knot theory with some hypothesis relating knots and vortex atoms. However, his theory was discarded years later. From there, many mathematicians were interested in classifying knots, and identifying significant properties.

In 1923, James Alexander introduced the first knot polynomial. This lies in the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$. Another important knot polynomial was discovered by Vaughan Jones in 1984 [5]. Years later, the bracket polynomial was discovered by Louis Kauffman in 1987 [6]. It is a link/knot invariant polynomial of regular isotopy which belongs in the ring $\mathbb{Z}[A, A^{-1}]$.

Both polynomials (Jones and Alexander) have difficulties distinguishing some knots or links. We can find knots whose Alexander or Jones Polynomial are equal, but their topological structures are different. For this reason, other invariants have been introduced and studied as a way to provide more information to identify knots.

As an extension of the Jones polynomial, Mikhail Khovanov (1999) gave a categorification of the Jones polynomial; [7]. He introduced a new knot topological property, called Khovanov Homology, which gives rise to a Poincaré polynomial over two variables. To calculate, it requires the construction of a chain complex over graded vector spaces with two basis elements whose degrees are 1 and -1. This chain complex is constructed using the information provided by the knot or link diagram. Once the chain complex is constructed, we proceed to find the graded dimensions of its homology groups and the Poincaré polynomial of the chain complex; all of these are invariants [6].

Another useful topological invariant is the Reidemeister Torsion (R-Torsion) which is defined for chain complexes over an associative ring with multiplicative identity [13]. The R-Torsion was originally introduced as a topological invariant of 3-manifolds. However, it has been adapted to a variety of contexts included knot theory. In 2007, Ortiz used the R-Torsion to demonstrate the existence of a volume form of the Khovanov Chain Complex which is also a knot invariant.

1.3 Objectives

This research is interested in applying the theory of torsion in the study of a specific knot class. To achieve the purpose of this thesis, the following objectives are presented:

1. Analyze the Khovanov Homology of (2,n)-torus knots
2. Analyze (2, n)-torus knots using Reidemeister Torsion.

   (a) Calculate homology groups H^{-n}, H^{-n+1}, H^{-1} and H^0.

   (b) Calculate R-Torsion on (2, n)-torus knots.

   (c) Identify patterns in the information provided by the R-Torsion applied on knot theory.

   (d) Find a formula for R-Torsion on (2, n)-torus knots.

CHAPTER 2
ALGEBRAIC ASPECTS

Reidemeister Torsion was originally introduced as a topological invariant of 3-manifolds. However, it has been adapted to a variety of contexts including knot theory. In this chapter, the reader will see the definitions and the facts concerning Reidemeister Torsion and how to calculate it. Moreover, we show some basic concepts about Smith Normal Form.

2.1 R-Torsion for acyclic chain complexes

Let $C_0, C_1, ..., C_m$ be finite dimensional vector spaces over F and $\partial_i : C_i \longrightarrow C_{i-1}$, $i \in \{0, ..., m\}$ be linear homomorphisms, where $C_{-1} = 0$ and $C_{m+1} = 0$. The sequence of vector spaces and homomorphisms

$$C : (\ 0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \longrightarrow \ldots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0\ )$$

is called a chain complex of length m, if $Im(\partial_i) \subset \ker(\partial_{i-1})$ for all $i \in \{1, 2, 3, \ldots, m\}$. This is equivalent to say $\partial_{i-1} \circ \partial_i = 0$.

Definition 2.1.1. [13] The vector space $H_i(C) = Ker(\partial_i)/Im(\partial_{i+1})$ is called the i-th homomology of th chain complex C.

In Definition 2.11, note that if $i \in \{0, ..., m\}$ note that $Ker(\partial_0) = C_0$ and $Im(\partial_{m+1}) = \emptyset$.

Definition 2.1.2. [13] The chain complex C is called acyclic, if $H_i(C) = 0$ for all i.

Definition 2.1.3. [13] The chain complex C is said to be based, if each C_i has a distinguished basis c_i.

Let C be an acyclic chain complex over F. Set $B_i = Im(\partial_i : C_i \longrightarrow C_{i-1}) \subset C_{i-1}, i \geq 0$. Since C is acyclic,

$$C_i/B_{i+1} = C_i/Ker(\partial_i) \cong Im(\partial_i) = B_i.$$

Set

$$\tilde{B}_i = C_i/B_{i+1}.$$

Choose a basis b_i of B_i and $\tilde{b}_i$ of $\tilde{B}_i$ for all i. Then $b_{i+1}\tilde{b}_i$ is a basis for C_i which can be compared with the distinguished basis c_i of C_i. Suppose dim $C_i = k$ and $(a_{i,j}) \in M_{k\times k}(F)$ to be the transition matrix from $b_{i+1}\tilde{b}_i$ to c_i.

We called $\tilde{e}_i$ as the "pullback" of b_i. That means $\partial_i(\tilde{e}_i) = e_i$.

Definition 2.1.4. [13] The torsion of C is

$$\tau(C) = \prod_{i=0}^{m} [b_{i+1}\tilde{b}_i/c_i]^{(-1)^{i+1}} \in F^* \tag{2.1}$$

where $[b_{i+1}\tilde{b}_i/c_i]$ =det$(a_{i,j}) \in F^*$.

2.2 R-Torsion for non-acyclic chain complex

Let C be a based chain complex over F, ie. $C = 0 \longrightarrow C_m \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$ with their respective boundary operators.

Set $H_i(C) = Ker(\partial_i)/Im(\partial_{i+1})$, every H_i is also based. Set $B_i = Im(\partial_i) \subset C_{i-1}$ and $Z_i = Ker(\partial_i)$. Then $0 \subset B_i \subset Z_{i-1} \subset C_{i-1}$ and $Z_i/B_{i+1} = H_i(C), \tilde{B}_i = C_i/Z_i \cong B_i$.

Let c_i and h_i be the distinguished basis of C_i and $H_i(C)$, respectively. Choose any basis b_i for B_i and $\tilde{b}_i$ for $\tilde{B}_i$. Then $b_{i+1}h_i\tilde{b}_i$ is a basis of C_i. We set $\tilde{b}_0 = \emptyset = b_{m+1}$.

Definition 2.2.1. [13] The torsion of C is $\tau(C) = \prod_{i=0}^{m}[b_{i+1}h_i\tilde{b}_i/c_i]^{(-1)^{i+1}} \in F^*$.

2.3 Smith Normal Form

[8] Let R be a principal ideal domain and let A be a $p \times n$ matrix with entries in R. We say that A is in Smith normal form if there are nonzero $a_1, ..., a_n \in R$ such that a_i divides a_{i+1} for each $i < m$, and

$$A = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_m & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

Proposition 2.3.1. *[8] Suppose that A is a relation matrix for an R-module M. If there are invertible matrices P and Q for which*

$$PAQ = \begin{pmatrix} a_1 & 0 & \cdots & \\ 0 & a_2 & 0 & \cdots \\ \vdots & & \ddots & \\ & & & a_m \\ 0 & \cdots & & \end{pmatrix}$$

is a diagonal matrix, then $M \cong R/(a_1) \oplus \cdots \oplus R/(a_n)$.

Corollary. *[8] If M is a finitely generated module over a principal ideal domain R, then there are elements $a_1, ..., a_m \in R$ such that a_i divides a_{i+1} for each $i \in \{1, ..., m-1\}$, and an integer $t \geq 0$ such that $M \cong R/(a_1) \oplus \cdots \oplus R/(a_n) \oplus R^t$.*

CHAPTER 3
INTRODUCTION TO KNOTS

This chapter shows concepts about knot. Definitions about equivalence of knot diagrams and some knot polynomials are presented.

Definition 3.0.1. [10] A link L of m components is a subsets of S^3 or R^3 that consists of m disjoint, piecewise linear simple closed curves. A link of one component is a knot.

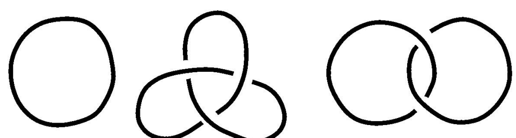


Figure 3.1: Unknot/Trefoil/Hopf link

One uses pictures to represent a knot, they are called **diagram projections** (or just projection). Places in which the knot crosses itself in the diagram projection are called as **crossings**. It is well known there are many differents projections that represent the same knot, therefore one needs tools to identify when two projection are equivalents.

3.1 Equivalence of knots

Definition 3.1.1. [4] Let $f, g : X \longrightarrow Y$ be continuous functions. A continuous function $F : X \times [0,1] \longrightarrow Y$such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$ is called an isotopy if $F|_{X\times\{t\}}$ is a homeomorphism for all $t \in [0,1]$.

Definition 3.1.2. [4] Let $f, g : Y \longrightarrow X$ be embeddings of Y into X. Then f and g are ambient isotopic, if there is an isotopy $F : X \times [0,1] \longrightarrow Y$ such that $F(x,0) = x$ for all $x \in X$ and $F(f(y),1) = g(y)$ for all $y \in Y$.

Definition 3.1.3. [6] Two knots are equivalent if they are ambient isotopic.

As example of two equivalent knots, see 3.2.

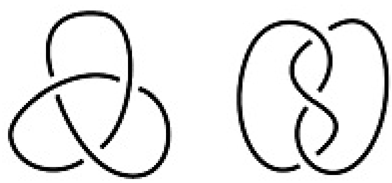


Figure 3.2: Two equivalent knots.

3.2 Reidemeister Moves

A Reidemeister Move is one of three ways to change a projection of the knot that will change the relation between crossings.[1]

TYPE 1. It allows us to put in or take out a twist in the knot.

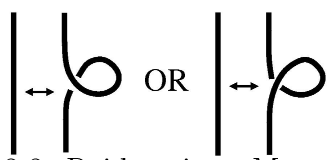


Figure 3.3: Reidemeister Move Type 1

TYPE 2. It allows us to either add two crossings or remove two crossings.

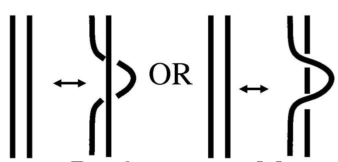


Figure 3.4: Reidemeister Move Type 2

TYPE 3. It allows us to slide a strand of the knot from one side of a crossing to the other side of the crossing.

Theorem 3.2.1. *[1] Let L_1 and L_2 be two diagrams for links. If we can get L_2 from L_1 through a finite sequence of Reidemeister moves, then we say that L_1 and L_2 are*

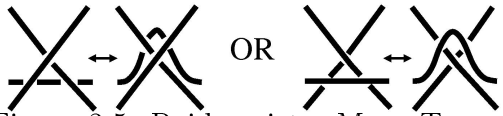


Figure 3.5: Reidemeister Move Type 3

ambient isotopic, and therefore, they are equivalent.

Definition 3.2.1. [6] Two links/knots are regularly isotopic, if they are ambient isotopic without the use of the Type I move.

Definition 3.2.2. A knot invariant is a property which assigns the same value to equivalent knots.

3.3 Torus knots

Definition 3.3.1. [1] A torus knot is a knot that lies on an unknotted torus, without crossing over or under themselves as it lies on the torus.

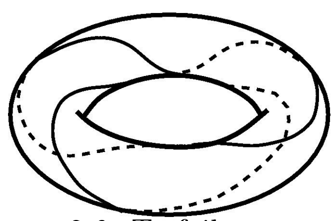


Figure 3.6: Trefoil on a torus

Definition 3.3.2. [1] A **meridian curve** is a curve that runs one the short way around the torus. A **longitude curve** is a curve that runs once around the torus in the long way.

To talk about torus knot, one denote a (p,q)-torus knot to the knot which every meridian(longitude) curve intersects to the knot in p(q) points. In fact, gcd(p,q)=1.

One of the purposes of knot theory is to identify the equivalence between knots and knot diagrams. Nowadays, there are knot invariants which make use of Laurent or Poinciaré polynomials and also by identifying homology groups from a specific chain complex.

3.4 Kauffman Bracket

Definition 3.4.1. [6] Let K be a diagram of an unoriented link. Let $\langle K \rangle$ be the element of the ring Z[A, B, d] defined by means of the rules:

1.$\langle O \rangle = 1$

2.$\langle O \cup K \rangle = d\langle K \rangle, K \neq \emptyset$

3. $\langle \times \rangle = A\langle \asymp \rangle + B\langle)(\rangle$

For our purposes, to make it invariant under regular isotopy, we set $A = 1$, $B = -q$ and $d = q + q^{-1}$.

Example 3.4.1. Consider the following oriented knot:

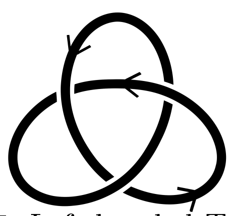


Figure 3.7: Left handed Trefoil knot

Therefore,

$$\langle \quad \rangle = q^{-3} - q - q^3 - q^5$$

3.5 Jones Polynomial

In 1984, Vaughan Jones discovered a Laurent polynomial invariant for oriented knots/links. Moreover, the Jones Polynomial is an invariant of the ambient isotopic type. See [5].

Let L be a link projection, χ the set of crossings of L and let $n = |\chi|$. Let us enumerate the elements of χ from 1 to n. Define $n_+(n_-)$ as the number of positive/(negative) crossing in χ. Note that $n = n_+ + n_-$.

Figure 3.8: Negative/Positive crossings

Definition 3.5.1. [2] The unnormalized Jones Polynomial of L is defined by:

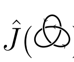
$$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle \tag{3.1}$$

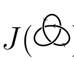
where $\langle L \rangle$ is the Kauffman Bracket.

Definition 3.5.2. [2, 5] The Jones Polynomial is given by:

$$J(l) = \hat{J}(L)/(q + q^{-1}). \tag{3.2}$$

Example 3.5.1. The Unnormalized and simple Jones Polynomial for figure 3.7 are:

$\hat{J}($$) = -q^{-9} + q^{-5} + q^{-3} + q^{-1}$

$J($$) = -q^{-8} + q^{-6} + q^{-2}$

CHAPTER 4
KHOVANOV HOMOLOGY

In 2008, Khovanov presented a new theory for knots, in which he introduced a new invariant. This theory enhances the idea of polynomials and make use of graded vector spaces, to find a chain complex and establish its homology groups as an invariant for links.

4.1 Preliminaries

Definition 4.1.1. [2] Let $W = \oplus_m W_m$ be a graded vector space with homogeneous components $\{W_m\}$. The graded dimension of W is the power series *qdim* $W = \sum_m q^m dim\ W_m$.

Definition 4.1.2. [2] Let $\{l\}$ be the "degree shift" operation on graded vector spaces. That is, if the graded dimension of W is the power series $qdim\ W = \sum_m q^m dim$ W_m where W is a graded vector space, we set $W\{L\}_m = W_{m-l}$, so that qdim $W\{l\} = q^l qdim\ W$.

Definition 4.1.3. [2] Let $[s]$ be the "height shift" operation on chain complexes. That is, if $\bar{C}$ is a chain complex $\ldots \longrightarrow \bar{C}^r \xrightarrow{\partial^r} \bar{C}^{r+1} \longrightarrow \ldots$ of (possible graded) vector spaces and if $C = \bar{C}[s]$, then $C^r = \bar{C}^{r-s}$.

4.2 Contructing the Khovanov Chain Complex

Let L, χ, n and $n_\pm$ as before. We name  the 0-smoothing and  the 1-smoothing of . From this, we set $\alpha \in \{0,1\}^\chi$ as a vertex of the n-dimensional cube, corresponding to a complete smoothing S_α [2].

We must construct the vector spaces which will allow us to establish the struture to find the Khovanov invariant. Let V be the graded vector space spanned by $1, x$ whose degree are $+1$ and -1 respectively, so that $qdim\ V = q + q^{-1}$. Let $V_\alpha(L) = V^{\oplus k}\{r\}$, where k is the number of cycles in the complete smoothing S_α, and $r = |\alpha| = \sum_i \alpha_i$, $|\alpha|$ is called the height of α. Finally, one defines $[\![L]\!]^r = \oplus_{\alpha:r=|\alpha|} V_\alpha(L)$, $0 \leq r \leq n$, as the r^{th} chain group for $C(L) = [\![L]\!][-n_-]\{n_+ - 2n_-\}$.

Theorem 4.2.1. *[2] The graded Euler Characteristc of C(L) is the unnormalized Jones polynomial of L.*

Now, we have to define the boundary operators to make $C(L)$ a chain complex. Let us label the edges of the cube $\{0,1\}^\chi$ by sequences $\xi \in \{0,1,\star\}^\chi$ such that ξ just has one $\star$ and $|\xi| = |\alpha|$, where α is the vertex in the tail of the edge. The element $\star$ is defined by $\star \longrightarrow 0$ in the tail and $\star \longrightarrow 1$ in the head. If ∂_ξ is the map on the edge ξ of the cube, then one defines $\partial^r = \sum_{\xi:|\xi|=r} (-1)^\xi \partial_\xi$ with $(-1)^\xi = (-1)^{\sum_{i<j} \xi_i}$ and where j is the location of $\star$ in ξ. These boundary operators make C(L) a chain complex.

Define $m : V \otimes V \longrightarrow V$ and $\Delta : V \longrightarrow V \otimes V$ as follows:

$$m = \begin{cases} 1 \otimes 1 \mapsto 1 \\ 1 \otimes x \mapsto x \\ x \otimes 1 \mapsto x \\ x \otimes x \mapsto 0 \end{cases} \tag{4.1}$$

$$\Delta = \begin{cases} 1 \mapsto 1 \otimes x + x \otimes 1 \\ x \mapsto x \otimes x \end{cases} \tag{4.2}$$

∂_ξ works as m or Δ on the cycles involved by $\star$; else, it works as identity.

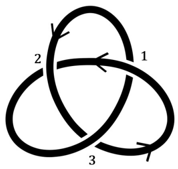
Definition 4.2.1. [7] Let H^r be the r-th cohomology of the chain complex $C(L)$. It is a graded vector space depending on the link projection L. Let $Kh(L)$ denote the graded Poincare polynomial of the complex $C(L)$ in the variable, i.e., let

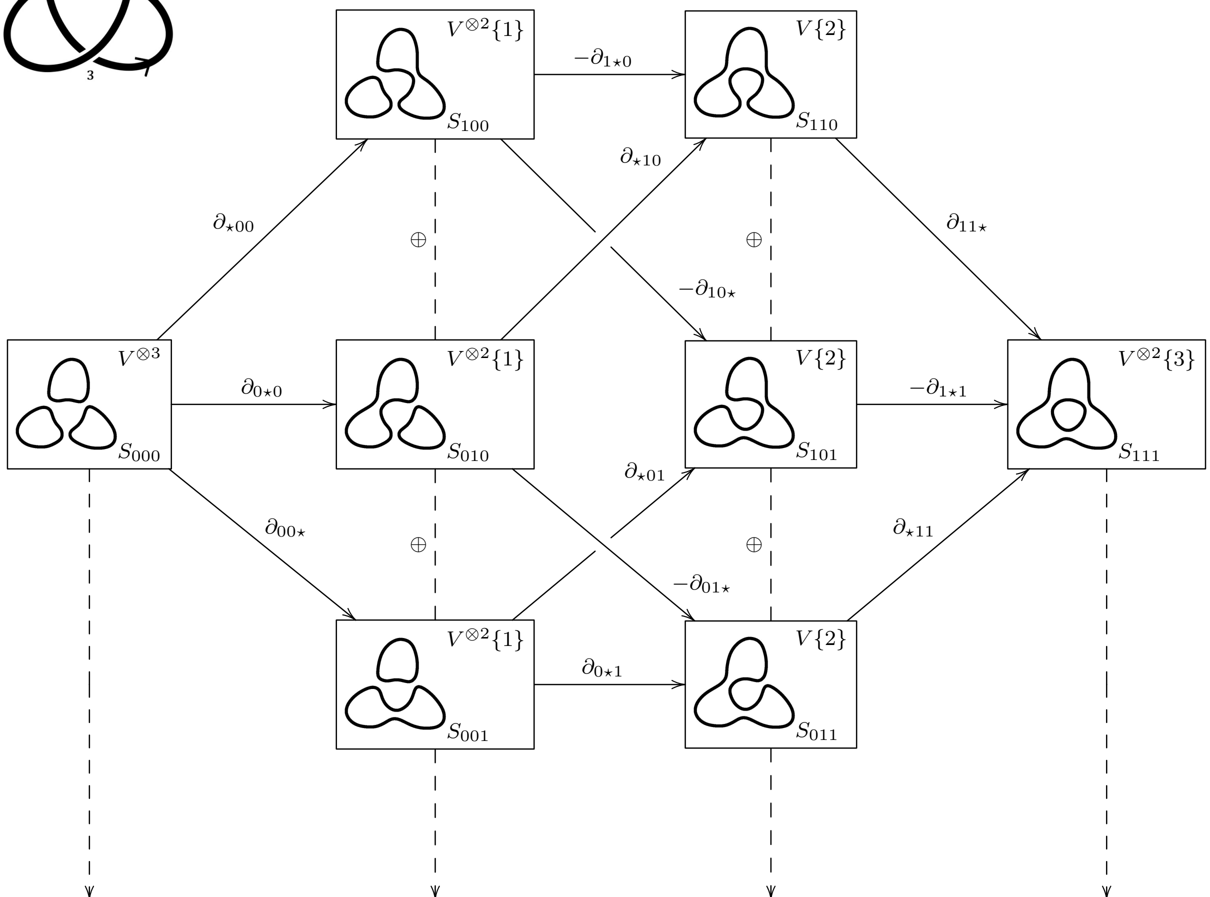
$$Kh(L) := \sum_r t^r qdim H^r(L)$$

Theorem 4.2.2. *[7] The graded dimension of the homology groups $H^r(L)$ are link invariants, and hence $Kh(L)$, a polynomial in the variables t and q, is a link invariant that specializes to the unnormalized Jones polynomial at t=-1.*

4.3 Example

Let K be the left-handed trefoil knot (Figure 3.7), $n_+ = 0$, $n_- = 3$, then we are considering the 3-dimensional cube.





$$[\![K]\!]^0 \xrightarrow{\partial^0} [\![K]\!]^1 \xrightarrow{\partial^1} [\![K]\!]^2 \xrightarrow{\partial^2} [\![K]\!]^3$$

$$= [\![K]\!] \xrightarrow[n_+=0, n_-=3]{[-n_-]\{n_+-2n_-\}} C(K)$$

Applying the height shift we obtain:

$$C(K) : (\ [\![K]\!]^{-3} \xrightarrow{\partial^{-3}} [\![K]\!]^{-2} \xrightarrow{\partial^{-2}} [\![K]\!]^{-1} \xrightarrow{\partial^{-1}} [\![K]\!]^{0}\ )$$

To work in a more confortable way, the last chain complex can be seen as:

$$0 \longrightarrow V \otimes V \otimes V \xrightarrow{\partial^{-3}} \oplus_3 V^{\otimes 2}\{1\} \xrightarrow{\partial^{-2}} V \oplus V \oplus V\{2\} \xrightarrow{\partial^{-1}} V \otimes V\{3\} \longrightarrow 0$$

Where each vector space has the following structure:

1. $V \otimes V \otimes V = span\langle 1 \otimes 1 \otimes 1, 1 \otimes 1 \otimes x, 1 \otimes x \otimes 1, 1 \otimes x \otimes x, x \otimes 1 \otimes 1, x \otimes 1 \otimes x, x \otimes x \otimes 1, x \otimes x \otimes x \rangle$
2. $(V \otimes V) \oplus (V \otimes V) \oplus (V \otimes V) = span\langle (1 \otimes 1, 0, 0), (1 \otimes x, 0, 0), (x \otimes 1, 0, 0), (x \otimes x, 0, 0), (0, 1 \otimes 1, 0), (0, 1 \otimes x, 0), (0, x \otimes 1, 0), (0, x \otimes x, 0), (0, 0, 1 \otimes 1), (0, 0, 1 \otimes x), (0, 0, x \otimes 1), (0, 0x \otimes x) \rangle$
3. $V \oplus V \oplus V = span\langle (1, 0, 0), (x, 0, 0), (0, 1, 0), (0, x, 0), (0, 0, 1), (0, 0, x) \rangle$
4. $V \otimes V = span\langle 1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x \rangle$

Evaluating each element of the basis for each vector space in the corresponding boundary operator we get the following:

1. $\mathrm{Ker}(\partial^{-3}) = span\langle x \otimes x \otimes x \rangle$
2. $\mathrm{Im}(\partial^{-3}) = span\langle (1 \otimes 1, 1 \otimes 1, 1 \otimes 1), (x \otimes x, x \otimes x, 0), (0, x \otimes x, x \otimes x), (x \otimes x, 0, x \otimes x), (x \otimes 1, x \otimes 1, x \otimes 1), (x \otimes 1, 1 \otimes x, 1 \otimes x), (1 \otimes x, x \otimes 1, 1 \otimes x) \rangle$
3. $\mathrm{Ker}(\partial^{-2}) = span\langle (1 \otimes 1, 1 \otimes 1, 1 \otimes 1), (x \otimes x, 0, 0), (0, x \otimes x, 0), (0, 0, x \otimes x), (1 \otimes x, 1 \otimes x.1 \otimes x), (x \otimes 1, x \otimes 1, 1 \otimes x), (1 \otimes x - x \otimes 1, 0, 0), (0, 1 \otimes x - x \otimes 1, 0), (0, 0, 1 \otimes x - x \otimes 1) \rangle$
4. $\mathrm{Im}(\partial^{-2}) = span\langle (1, 1, 0), (0, 1, 1), (1, 0, -1), (x, x, 0), (0, x, x), (x, 0, -x) \rangle$
5. $\mathrm{Ker}(\partial^{-1}) = span\langle (x, x, 0), (0, x, x), (x, 0, -x), (1, 1, 0), (0, 1, 1), (1, 0, -1) \rangle$
6. $\mathrm{Im}(\partial^{-1}) = span\langle 1 \otimes x + x \otimes 1, x \otimes x \rangle$

Therefore,

$$H^{-3}(K) \cong span\langle x \otimes x \otimes x \rangle \cong \mathbb{Z}$$

$$H^{-2}(K) \cong span\langle (x \otimes 1, x \otimes 1, x \otimes 1) \rangle \bigoplus \ldots$$

$$\frac{span\langle (x \otimes x, 0, 0), (0, x \otimes x, 0), (0, 0, x \otimes x) \rangle}{span\langle (x \otimes x, x \otimes x, 0), (x \otimes x, 0, x \otimes x), (0, x \otimes x, x \otimes x) \rangle} \cong \mathbb{Z} \bigoplus \mathbb{Z}_2$$

$$H^{-1}(K) \cong 0$$

$$H^{0}(K) \cong span\langle 1 \otimes 1, 1 \otimes x \rangle \cong \mathbb{Z} \bigoplus \mathbb{Z}$$

Therefore, the Khovanov polynomial invariant is:

$$Kh(K) = t^{-3}q^{-9} + t^{-2}q^{-5} + t^{0}q^{-3} + t^{0}q^{-1}$$

and replacing $t = -1$

$$\chi(C) = -q^{-9} + q^{-5} + q^{-3} + q^{-1} = \hat{J}(K)$$

This polynomial comes from the free homology groups of the chain complex, but H^{-2} is a direct sum of a free group and a torsion group. Let $(x \otimes x, 0, 0) + span\langle (x \otimes x, x \otimes x, 0), (x \otimes x, 0, x \otimes x), (0, x \otimes x, x \otimes x) \rangle = [(x \otimes x, 0, 0)]$ be a representative element of that quotient group. From it, one gets the following Khovanov polynomial mod 2: $t^{-2}q^{-7}$, $Kh(K)(mod 2) = t^{-2}q^{-7}$.

It is important to note that Khovanov invariant is stronger than Jones polynomial because it categorizes the Jones polynomial. Khovanov bracket generalizes the concept of Kauffman bracket upon which Jones polynomial is based. It is an exact complex chain that was obtained by the knot diagram and whose homology is called the Khovanov invariant or just Khovanov homology. Moreover, when we substitute $t = -1$ in the Khovanov polynomial then we get the unnormalized Jones polynomial of the respective knot.

4.4 R-torsion of the Khovanov homology

Ortiz (2008) established that Khovanov Homology has a volume form which is exactly the R-torsion of the chain complex constructed for finding the Khovanov invariant.

Theorem 4.4.1. *[9] The Khovanov homology has a volume form which is invariant for knots and links.*

Example 4.4.1. R-torsion for the trefoil is presented in the following table

	$h=-3$	$h=-2$	$h=-1$	$h=0$	τ
$pd=-1$				1	1
$pd=-3$	1	1	1	1	1
$pd=-5$	1	1	1		1
$pd=-7$	1	2			1/2
$pd=-9$	1				1

Table 4.1: R-Torsion for (2,3)-torus knot

where "h" means the height of the respective component of the Khovanov chain complex, "pd" means the polynomial degree of the chain subcomplex and τ is the R-torsion.

CHAPTER 5
MAIN ASPECTS

This thesis is focused in (2,n)-torus knots. Some properties about this knot class are presented. We are going to use the ones whose crossing are in negative way. Moreover, every crossing is enumerated in counterclockwise, taking as first cross the one which is located in the top right.

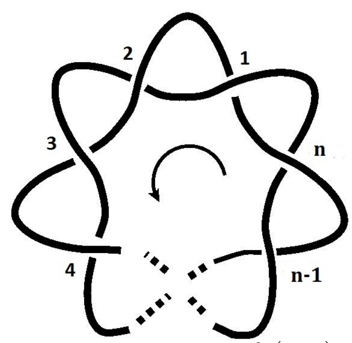


Figure 5.1: Representation of (2,n)-torus knots.

One knows the vector space V has $\{1, x\}$ as a basis. Then, we are going to use lexicographic order to get the following ordered basis for $V^{\otimes n}$:

$$1 \otimes 1 \otimes 1 \otimes ... \otimes 1 \otimes 1$$

$$1 \otimes 1 \otimes 1 \otimes ... \otimes 1 \otimes x$$

$$1 \otimes 1 \otimes 1 \otimes ... \otimes x \otimes 1$$

$$1 \otimes 1 \otimes 1 \otimes ... \otimes x \otimes x$$

$$\vdots$$

$$x \otimes x \otimes x \otimes ... \otimes 1 \otimes 1$$

$$x \otimes x \otimes x \otimes ... \otimes 1 \otimes x$$

$$x \otimes x \otimes x \otimes ... \otimes x \otimes 1$$

$$x \otimes x \otimes x \otimes ... \otimes x \otimes x$$

5.1 Khovanov homology on (2,n)-torus knots

Definition 5.1.1. Let K be the (2,n)-torus knot, and take χ and $|\chi|$ as before. Suppose that $|\chi| = n$, then we define $\Lambda_\beta = \{\xi \in \{0, 1, \star\}^\chi : |\xi| = \beta\}$, $\beta \in \{0, ..., n\}$.

Example 5.1.1. Suppose $|\chi| = 3$, then $\Lambda_0 = \{\star 00, 0 \star 0, 00\star\}$, $\Lambda_1 =$ $\{1 \star 0, 10\star, \star 10, 01\star, \star 01, 0 \star 1\}$, etc.

Proposition 5.1.1. *Let K be the (2,n)-torus knots, then $H^{-n}(K) \cong \mathbb{Z}$ and $t^{-n}qdim$ $H^{-n}(K) = t^{-n}q^{-3n}$,*

Proof. Applying the 0-smoothing to all n crossings we get $[\![K]\!]^{-n} = V^{\otimes n}$. This graded vector space is mapping onto $[\![K]\!]^{-n+1} = \bigoplus_n V^{\otimes 2}$ through $\partial^{-n} = \sum_{\xi \in \Lambda_0} (-1)^\xi \partial_\xi$ $= \sum_{\xi \in \Lambda} m_\xi$. Since, for all $\xi \in \Lambda_0$, $m_\xi(x \otimes x \otimes ... \otimes x \otimes x) = 0$, then we get $\partial^{-n}(x \otimes x \otimes ... \otimes x \otimes x) = 0$. Moreover, it is obvious that for all $y \in \bigotimes_n V, y \neq \otimes_n x$, $\partial^{-n}(y) \neq 0$ then $Ker(\partial^{-n}) = \langle \otimes_n x \rangle$.
Therefore, $H^{-n}(K) = Ker(\partial^{-n})/\langle 0 \rangle = \langle x \otimes x \otimes ... \otimes x \otimes x \rangle \cong \mathbb{Z}$. Since degree shift $\{n_+ - 2n_-\} = \{-2n\}$, then $qdim(H^{-n}(K)) = q^{-2n} \cdot q^{-n} = q^{-3n}$. Thus, $t^{-n}qdim(H^{-n}(K)) = t^{-n}q^{-3n}$ for all $n \geq 0$, $gcd(2, n) = 1$.

□

Proposition 5.1.2. *Let K be the (2,n)-torus knots, then $H^0(K) \cong \mathbb{Z} \bigoplus \mathbb{Z}$ and $t^0qdim\ H^0(K) = q^{-n} + q^{-n+2}$.*

Proof. From definition of Δ, one deduces $Im(\partial^{-1}) = \langle x \otimes x, 1 \otimes x + x \otimes 1 \rangle$. Thus, $H^0(K) = [\![K]\!]^0 / Im(\partial^{-1}) = \langle x \otimes 1, 1 \otimes 1 \rangle \cong \mathbb{Z} \bigoplus \mathbb{Z}$.

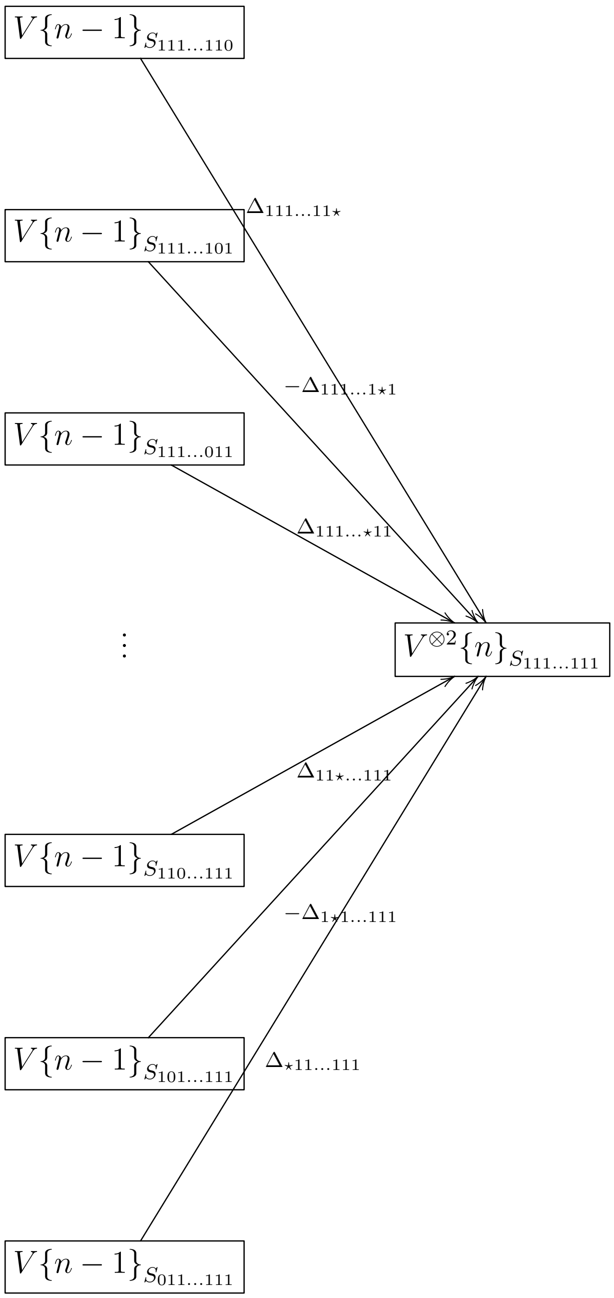
Since $\{n\}\{-2n_-\} = \{-n\}$, then $qdimH^0(K) = q^{-n} + q^{-n+2}$. Therefore,

$$t^0 qdimH^0(K) = t^0 q^{-n} + t^0 q^{-n+2} = q^{-n} + q^{-n+2}$$

□

Proposition 5.1.3. *Let K be the (2,n)-torus knot, then $H^{-1}(K) = \langle 0 \rangle$.*

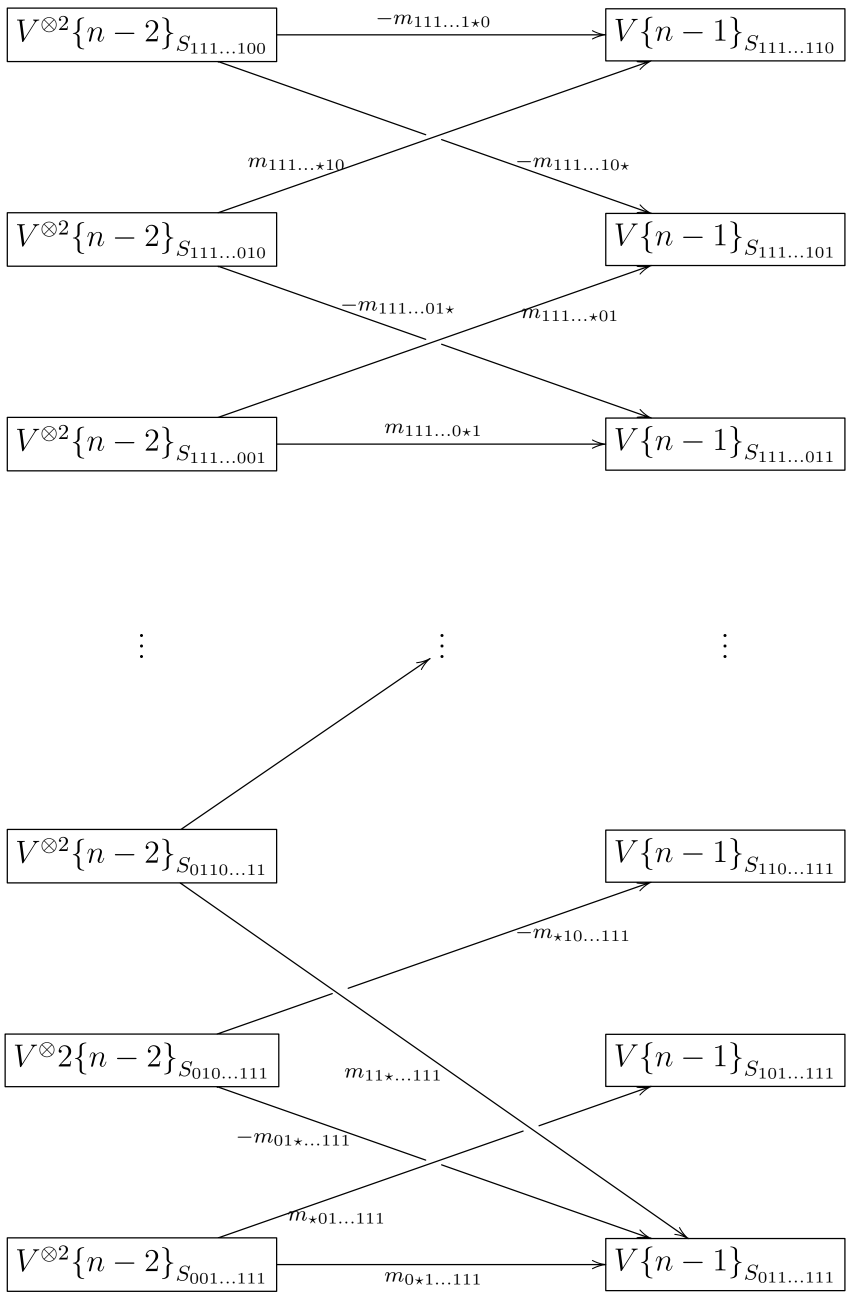
Proof. Let us check the edge operators that correspond to ∂^{-1}



As we see in the previous diagram every operator is alternating its sign. Since every Δ_ξ, $\xi \in \Lambda_{n-1}$, is mapping from a copy of V to $V^{\otimes 2}$ and every two consecutive ones have different signs. Then ones see each element on the set $\{(1,1,0,...,0,0,0),(0,1,1,...,0,0,0),$ $(0,0,0,...,0,1,1),(x,x,0,...,0,0,0),(0,x,x,...,0,0,0),...,(0,0,0,...,x,x,0),$ $(0,0,0,...,0,x,x)\}$ vanishes on the boundary operator, note this set is linearly independent and therefore this set is a basis for the kernel of ∂^{-1}.

Therefore, $Ker(\partial^{-1}) = span\langle(\pm1,\pm1,0,...,0,0,0),(0,\pm1,\pm1,...,0,0,0),...,$
$(0,0,0,...,\pm1,\pm1,0),(0,0,0,...,0,\pm1,\pm1),(\pm x,\pm x,0,...,0,0,0),$
$(0,\pm x,\pm x,...,0,0,0),...,(0,0,0,...,\pm x,\pm x,0),(0,0,0,...,0,\pm x,\pm x)\rangle$

Now, our interest is to verify what is happening on the image of ∂^{-2}.



Note that every component in $[\![K]\!]^{-2}$ map to two components of $[\![K]\!]^{-1}$. This happens because each smoothing in $[\![K]\!]^{-2}$ contains two zeros which will be changed by $* : 0 \longrightarrow 1$.

Let $\alpha \in \{0,1\}^{\chi}$ such that α contains two zeros. If both zeros are in even or odd positions, then S_α will be mapped by two operators with different sign. In the other side, if both zeros are in different position then that operator will have the same sign.

Therefore, If $x \in c_{-2}$ then $\partial^{-2}(x) \in \{(\pm1, \pm1, 0, ..., 0, 0, 0), (\pm1, 0, \mp1, ..., 0, 0, 0), ...,$ $(\pm1, 0, 0, ..., 0, 0, \mp1), (0, \pm1, \pm1, ..., 0, 0, 0), ..., (0, \pm1, 0, ..., 0, 0, \pm1), ...,$ $(0, 0, 0, ..., 0, \pm1, \pm1), (\pm x, \pm x, 0, ..., 0, 0, 0), (\pm x, 0, \mp x, ..., 0, 0, 0), ...,$ $(\pm x, 0, 0, ..., 0, 0, \mp x), (0, \pm x, \pm x, ..., 0, 0, 0), ..., (0, \pm x, 0, ..., 0, 0, \pm x), ...,$ $(0, 0, 0, ..., 0, \pm x, \pm x)\}$.

Thus, if $x \in [\![K]\!]^{-2}$, then $\partial^{-2}(x) \in \langle(\pm1, \pm1, 0, ..., 0, 0, 0), (\pm1, 0, \mp1, ..., 0, 0, 0), ...,$ $(\pm1, 0, 0, ..., 0, 0, \mp1), (0, \pm1, \pm1, ..., 0, 0, 0), ..., (0, \pm1, 0, ..., 0, 0, \pm1), ...,$ $(0, 0, 0, ..., 0, \pm1, \pm1), (\pm x, \pm x, 0, ..., 0, 0, 0), (\pm x, 0, \mp x, ..., 0, 0, 0), ...,$ $(\pm x, 0, 0, ..., 0, 0, \mp x), (0, \pm x, \pm x, ..., 0, 0, 0), ..., (0, \pm x, 0, ..., 0, 0, \pm x), ...,$ $(0, 0, 0, ..., 0, \pm x, \pm x)\rangle$.

However, some vectors can be obtained by linear combination of the following vectors: $(\pm1, \pm1, 0, ..., 0, 0, 0), (0, \pm1, \pm1, ..., 0, 0, 0), ..., (0, 0, 0, ..., \pm1, \pm1, 0),$ $(0, 0, 0, ..., 0, \pm1, \pm1), (\pm x, \pm x, 0, ..., 0, 0, 0), (0, \pm x, \pm x, ..., 0, 0, 0), ...,$ $(0, 0, 0, ..., \pm x, \pm x, 0), (0, 0, 0, ..., 0, \pm x, \pm x)$.

Therefore, $Im(\partial^{-2}) = \langle(\pm1, \pm1, 0, ..., 0, 0, 0), (0, \pm1, \pm1, ..., 0, 0, 0), ...,$ $(0, 0, 0, ..., \pm1, \pm1, 0), (0, 0, 0, ..., 0, \pm1, \pm1), (\pm x, \pm x, 0, ..., 0, 0, 0),$ $(0, \pm x, \pm x, ..., 0, 0, 0), ...,$ $(0, 0, 0, ..., \pm x, \pm x, 0), (0, 0, 0, ..., 0, \pm x, \pm x)\rangle$.

Since $Im(\partial^{-2}) = Ker(\partial^{-1})$, we get $H^{-1}(K) \cong \{0\}$

$\square$

Definition 5.1.2. If C is the Khovanov Complex of any K=(2,n)-torus knot, then we say C_α is the subcomplex whose components are vector spaces spanned by all elements of polynomial degree α from components of C. Moreover, we say c_α is a basis for C_α.

Similarly, we use the same notation to talk about basis of any homology group, kernel or image of boundary operators. For example, h_α^{-2} is a basis for H_α^{-2}.

Proposition 5.1.4. *H^{-n+1} has a torsion group.*

Proof. By definition of the chain complex, one knows that

$$H^{-n+1}(K) \cong H_{-n}^{-n+1} \bigoplus H_{-n-2}^{-n+1} \bigoplus ... \bigoplus H_{-3n+4}^{-n+1} \bigoplus H_{-3n+2}^{-n+1}$$

Now, we are going to focus on H_{-3n+2}^{-n+1}

By definition of the boundary operator m, we get $Ker(\partial_{-3n+2}^{-n+1}) = \langle (x \otimes ... \otimes x, 0, ..., 0, 0), (0, x \otimes ... \otimes x, ..., 0, 0), ..., (0, 0, ..., x \otimes ... \otimes x, 0), (0, 0, ..., 0, x \otimes ... \otimes x) \rangle$.

Moreover, by doing calculations on the following diagram

$$
\begin{tikzcd}
 & & \boxed{V^{\otimes n-1}\{1\}_{S_{100\ldots000}}} \\
 & & \boxed{V^{\otimes n-1}\{1\}_{S_{010\ldots000}}} \\
 & & \boxed{V^{\otimes n-1}\{1\}_{S_{001\ldots000}}} \\
\boxed{V^{\otimes n}\{0\}_{S_{000\ldots000}}} \arrow[uuurr, "m_{\star00\ldots000}"] \arrow[uurr, "m_{0\star0\ldots000}"] \arrow[urr, "m_{00\star\ldots000}"] \arrow[drr, "m_{000\ldots\star00}"] \arrow[ddrr, "m_{000\ldots0\star0}"] \arrow[dddrr, "m_{000\ldots00\star}"] & \vdots & \vdots \\
 & & \boxed{V^{\otimes n-1}\{1\}_{S_{000\ldots100}}} \\
 & & \boxed{V^{\otimes n-1}\{1\}_{S_{000\ldots010}}} \\
 & & \boxed{V^{\otimes n-1}\{1\}_{S_{000\ldots001}}}
\end{tikzcd}
$$

one gets:

$$\partial^{-n}(1 \otimes x \otimes ... \otimes x \otimes x) = (x \otimes ... \otimes x, x \otimes ... \otimes x, 0, ..., 0, 0, 0)$$

$$\partial^{-n}(x \otimes 1 \otimes ... \otimes x \otimes x) = (0, x \otimes ... \otimes x, x \otimes ... \otimes x, ..., 0, 0, 0)$$

$$\vdots$$

$$\partial^{-n}(x \otimes x \otimes ... \otimes 1 \otimes x) = (0, 0, 0, ..., x \otimes ... \otimes x, x \otimes ... \otimes x, 0)$$

$$\partial^{-n}(x \otimes x \otimes ... \otimes x \otimes 1) = (0,0,0,...,0,x \otimes ... \otimes x, x \otimes ... \otimes x)$$

$Im(\partial^{-n}_{-3n+2}) = \langle (x\otimes ...\otimes x, x\otimes ...\otimes x, 0, ..., 0, 0), (0, x\otimes ...\otimes x, x\otimes ...\otimes x..., 0, 0),$ $..., (0,0,0,...,x \otimes ... \otimes x, x \otimes ... \otimes x, 0), (0,0,0,...,0,x \otimes ... \otimes x, x \otimes ... \otimes x)\rangle \cong$ $\langle (1,0,...,0,0),(0,1,...,0,0),...,(0,0,...,1,0),(0,0,...,0,1)\rangle / \langle (1,1,0,...,0,0,0),...$ $(0,1,1,...,0,0,0),...,(0,0,0,...,1,1,0),(0,0,...,0,1,1)\rangle$

$H^{-n+1}_{-3n+2} = Ker(\partial^{-n+1})_{-3n+2}/Im(\partial^{-n})_{-3n+2} = \langle (x \otimes ... \otimes x, 0, ..., 0, 0), (0, x \otimes ... \otimes x, ..., 0, 0), ..., (0,0,...,x \otimes ... \otimes x, 0), (0,0,...,0,x \otimes ... \otimes x)\rangle / \langle (x \otimes ... \otimes x, x \otimes ... \otimes x, 0, ..., 0, 0), (0, x \otimes ... \otimes x, x \otimes ... \otimes x, ..., 0, 0), ..., (0,0,0,...,x \otimes ... \otimes x, x \otimes ... \otimes x, 0), (0,0,0,...,0,x\otimes ...\otimes x, x\otimes ...\otimes x)\rangle \cong \langle (1,0,...,0,0),(0,1,...,0,0),...,(0,0,...,1,0),$ $(0,0,...,0,1)\rangle / \langle (1,1,0,...,0,0,0),(0,1,1,...,0,0,0),...,(0,0,0,...,1,1,0),$ $(0,0,...,0,1,1)\rangle$ Applying the corollary of Proposition 2.3.1

$$snf\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \vdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \vdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 2 \end{pmatrix}$$

Therefore, $H^{-n+1}_{-3n+2} \cong \bigoplus_{n-1} \mathbb{Z}/(1) \bigoplus \mathbb{Z}/(2) \cong \mathbb{Z}_2$. From this, one deduces the qdimension for this homology group which is $t^{-n+1}q^{-3n+2}$.

$\square$

5.2 R-Torsion on (2,n)-torus knots

Proposition 5.2.1. *If C is the Khovanov Chain Complex associated to (2,n)-torus knot then:*

1. $\tau(C_{2-n}) = \pm 1$
2. $\tau(C_{-3n}) = \pm 1$
3. $\tau(C_{-3n+2}) = \pm 1/2$

Proof. To prove the first part, note that C^r has no elements of polynomial degree $2-n$ if $r \in \{-n, ..., -1\}$. On the other side, if $\alpha = 0$, $1 \otimes 1 \in C^0$ has polynomial degree $2-n$. In fact, by definition of the boundary operator Δ, $1 \otimes 1 \notin Im(\partial^{-1})$, then $b^{-1}_{2-n} = \emptyset$.

One can construct the subcomplex C_{2-n} as follows:

$$0 \xrightarrow{\partial^{-1}=0} C^0_{2-n} \xrightarrow{0} 0$$

Moreover, in proof of the Proposition 5.0.2. We saw $H^0(K) = \langle x \otimes 1, 1 \otimes 1 \rangle$, thus $h^0_{2-n} = \{\pm 1 \otimes 1\}$.

Therefore, $b^{-1}_{2-n} h^0_{2-n} / c^0_{2-n} = [\pm 1] \in M_{1\times 1}$, which implies $\tau(C_{2-n}) = \pm 1$.

Now, we are going to check the subcomplex of polynomial degree $-3n$. For every component of degree different to $-n$ there are no elements of polynomial degree $-3n$. This implies that the subcomponents $C^{\alpha}_{-3n} = \{0\}$, for $-n < \alpha \leq 0$, then our interest is study C^{-n}.

One can see C_{-3n} as follows

$$0 \xrightarrow{0} C^{-n}_{-3n} \xrightarrow{\partial_{-n}=0} 0$$

The proof of the Proposition 5.0.1, we got $H^{-n} = \langle \pm x \otimes ... \otimes x \rangle$, this homological group is spanned by a single element which has polynomial degree $-3n$.

Since $Im(0) = \langle 0 \rangle$, $\tilde{Im}(0) = \langle 0 \rangle$, thus $C^{-n}_{-3n} = span\langle h^{-3n}_{-n} \rangle = span\langle h^{-n} \rangle$.

Moreover, the unique element in basis of C^{-n} which has polynomial degree $-3n$ is $x \otimes ... \otimes x$, thus

$$h^{-n}_{-3}/c^{-n}_{-3n} = [\pm 1] \in M_{1\times 1}(\mathbb{Z})$$

which has determinant ± 1, then applying definition of R-Torsion we get $\tau(C_{-3n}) = \pm 1$

Since every component of C do not have elements of polynomial degree $-3n+2$, then $C^{r}_{-3n+2} = \emptyset$ for every $r \in \{-n+2, ..., 0\}$.

However, we can find elements with polynomial degree equal to $-3n+2$ in C^{-n} and C^{-n+1}. In fact,

$$C^{-n+1}_{-3n+2} = span\langle (x \otimes ... \otimes x, 0, ..., 0, 0), ..., (0, 0, ..., 0, x \otimes ... \otimes x)\rangle$$

and

$$C^{-n}_{-3n+2} = span\langle 1 \otimes ... \otimes x, ..., x \otimes ... \otimes 1\rangle$$

Then, C_{-3n+2} is the following:

$$0 \xrightarrow{0} C^{-n}_{-3n+2} \xrightarrow{\partial^{-n}} C^{-n+1}_{-3n+2} \xrightarrow{\partial^{-n+1}=0} 0$$

Let us analyze C^{-n}_{-3n+2}.

$H^{-n} = \langle \pm x \otimes ... \otimes x\rangle$ this implies $H^{-n}_{-3n+2} = \emptyset$, thus its basis $h^{-n}_{-3n+2} = \emptyset$. Also, the image of every element of C^{-n}_{-3n+2} is not 0, then $\tilde{b}^{-n}_{-3n+2} = \{1 \otimes ... \otimes x, ..., x \otimes ... \otimes 1\}$

$$\tilde{b}^{-n}_{-3n+2}/c^{-n}_{-3n+2} = \begin{pmatrix} \pm 1 & \cdots & 0 \\ : & \ddots & : \\ 0 & \cdots & \pm 1 \end{pmatrix}$$

thus, the determinant of this matrix is ± 1.

On the other side, by definition of subcomplex C_{-3n+2}, one has $\tilde{b}^{-n+1}_{-3n+2} = \emptyset$ and we already know

$b^{-n}_{-3n+2} = \{(x\otimes ...\otimes x, x\otimes ...\otimes x, 0, ..., 0, 0, 0), (0, x\otimes ...\otimes x, x\otimes ...\otimes x, ..., 0, 0, 0), ..., (0, 0, 0, ..., x\otimes ... \otimes x, x \otimes ... \otimes x, 0), (0, 0, 0, ..., 0, x \otimes ... \otimes x, x \otimes ... \otimes x)\}$

We will name X^{-n}_{-3n+2} to the matrix representation of ∂^{-n}_{-3n+2}. For which:

$$X^{-n}_{-3n+2} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ : & : & \cdots & : & : \\ 0 & 0 & : & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix}$$

Applying the Smith Normal Form to X^{-n}_{-3n+2}, one gets:

$$X^{-n}_{-3n+2} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & & \cdots & 0 & 0 \\ : & : & \cdots & : & : \\ 0 & 0 & : & 1 & 0 \\ 0 & 0 & \cdots & 0 & 2 \end{pmatrix}$$

ans its inverse is:

$$(X^{-n}_{-3n+2})^{-1} = \begin{pmatrix} 1/2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \vdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Then we directly extract the basis for homology from $(X^{-n+1}_{-3n+2})^{-1}$, which is $h^{-n+1}_{-3n+2} = \{(\pm 1/2x \otimes ... \otimes x, 0, ..., 0, 0)\}$.

Since C^{-n+1}_{-3n+2} is being mapped by the 0 operator,

$b^{-n}_{-3n+2}h^{-n+1}_{-3n+2} = \{(\pm x \otimes ... \otimes x, \pm x \otimes ... \otimes x, 0, ..., 0, 0, 0),$ $(0, \pm x \otimes ... \otimes x, \pm x \otimes, ..., \otimes x, ..., 0, 0, 0), ..., (0, 0, 0, ..., \pm x \otimes ... \otimes x, \pm x \otimes ... \otimes x, 0),$ $(0, 0, 0, ..., 0, \pm x \otimes ... \otimes x, \pm x \otimes ... \otimes x), (\pm 1/2x \otimes ... \otimes x, 0, 0, ..., 0, 0, 0)\}$ span C^{-n+1}_{-3n+2}.

and reducing by linear independent vectors, one gets a basis for C^{-n+1}_{-3n+2}

$b^{-n}_{-3n+2}h^{-n+1}_{-3n+2} = \{(\pm x \otimes ... \otimes \pm x, \pm x \otimes ... \otimes \pm x, 0, ..., 0, 0, 0), (0, \pm x \otimes ... \otimes \pm x, \pm x \otimes, ..., \otimes \pm x, ..., 0, 0, 0)$ $, ..., (0, 0, 0, ..., 0, \pm x \otimes ... \otimes \pm x, \pm x \otimes ... \otimes \pm x), (\pm x \otimes ... \otimes \pm x, 0, 0, ..., 0, 0, \pm x \otimes ... \otimes \pm x)\}$

The previous sets can be verified by reducing the following matrix:

$$\begin{pmatrix} \pm 1 & 0 & \cdots & 0 & \pm 1 & \pm 1/2 \\ \pm 1 & \pm 1 & \cdots & 0 & 0 & 0 \\ 0 & \pm 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & \pm 1 & 0 & 0 \\ 0 & 0 & \cdots & \pm 1 & \pm 1 & 0 \end{pmatrix}$$

And since $c^{-n+1}_{-3n+2} = \{(\pm x \otimes ... \otimes \pm x, 0, 0, ..., 0, 0, 0), (0, \pm x \otimes ... \otimes \pm x, 0, ..., 0, 0, 0)$ $, ..., (0, 0, 0, ..., 0, \pm x \otimes ... \otimes \pm x, 0), (0, 0, 0, ..., 0, 0, \pm x \otimes ... \otimes \pm x)\}$, one gets:

$$b^{-n}_{-3n+2} h^{-n+1}_{-3n+2} / c^{-n+1}_{-3n+2} = \begin{pmatrix} \pm 1 & 0 & \cdots & 0 & \pm 1 \\ \pm 1 & \pm 1 & \cdots & 0 & 0 \\ 0 & \pm 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \vdots & \pm 1 & 0 \\ 0 & 0 & \cdots & \pm 1 & \pm 1 \end{pmatrix}$$

Since determinant is invariant under adding a multiple row to another row, then:

$$[b^{-n}_{-3n+2}h^{-n+1}_{-3n+2}/c^{-n+1}_{-3n+2}] = \begin{vmatrix} \pm 1 & 0 & \cdots & 0 & \pm 1 \\ \pm 1 & \pm 1 & \cdots & 0 & 0 \\ 0 & \pm 1 & \cdots & 0 & 0 \\ : & : & \cdots & : & : \\ 0 & 0 & : & \pm 1 & 0 \\ 0 & 0 & \cdots & \pm 1 & \pm 1 \end{vmatrix}$$

$$= \begin{vmatrix} \pm 1 & 0 & \cdots & 0 & 0 \\ 0 & \pm 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ : & : & \cdots & : & : \\ 0 & 0 & : & \pm 1 & 0 \\ 0 & 0 & \cdots & 0 & \pm 2 \end{vmatrix} = \pm 2$$

Applying the definition of R-Torsion we get $\tau(C_{-n+1}) = \pm 1/2$ $\square$

CHAPTER 6
CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

Our interest was to analyze the (2,n)-torus knots via Khovanov homology and R-Torsion on the Khovanov Chain Complex for the knot.

In this work, we proved congruence of the homology groups with height $-n$, -1 and 0. As results showed, they are isomorphic to $\mathbb{Z}$, (0) and $\mathbb{Z} \bigoplus \mathbb{Z}$, respectively. Also, a torsion group of polynomial degree $-3n+2$ was calculated in the homology group of height $-n+1$.

Moreover, we calculated the R-torsion for the chain subcomplexes of polynomial degrees $2-n$, $-3n$ and $-3n+2$, which are ± 1, ± 1 and $\pm 1/2$, respectively. One curious thing about the R-Torsion $\pm 1/2$ is that it corresponds to a chain subcomplex which has a torsion group which is isomorphic to $\mathbb{Z}_2$. This aspect allows to create some hypothesis related to the Khovanov Homology Groups.

6.2 Future work

1. To complete the study for all homology groups for (2,n)-torus knots.
2. To obtain the complete Khovanov Polynomial for (2,n)-torus knots.
3. To prove $Kh(T(2,n))(mod 2) = \sum_{i=1}^{n-1/2} = q^{-n-4i}t^{-2i}$
4. To study possible relations between R-Torsion $\pm 1/2$ and $\mathbb{Z}_2$ at the Khovanov chain subcomplexes for (2,n)-torus knots.
5. To prove that $\pm 1/2$ is the R-Torsion for chain subcomplexes which have torsion groups in one of its homology groups.

6. To get the R-torsion for all chain subcomplexes for (2,n)-torus knots.

CHAPTER 7
APPENDIX

In this chapter, we present the Khovanov polynomial and Khovanov homology groups for the (2,n)-torus knot, $n < 16$. This information can be obtained by KhoHo [12] and the Knot Atlas [3].

(2,3)-torus knot

$Kh(T(2,3)) = q^{-9}t^{-3} + q^{-5}t^{-2} + q^{-3} + q^{-1}$

$Kh(T(2,3))(mod2) = q^{-7}t^{-2}$

dim $q^{2r+i}H^r_{\mathbb{Z}}$	$i = -3$	$i = -1$
$r = -3$	$\mathbb{Z}$	
$r = -2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r = -1$		
$r = 0$	$\mathbb{Z}$	$\mathbb{Z}$

Table 7.1: Khovanov Homology groups for (2,3)-torus knot

(2,5)-torus knot

$Kh(T(2,5)) = q^{-15}t^{-5} + q^{-11}t^{-4} + q^{-11}t^{-3} + q^{-7}t^{-2} + q^{-5} + q^{-3}$

$Kh(T(2,5))(mod2) = q^{-13}t^{-4} + q^{-9}t^{-2}$

dim $q^{2r+i}H^r_{\mathbb{Z}}$	$i=-5$	$i=-3$
$r=-5$	$\mathbb{Z}$	
$r=-4$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-3$	$\mathbb{Z}$	
$r=-2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-1$		
$r=0$	$\mathbb{Z}$	$\mathbb{Z}$

Table 7.2: Khovanov Homology groups for (2,5)-torus knot

(2,7)-torus knot

$Kh(T(2,7)) = q^{-21}t^{-7} + q^{-17}t^{-6} + q^{-17}t^{-5} + q^{-13}t^{-4} + q^{-13}t^{-3} + q^{-9}t^{-2} + q^{-7} + q^{-5}$

$Kh(T(2,7))(mod2) = q^{-19}t^{-6} + q^{-15}t^{-4} + q^{-11}t^{-2}$

dim $q^{2r+i}H^r_{\mathbb{Z}}$	$i=-7$	$i=-5$
$r=-7$	$\mathbb{Z}$	
$r=-6$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-5$	$\mathbb{Z}$	
$r=-4$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-3$	$\mathbb{Z}$	
$r=-2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-1$		
$r=0$	$\mathbb{Z}$	$\mathbb{Z}$

Table 7.3: Khovanov Homology groups for (2,7)-torus knot

(2,9)-torus knot

$Kh(T(2,9)) = q^{-27}t^{-9} + q^{-23}t^{-8} + q^{-23}t^{-7} + q^{-19}t^{-6} + q^{-19}t^{-5} + q^{-15}t^{-4} + q^{-15}t^{-3} +$
$q^{-11}t^{-2} + q^{-9} + q^{-7}$

$Kh(T(2,9))(mod2) = q^{-25}t^{-8} + q^{-21}t^{-6} + q^{-17}t^{-4} + q^{-13}t^{-2}$

dim $q^{2r+i}H^r_{\mathbb{Z}}$	$i=-9$	$i=-7$
$r=-9$	$\mathbb{Z}$	
$r=-8$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-7$	$\mathbb{Z}$	
$r=-6$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-5$	$\mathbb{Z}$	
$r=-4$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-3$	$\mathbb{Z}$	
$r=-2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-1$		
$r=0$	$\mathbb{Z}$	$\mathbb{Z}$

Table 7.4: Khovanov Homology groups for (2,9)-torus knot

(2,11)-torus knot

$Kh(T(2,11)) = q^{-33}t^{-11}+q^{-29}t^{-10}+q^{-29}t^{-9}+q^{-25}t^{-8}+q^{-25}t^{-7}+q^{-21}t^{-6}+q^{-21}t^{-5}+$
$q^{-17}t^{-4}+q^{-17}t^{-3}+q^{-13}t^{-2}+q^{-11}+q^{-9}$

$Kh(T(2,11))(mod2) = q^{-31}t^{-10}+q^{-27}t^{-8}+q^{-23}t^{-6}+q^{-19}t^{-4}+q^{-15}t^{-2}$

dim $q^{2r+i}H^r_{\mathbb{Z}}$	$i=-11$	$i=-9$
$r=-11$	$\mathbb{Z}$	
$r=-10$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-9$	$\mathbb{Z}$	
$r=-8$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-7$	$\mathbb{Z}$	
$r=-6$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-5$	$\mathbb{Z}$	
$r=-4$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-3$	$\mathbb{Z}$	
$r=-2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-1$		
$r=0$	$\mathbb{Z}$	$\mathbb{Z}$

Table 7.5: Khovanov Homology groups for (2,11)-torus knot

(2, 13)-torus knot

$Kh(T(2,13)) = q^{-39}t^{-13} + q^{-35}t^{-12} + q^{-35}t^{-11} + q^{-31}t^{-10} + q^{-31}t^{-9} + q^{-27}t^{-8} + q^{-27}t^{-7} + q^{-23}t^{-6} + q^{-23}t^{-5} + q^{-19}t^{-4} + q^{-19}t^{-3} + q^{-15}t^{-2} + q^{-13} + q^{-11}$

$Kh(T(2,13))(mod2) = q^{-37}t^{-12} + q^{-33}t^{-10} + q^{-29}t^{-8} + q^{-25}t^{-6} + q^{-21}t^{-4} + q^{-17}t^{-2}$

dim $q^{2r+i}H^r_{\mathbb{Z}}$	$i=-13$	$i=-11$
$r=-13$	$\mathbb{Z}$	
$r=-12$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-11$	$\mathbb{Z}$	
$r=-10$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-9$	$\mathbb{Z}$	
$r=-8$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-7$	$\mathbb{Z}$	
$r=-6$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-5$	$\mathbb{Z}$	
$r=-4$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-3$	$\mathbb{Z}$	
$r=-2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-1$		
$r=0$	$\mathbb{Z}$	$\mathbb{Z}$

Table 7.6: Khovanov Homology groups for (2,13)-torus knot

(2, 15)-torus knot

$Kh(T(2,15)) = q^{-45}t^{-15} + q^{-41}t^{-14} + q^{-41}t^{-13} + q^{-37}t^{-12} + q^{-37}t^{-11} + q^{-33}t^{-10} + q^{-33}t^{-9} + q^{-29}t^{-8} + q^{-29}t^{-7} + q^{-25}t^{-6} + q^{-25}t^{-5} + q^{-21}t^{-4} + q^{-21}t^{-3} + q^{-17}t^{-2} + q^{-15} + q^{-13}$

$Kh(T(2,15))(mod2) = q^{-43}t^{-14} + q^{-39}t^{-12} + q^{-35}t^{-10} + q^{-31}t^{-8} + q^{-27}t^{-6} + q^{-23}t^{-4} + q^{-19}t^{-2}$

dim $q^{2r+i}H^r_{\mathbb{Z}}$	$i=-15$	$i=-13$
$r=-15$	$\mathbb{Z}$	
$r=-14$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-13$	$\mathbb{Z}$	
$r=-12$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-11$	$\mathbb{Z}$	
$r=-10$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-9$	$\mathbb{Z}$	
$r=-8$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-7$	$\mathbb{Z}$	
$r=-6$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-5$	$\mathbb{Z}$	
$r=-4$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-3$	$\mathbb{Z}$	
$r=-2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$r=-1$		
$r=0$	$\mathbb{Z}$	$\mathbb{Z}$

Table 7.7: Khovanov Homology groups for (2,15)-torus knot

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