Interior Operator Theory in Topology and Algebra

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Abstract

A categorical notion of interior operator is used in topology to define connectedness and disconnectedness with respect to an interior operator. A commutative diagram of Galois connections is used to show a relationship between these notions and the notions of connectedness and disconnectedness with respect to a subclass of topological spaces introduced by Arhangel'skii and Wiegandt.

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Interior Operator Theory in Topology and Algebra

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Resumen

Una noción categorica de operadores de interior se utiliza en la categoria de espacios topológicos para definir espacios conectados y desconectados con respecto a un operador de interior. Se presenta un diagrama conmutativo de conecciones de Galois para describir la relación entre las ideas anteriores y las nociones de conección y desconección con respecto a una subclase de espacios topológicos introducidas por Arhangel'skii and Wiegandt.

Never knowing what the waves will bring, but riding them nonetheless.

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Contents

Introduction

A general notion of closure operator was previously used to export classical topological notions to an arbitrary category. The reader interested in this topic could consult [3] or [4]. The idea of this work is to try to do something similar but using the notion of interior operator. In Chapter 1 we introduce the reader to the notion of a Galois connection between pre-ordered classes and some of its properties. In Chapter 2 we define the notion of an interior operator in the category of topological spaces and also present some examples and properties. In the following chapter we define the notions of connectedness and disconnectedness with respect to an interior operator and we study their basic properties. Chapter 4 is devoted to the construction of the discrete and indiscrete Galois connections and to the main theorem that describes their composition. Moreover, we also relate our notions of connectedness and disconnectedness with respect to an interior operator to the ones of connectedness and disconnectedness with respect to a subclass of topological spaces introduced in [2]. Several examples that illustrate the theory are also included. In Chapter 5 we make some comments to the notion of interior operator in the category of Groups and give an example.

Chapter 1

Preliminary Notions

In this chapter we include definition and properties of Galois connections that will be useful throughout this work.

Definition 1.1

A relation $\leq$ on a class $\mathcal{P}$ is called a "pre-order" if:

1. $a \leq a$, $\forall a \in \mathcal{P}$.

2. $a \leq b$ and $b \leq c$ implies $a \leq c$, $\forall a, b, c \in \mathcal{P}$.

   If in addition to 1 and 2 also the following holds:

3. $a \leq b$ and $b \leq a$ implies $a = b$, $\forall a, b \in \mathcal{P}$ then the relation $\leq$ is called a "partial order".

Let $(\mathcal{P}, \leq)$ and $(\mathcal{Q}, \sqsubseteq)$ be two partially ordered classes. A Galois connection $\mathcal{P} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{Q}$ between them consists of two order preserving functions: $f^* : \mathcal{P} \to \mathcal{Q}$ and $f_* : \mathcal{Q} \to \mathcal{P}$, such that for all $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$, we have that $f^*(P) \sqsubseteq Q$ if and only if $P \leq f_*(Q)$. Throughout this work we will use certain results that appear in the book by G. Castellini [3] :

Classical Results

Proposition 1.1

The composition of two Galois connections is a Galois connection.

Proof. Let $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ and $Y \underset{k}{\overset{h}{\rightleftarrows}} Z$ be two Galois connections. First let us recall that the functions $h \circ f$ and $g \circ k$ are order preserving since the functions f, g, h, k have the same property. Let $x \in X$, then $f(x) \subseteq k(h(f(x)))$ and so $x \subseteq g(f(x)) \subseteq g(k(h(f(x)))) = (g \circ k)((h \circ f)(x))$. Letting $z \in Z$, we have that $f(g(k(z))) \subseteq k(z)$ and so $h(f(g(k(z)))) \subseteq h(k(z)) \subseteq z$. Hence $(h \circ f)((g \circ k)(z)) \subseteq z$. Therefore $X \underset{g \circ k}{\overset{h \circ f}{\rightleftarrows}} Z$ is a Galois connection. $\square$

Proposition 1.2

Let $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ be a Galois connection, then the functions f and g uniquely determine each other.

Proof. Let $g' : Y \to X$ be a function such that $X \underset{g'}{\overset{f}{\rightleftarrows}} Y$ is a Galois connection. By applying g' to $f(g(y)) \leq y$, we obtain that $g(y) \leq g'(f(g(y))) \leq g'(y)$. Moreover, if we apply g to $f(g'(y)) \leq y$ we get that $g'(y) \leq g(f(g'(y))) \leq g(y)$. Hence we conclude that $g(y) = g'(y)$ for every $y \in Y$. Now, let $f' : X \to Y$ be such that $X \underset{g}{\overset{f'}{\rightleftarrows}} Y$ is a Galois Connection. Let us apply f' to $x \leq g(f(x))$. This gives us that $f'(x) \leq f'(g(f(x))) \leq f(x)$. Then by applying f to $x \leq g(f'(x))$ we obtain that $f(x) \leq f(g(f'(x))) \leq f'(x)$. Hence we conclude that $f(x) = f'(x)$ for every $x \in X$. Thus f and g uniquely determine each other. $\square$

Definition 1.2

Let $(\mathcal{P}, \leq)$ be a pre-ordered class and let $\{x_i\}_{i\in I}$ be a family of elements of $\mathcal{P}$. An element $x_0 \in \mathcal{P}$ is called the supremum of the family $\{x_i\}_{i\in I}$ ($x_0 = \bigvee_{i\in I}\{x_i\}$) if:

1. $x_i \leq x_0$, $\forall i \in I$.

2. If $x \in \mathcal{P}$ satisfies $x_i \leq x$ $\forall i \in I$, then $x_0 \leq x$.

Similarly $y_0 \in \mathcal{P}$ is called the infimum of the family $\{x_i\}_{i\in I}$ ($y_0 = \bigwedge_{i\in I}\{x_i\}$) if:

1. $y_0 \leq x_i$, $\forall i \in I$.

2. If $y \in \mathcal{P}$ satisfies $y \leq x_i$ $\forall i \in I$, then $y \leq y_0$.

Proposition 1.3

Let X and Y be two pre-ordered classes and assume that suprema exist in X. Let $f : X \to Y$ be a suprema preserving function. Define $g : Y \to X$ as follows: for every $y \in Y$, $g(y) = \bigvee\{x \in X : f(x) \leq y\}$. Then $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ is a Galois Connection.

Proof. Let $x_1 \leq x_2 \in X$. Since f preserves suprema, we have that $f(x_1) \leq f(x_2)$. Let $y_1 \leq y_2 \in Y$. Clearly we have that $\{x \in X : f(x) \leq y_1\} \subseteq \{x \in X : f(x) \leq y_2\}$. Thus by taking the supremum we obtain that $g(y_1) \leq g(y_2)$. Now, let $x' \in X$. By applying the definition of g we obtain that $g(f(x')) = \bigvee\{x \in X : f(x) \leq f(x')\} \geq x'$. Finally, let $y' \in Y$. Since f preserves suprema, we have that $f(g(y')) = f(\bigvee\{x \in X : f(x) \leq y'\} = \bigvee\{f(x) \in Y : f(x) \leq y'\} \leq y'$. $\square$

Proposition 1.4

Let X, Y be two pre-ordered classes and assume that infima exist in Y. Let $g : Y \to X$ be an infima preserving function. Define $f : X \to Y$ as follows: for every $x \in X$, $f(x) = \bigwedge\{y \in Y : x \leq g(y)\}$. Then $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ is a Galois Connection.

Proof. Let $y_1 \leq y_2 \in Y$. Since g preserves infima we have that $g(y_1) \leq g(y_2)$. Now let $x_1 \leq x_2 \in X$ then $\{y \in Y : x_1 \leq g(y)\} \supseteq \{y \in Y : x_2 \leq g(y)\}$. Thus taking infima we obtain that $f(x_1) \leq f(x_2)$. Now if $y' \in Y$, by applying f we obtain that $f(g(y')) = \bigwedge\{y \in Y : g(y') \leq g(y)\} \leq y'$. Lastly since g preserves infima if $x' \in X$ we obtain that $g(f(x')) = g(\bigwedge\{y \in Y : x' \leq g(y)\}) = \bigwedge\{g(y) \in X : x' \leq g(y)\} \geq x'$. $\square$

Chapter 2

Interior Operators in Topology

The aim of category theory is to identify common ideas among different areas in mathematics. Roughly speaking, a category consists of objects $(X, Y, Z, ...)$ and morphisms $(X \xrightarrow{f} Y,\ Y \xrightarrow{g} Z,\ ...)$ with an operation of composition among morphisms (that is not always defined). The most common examples of a category are: Top, the category of topological spaces and continuous functions, Grp, the category of groups and groups homomorphisms, Vec, the category of vector spaces and linear transformations, Set, the category of sets and functions. Since the work presented here is done mostly in the category of topological spaces we will not present any further details about the concept of category and we refer the interested reader to the book *Abstract and Concrete Categories* by J. Adámek, H Herrlich and G.E. Strecker [1], for instance. The notion of closure operator in an arbitrary category was used to generalize most notions of topological nature to an arbitrary category $\underline{X}$. We refer the interested reader to the books *Categorical Closure Operators* by G. Castellini [3] and *Categorical Structure of Closure Operators* by Dikranjan-

Tholen [4]. The purpose of this thesis is to try a similar approach but by using a "dual" notion of interior operator instead. For the time being we will confine ourselves to the category Top of topological spaces.

A general notion of interior operator on an arbitrary category was introduced by S.J.R. Voster [6]. We specialize it here to the category of topological spaces, Top.

Definition 2.1

An *interior operator* I on Top is a family of functions $\{i_X\}_{X \in Top}$ on the subset lattices of Top with the following properties for every $X \in Top$:

Contractiveness: For $M \subseteq X$ then $i_X(M) \subseteq M$.

Order Preservation: For each pair of subsets M_1, M_2 of X with $M_1 \subseteq M_2$ then $i_X(M_1) \subseteq i_X(M_2)$.

Continuity: For every continuous function $f : X \to Y$ and any subset $M \subseteq Y$ then $f^{-1}(i_Y(M)) \subseteq i_X(f^{-1}(M))$.

The subscript X in $i_X(M)$ is omitted whenever no confusion is possible.

Examples of Interior Operators

Here we present a few examples of interior operators. Let $X \in Top$:

Example 2.1

For $M \subseteq X$ define $k_X(M) = \bigcup\{O \text{ open in } X \text{ with } O \subseteq M\}$. That is, $k_X(M)$ is the union of all open subsets O of X that are contained in M.

Proof. Clearly by definition we have that $k_X(M) \subseteq M$. Now for order preservation, let $M_1 \subseteq M_2$ be subsets of X. Then $k_X(M_1)$ is the union of all open sets O_1 contained in M_1 and by transitivity this union is also contained in M_2. Since $k_X(M_2) = \bigcup\{O$ open in X with $O \subseteq M_2\}$ we have that already all the open sets inside M_1 are included and therefore $k_X(M_1) \subseteq k_X(M_2)$. Lastly, let $f\colon X \to Y$ be a continuous function and let $M \subseteq Y$. Then, $f^{-1}(k_X(M)) = f^{-1}(\bigcup\{O$ open in Y with $O \subseteq M\}) = \bigcup\{f^{-1}(O) : O$ open in Y with $O \subseteq M\} \subseteq \bigcup\{K$ open in X: $K \subseteq f^{-1}(M)\} = k_X(f^{-1}(M))$, since inverse images of continuous functions preserves open sets. Therefore we obtain that $f^{-1}(k_X(M)) \subseteq k_X(f^{-1}(M))$. We have just proved that K is indeed an interior operator. $\square$

Example 2.2

For $M \subseteq X$ define $q_X(M) = \bigcup\{C$ clopen in $X : C \subseteq M\}$, namely $q_X(M)$ is the union of all clopen (closed and open) subsets C of X that are contained in M. This is known as the clopen interior operator.

Proof. By definition we have that $q_X(M) \subseteq M$. Let $M_1 \subseteq M_2 \subseteq X$, then $q_X(M_1) = \bigcup\{C_1$ clopen in $X : C_1 \subseteq M_1\}$. Now, every clopen subset C_1 contained in M_1 is also contained in M_2 and therefore $q_X(M_1) \subseteq q_X(M_2)$. To check continuity, let $f : X \to Y$ be a continuous function and let $M \subseteq Y$. Then, $f^{-1}(q_X(M)) = f^{-1}(\bigcup\{C$ clopen in $Y : C \subseteq M\}) = \bigcup\{f^{-1}(C)$ clopen in $X : f^{-1}(C) \subseteq f^{-1}(M)\} \subseteq \bigcup\{K$ clopen in $X : K \subseteq f^{-1}(M)\} = q_X(f^{-1}(M))$. Here we have used the fact that open and closed subsets are preserved by inverse images of continuous functions. Therefore Q is an interior operator in Top. $\square$

Example 2.3

For $M \subseteq X$, define $h_{_X}(M) = \bigcup\{C \text{ closed in } X : C \subseteq M\}$.

Proof. The proof of this case can be obtained from the one in Example 2.1 by replacing "open" with "closed" since closed subsets are preserved by inverse images of continuous functions. Therefore, we omit the details. $\square$

Example 2.4

For $M \subseteq X$, define $l_{_X}(M) = \{x \in X : C_x \subseteq M\}$ where C_x is the connected component of x in X.

Proof. First we have that $l_{_X}(M) \subseteq M$ since every element x belongs to its connected component. Next, let $M_1 \subseteq M_2 \subseteq X$. Then, we have that $l_{_X}(M_1) \subseteq l_{_X}(M_2)$ because for any $x \in l_{_X}(M_1)$, $C_x \subseteq M_1 \subseteq M_2$. Now, let $f : X \to Y$ be a continuous function and let N be a subset of the topological space Y. If $x \in f^{-1}(l_{_Y}(N))$ then, $f(x) \in l_{_Y}(N)$ and so $C_{f(x)} \subseteq N$. We wish to conclude that $x \in l_{_X}(f^{-1}(N))$. Suppose this is not the case, then $C_{_X} \bigcap (X - f^{-1}(N)) \neq \phi$ and consequently $f(C_{_X}) \bigcap (Y - N) \neq \phi$. Clearly this is true otherwise if $f(C_{_X}) \subseteq N$ then $C_{_X} \subseteq f^{-1}(f(C_{_X})) \subseteq f^{-1}(N)$, which is a contradiction. Now, continuity of f implies that $f(C_{_X}) \subseteq C_{f(x)}$ and consequently $C_{f(x)} \bigcap (Y - N) \neq \phi$ which contradicts the fact that $C_{f(x)} \subseteq N$. Therefore $x \in l_{_X}(f^{-1}(N))$ and we conclude that $f^{-1}(l_{_Y}(N)) \subseteq l_{_X}(f^{-1}(N))$. Therefore L is an interior operator in Top. $\square$

Example 2.5

For $M \subseteq X$, define $\theta_{_X}(M) = \{x \in M : \exists$ a nbhd U_x of x such that $\overline{U_x} \subset M\}$. Here, $\overline{U_x}$ denotes the usual kuratowski closure of the neighborhood U_x.

Proof. The first two conditions are straightforward therefore we present the proof for the last one. Let $f : X \to Y$ be a continuous function and let $N \subseteq Y$. We have that: $f^{-1}(\theta_{_Y}(N)) = f^{-1}(\{y \in N : \exists$ a nbhd U_y of y: $\overline{U_y} \subseteq N\})$. Now, if $x \in f^{-1}(\theta_{_Y}(N))$, then $f(x) \in \theta_{_Y}(N)$ that is, there is a neighborhood $U_{_{f(x)}}$ such that $\overline{U_{_{f(x)}}} \subseteq N$. Consequently, $x \in f^{-1}(\overline{U_{_{f(x)}}}) \subseteq f^{-1}(N)$. This implies that $x \in \theta_{_X}(f^{-1}(N))$, since $f^{-1}(U_{_{f(x)}})$ is a neighborhood of x and $f^{-1}(\overline{U_{_{f(x)}}}) \supset \overline{f^{-1}(U_{_{f(x)}})}$ since $f^{-1}(\overline{U_{_{f(x)}}})$ is closed. Hence, $f^{-1}(\theta_{_Y}(N)) \subseteq \theta_{_X}(f^{-1}(N))$. Thus, Θ is an interior operator. □

We denote the collection of all interior operators on Top by $IN(Top)$ preordered as follows: $I \sqsubseteq J$ if $i_{_X}(M) \leq j_{_X}(M)$ for all $M \subseteq X \in Top$.

The following propositions show that arbitrary suprema and infima exist in $IN(Top)$. Let $\{I_k\}_{k\in K}$ be a family of interior operators belonging to $IN(Top)$.

Proposition 2.1

For every $M \subseteq X \in Top$ define $\bigwedge_{k\in K} I_k$ as follows $i_{\bigwedge_{I_k}}(M) = \bigcap_{_{k\in K}} i_k(M)$. Then, $\bigwedge_k I_k$ belongs to $IN(Top)$ and is the infimum of $\{I_k\}_{k\in K}$

Proof. We prove that $\bigwedge_k I_k$ is an interior operator. Let $M \subseteq X$, then $i_{\bigwedge_{I_k}}(M) = \bigcap_{_{k\in K}} i_k(M)$ with $I_k \in IN(Top)$. Then, since for every $k \in K$, I_k is an interior operator we have that $i_k(M) \subseteq M$, $\forall k \in K$. Consequently we have that $\bigcap_{k\in K} i_k(M) \subseteq i_k(M) \subseteq M$ and therefore $i_{\bigwedge_{I_k}}(M) \subseteq M$. Now let $M_1 \subseteq M_2 \subseteq X$. Since for every $k \in K$, $I_k \in IN(Top)$, we have that $i_k(M_1) \subseteq i_k(M_2)$, it follows that $\bigcap_{k\in K} i_k(M_1) \subseteq \bigcap_{k\in K} i_k(M_2)$ and so $i_{\bigwedge_{I_k}}(M_1) \subseteq i_{\bigwedge_{I_k}}(M_2)$. Lastly, let $f : X \to Y$ be a continuous function and let $M \subseteq Y$. Since for all $k \in K$, $f^{-1}(i_k(M)) \subseteq i_k(f^{-1}(M))$ then $f^{-1}(i_{\bigwedge_{I_k}}(M)) = f^{-1}(\bigcap_{k\in K} i_k(M)) = \bigcap_{k\in K} f^{-1}(i_k(M)) \subseteq \bigcap_{k\in K} i_k(f^{-1}(M)) = i_{\bigwedge_{I_k}}(f^{-1}(M))$. Therefore $\bigwedge_{k\in K} I_k \in IN(Top)$.

Let $I_\alpha \in IN(Top)$ be such that for $M \subseteq X$ it satisfies that for all $j \in K$, $i_\alpha(M) \subseteq i_j(M)$. This means that I_α is a lower bound of the family $\{I_k\}$, and by taking intersection yields $i_\alpha(M) \subseteq \bigcap_{j\in K} i_j(M) = i_{\bigwedge I_k}(M)$. Therefore $\bigwedge I_k$ is the greatest lower bound, i.e. the infimum, of the family $\{I_k\}_{k\in K}$. $\square$

Proposition 2.2

For every $M \subseteq X \in Top$, define $\bigvee_{k\in K} I_k$ as follows: $i_{\bigvee I_k}(M) = \bigcup_{k\in K} i_k(M)$. Then, $\bigvee_k I_k$ belongs to $IN(Top)$ and is the supremum of the family $\{I_k\}_{k\in K}$.

Proof. Let $M \subseteq X$, then $i_{\bigvee I_k}(M) = \bigcup_{k\in K} i_k(M)$. Since for every $k \in K$, $i_k(M) \subseteq M$, it follows that $\bigcup_{k\in K} i_k(M) \subseteq M$. This implies that $i_{\bigvee I_k}(M) \subseteq M$. Now if we have $M_1 \subseteq M_2 \subseteq X$, $i_{\bigvee I_k}(M_1) = \bigcup_{k\in K} i_k(M_1) \subseteq \bigcup_{k\in K} i_k(M_2) = i_{\bigvee I_k}(M_2)$ since $\forall k \in K$, $i_k(M_1) \subseteq i_k(M_2)$ due to the fact that all I_k's are interior operators. Therefore $i_{\bigvee I_k}(M_1) \subseteq i_{\bigvee I_k}(M_2)$. Finally, let $f : X \to Y$ be a continuous function with $M \subseteq Y$. Then $f^{-1}(i_{\bigvee I_k}(M)) = f^{-1}(\bigcup_{k\in K} i_k(M)) = \bigcup_{k\in K} f^{-1}(i_k(M)) \subseteq \bigcup_{k\in K} i_k(f^{-1}(M)) = i_{\bigvee I_k}(f^{-1}(M))$. Since $\bigvee_{k\in K} I_k$ satisfies all conditions of an interior operator we conclude that $\bigvee_{k\in K} I_k \in IN(Top)$.

Now to prove $\bigvee_{k\in K} I_k$ is the supremum of the family $\{I_k\}_{k\in K}$, let $I_\beta \in IN(Top)$ be such that for every $j \in K$ and $M \subseteq X$, $i_j(M) \subseteq i_\beta(M)$. Then by taking the union we obtain that $\bigcup_{j\in K} i_j(M) \subseteq i_\beta(M)$ implying that $i_{\bigvee I_k}(M) \subseteq i_\beta(M)$. Therefore $\bigvee_{k\in K} I_k$ is the least upper bound, or the supremum, of the family $\{I_k\}_{k\in K}$. □

Definition 2.2

Let I be an interior operator. Then for $M \subseteq X$ we say:

- M is I-open if $i_X(M) = M$
- M is I-thin if $i_X(M) = \phi$
- X is I-discrete if for every $M \subseteq X$, $i_X(M) = M$
- X is I-indiscrete if for every proper subset $M \subset X$, $i_X(M) = \phi$

Proposition 2.3

Let $f : X \to Y$ be a continuous function and let I be an interior operator on *Top*. If $N \subseteq Y$ is I-open then so is $f^{-1}(N)$.

Proof. We prove that $f^{-1}(N)$ is I-open namely that $i_X(f^{-1}(N)) = f^{-1}(N)$. Since N is I-open and f is continuous we have that $f^{-1}(N) = f^{-1}(i_Y(N)) \subseteq i_X(f^{-1}(N))$ and also $i_X(f^{-1}(N)) \subseteq f^{-1}(N)$ by contractiveness of I. Therefore, $f^{-1}(N) = i_X(f^{-1}(N))$ implying that $f^{-1}(N)$ is I-open whenever N is I-open. □

Chapter 3

Connectedness and Disconnectedness

In this chapter we introduce the notions of connectedness and disconnectedness with respect to an interior operator and study their main properties.

We recall that a function $f : X \to Y$ is said to be constant if $\forall x_1, x_2 \in X$, $f(x_1) = f(x_2)$. Notice that whenever $X \neq \phi$, this is equivalent to $f(X) = y_0 \in Y$.

Definition 3.1

Given an interior operator I, we say that:

1. $X \in Top$ is I-connected if every continuous function from X into an I-discrete topological space Y is constant.

2. $X \in Top$ is I-disconnected if for every continuous function from an I-indiscrete topological space Y into X is constant.

Next we present some properties of I-connectedness and I-disconnectedness.

Definition 3.2

A subset M of $X \in Top$ is I-dense in X if $i_X(X - M) = \phi$.

Properties of I-connectedness

Proposition 3.1

Let $M \neq \phi$ be I-dense in $X \in Top$. If M is I-connected then so is X.

Proof. Let $f : X \to Y$ be a continuous function with Y I-discrete. We have by assumption that $f(M) = y_0 \in Y$. Now, let $x \in X - M$ and assume that $f(x) = y \neq y_0$. Then, $f^{-1}(Y - \{y_0\}) = f^{-1}(i_{_Y}(Y - \{y_0\}) \subseteq i_{_X}(f^{-1}(Y - \{y_0\})) \subseteq i_{_X}(X - M) = \phi$, which is a contradiction. Hence, f is constant and so X is I-connected. □

Proposition 3.2

Let $\{M_i\}_{i \in I}$ be a family of subspaces of $X \in Top$ such that $\bigcap_{i \in I} M_i \neq \phi$. If each M_i is I-connected then so is $\bigcup_{i \in I} M_i$.

Proof. Let $f : \bigcup_{i \in I} M_i \to Y$ be a continuous function with Y an I-discrete topological space. Since each M_i is I-connected, we have that the restriction of f to each M_i is constant into $y_i \in Y$. Now , let $i_0, i \in I$ with $i_0 \neq i$. We have that $f(M_{i_0}) = y_{i_0} \in Y$ and $f(M_i) = y_i \in Y$. By hypothesis, there exists $x_0 \in \bigcap_{i \in I} M_i$. In particular, $x_0 \in M_{i_0} \cap M_i$ and so $y_{i_0} = f(x_0) = y_i$. Since this is true for every $i \in I$ such that $i \neq i_0$ we conclude that f is constant, that is $\bigcup_{i \in I} M_i$ is I-connected. □

Proposition 3.3

Let $f : X \to Y$ be a surjective continuous function. If X is I-connected then so is Y.

Proof. First we observe that if $X = \phi$ then so is Y and the statement is true. Now let $X \neq \phi$ and let $g : Y \to Z$ be a continuous function with Z an I-discrete topological space. Since $g \circ f$ is continuous and X is I-connected we have that $g \circ f$ is constant, that is $g(Y) = g(f(X)) = z_0 \in Z$. Hence g is constant and so Y is I-connected. $\square$

Properties of I-disconnectedness

Proposition 3.4

Let $M \subseteq X$ in Top. If X is I-disconnected, then so is M.

Proof. Let $m : M \to X$ denote the inclusion of the subspace M into X and let $f : Y \to M$ be a continuous function with Y I-indiscrete. As it can be seen, the fact that $m \circ f$ is constant and that m is injective implies that f is constant, therefore M is I-disconnected. $\square$

Proposition 3.5

The product of a family of non-empty I-disconnected topological spaces is I-disconnected.

Proof. Let $\{X_i\}_{i \in I}$ denote a family of I-disconnected topological spaces, let $f : Y \to \prod_{i \in I} X_i$ be a continuous function with Y I-indiscrete and let $(\pi_i)_{i \in I}$ denote the usual projections. Since each X_i is I-disconnected, we have that $\pi_i \circ f$ is constant for every $i \in I$. Hence $\forall i \in I$ there exists $x_{i_0} \in X_i$ such that $(\pi_i \circ f)(x) = x_{i_0}$. This implies that $f(x) = \{x_{i_0}\}_{i \in I}$, that is f is constant and so $\prod_{i \in I} X_i$ is I-disconnected. $\square$

Chapter 4

A factorization of the connected-disconnected Galois connection

In this chapter we are going to construct two Galois connections between the class of all interior operators in Top ($IN(Top)$) and the collection of all subclasses of Top ($S(Top)$) ordered via inclusion. Next we will describe their composition.

Discrete Galois connection

Proposition 4.1

The function $D : IN(Top) \to S(Top)$ defined by

$D(I) = \{X \in Top : X \text{ is } I\text{-discrete}\}$ preserves infima.

Proof. We first prove that D is order-preserving. So, let $I_1 \sqsubseteq I_2$ and let $M \subseteq X \in Top$. $i_1(M) \subseteq i_2(M) \subseteq M$ implies that if M is I_1-open then it is also I_2-open. Consequently, if X is I_1-discrete then X is also I_2-discrete that is $D(I_1) \subseteq D(I_2)$. Now let $\{I_k\}_{k\in K}$ be a family of interior operators in Top. Since $\bigwedge I_k \sqsubseteq I_k$ for each $k \in K$, order preservation of D implies that $D(\bigwedge I_k) \subseteq D(I_k), \forall k \in K$ and so $D(\bigwedge I_k) \subseteq \bigcap_{k\in K} D(I_k)$. Now let $X \in \bigcap D(I_k)$. Then, $\forall\ M \subseteq X$, $i_k(M) = M$, $\forall k \in K$, and so by Proposition 2.1, $i_{\bigwedge_{I_k}}(M) = \bigcap i_k(M) = \bigcap_{k\in K} M = M$. Hence, $X \in D(\bigwedge I_k)$. Since $D(\bigwedge I_k) \subseteq \bigcap_{k\in K} D(I_k)$ and $\bigcap_{k\in K} D(I_k) \subseteq D(\bigwedge I_k)$, we conclude that $\bigcap_{k\in K} D(I_k) = D(\bigwedge I_k)$, implying that the function D preserves infima. $\square$

Let $IN(Top)^{op}$ and $S(Top)^{op}$ denote the same classes as $IN(Top)$ and $S(Top)$ but with the order reversed. In particular, $\underline{A} \leq \underline{B}$ in $S(Top)^{op}$ means $\underline{B} \subseteq \underline{A}$. As a consequence of Proposition 4.1 we have that the function $D^{op} : IN(Top)^{op} \to S(Top)^{op}$ defined by $D^{op}(I) = D(I)$ preserves suprema. Then, from Proposition 1.3, there is a function $T : S(Top)^{op} \to IN(Top)^{op}$, defined for $\underline{A} \in S(Top)^{op}$ by $T(\underline{A}) = \bigvee\{I \in IN(Top) : D(I) \leq \underline{A}\}$ such that the diagram

$$\begin{tikzcd} IN(Top)^{op} \arrow[r, shift left, "D"] & S(Top)^{op} \arrow[l, shift left, "T"] \end{tikzcd}$$

forms a Galois connection.

The above definition of the function T is too general to be useful in concrete situations. However, we have the following more practical characterization of T.

Proposition 4.2

For every subclass $\underline{A}$ of Top and subset M of $X \in Top$, we have that $i_{T(A)}(M) = \bigcup\{f^{-1}(N) \subseteq M | f : X \to Y \text{ continuous}, Y \in \underline{A}, N \subseteq Y\}$.

Proof. Define $i^*(M) = \bigcup\{f^{-1}(N) \subseteq M | f : X \to Y$ continuous, $Y \in \underset{\sim}{A}, N \subseteq Y\}$. We are going to show that the assignment I^* that to each subset M of X associates $i^*(M)$ is an interior operator that coincides with $T(\underset{\sim}{A})$. Firstly for $M \subseteq X$ we have that $i^*(M) \subseteq M$ because it is the union of all inverse images contained in M. Now if $M_1, M_2 \subseteq X$ with $M_1 \subseteq M_2$ we have by mere inspection that $i^*(M_1) \subseteq i^*(M_2)$, since inverse images contained in M_1 are also contained in M_2 by transitivity.

Let $f : Z \to X$ be a continuous function and let $M \subseteq X$. Now, $f^{-1}(i^*(M)) = f^{-1}(\bigcup\{g^{-1}(N) | g : X \to Y$ continuous, $Y \in \underset{\sim}{A}, N \subseteq Y$ and $g^{-1}(N) \subseteq M\}) = \bigcup\{f^{-1}(g^{-1}(N)) | g : X \to Y$ is continuous, $Y \in \underset{\sim}{A}, N \subseteq Y$ and $g^{-1}(N) \subseteq M\} = \bigcup\{(g \circ f)^{-1}(N) | g : X \to Y$ is continuous, $Y \in \underset{\sim}{A}, N \subseteq Y$ and $g^{-1}(N) \subseteq M\}$, since inverse images and unions commute . Notice that $(g \circ f)^{-1}(N) = f^{-1}(g^{-1}(N)) \subseteq f^{-1}(M)$ and so it occurs in the construction of $i^*(f^{-1}(M))$. Moreover, since not all continuous functions from Z to Y are of the form $g \circ f$ we obtain that $f^{-1}(i^*(M)) \subseteq \bigcup\{l^{-1}(N) | l : Z \to Y$ continuous, $Y \in \underset{\sim}{A}$, $N \subseteq Y$ and $l^{-1}(N) \subseteq f^{-1}(M)\} = i^*(f^{-1}(M))$.Therefore we have proved that I^* is indeed an interior operator.

Let us now verify that $T(\underset{\sim}{A}) = I^*$. Let $\underset{\sim}{A} \in S(Top)$ and let $X \in \underset{\sim}{A}$. The existence of the identity function $id_X : X \to X$ implies that for every subset N of X, $i^*(N) = N$. Consequently $\underset{\sim}{A} \subseteq D(I^*)$, in $S(Top)$ which means $D(I^*) \leq \underset{\sim}{A}$ in $S(Top)^{op}$. Hence by definition of T, this implies that $I^* \sqsubseteq T(\underset{\sim}{A})$ in $IN(Top)^{op}$. Now, since D and T form a Galois connection, we have that $D(T(\underset{\sim}{A})) \leq \underset{\sim}{A}$ in $S(Top)^{op}$, namely if we apply T and then D the image shrinks. Now, in $S(Top)$ this means that $A \subseteq D(T(\underset{\sim}{A}))$ and this tells us that the "objects" in $\underline{A}$ are discrete with respect to $T(\underline{A})$. Consequently if $N \subseteq Y \in \underline{A}$, we have that N is $T(\underline{A})$-open. From Proposition 2.3 we have that $f^{-1}(N)$ is $T(\underline{A})$-open too. Moreover if $f^{-1}(N) \subseteq M$, then $f^{-1}(N) = i_{T(A)}(f^{-1}(N)) \subseteq i_{T(A)}(M)$. Now since $i^*(M) = \bigcup\{f^{-1}(N) \subseteq M | f : X \to Y$ continuous, $Y \in \underline{A}, N \subseteq Y\} \subseteq i_{T(A)}(M)$ in $S(Top)$, applying the *op* we obtain that $T(\underline{A}) \sqsubseteq I^*$ in $IN(Top)^{op}$ which together with $I^* \sqsubseteq T(\underline{A})$ yields $I^* = T(\underline{A})$. □

Indiscrete Galois connection

Proposition 4.3

The function $C : IN(Top) \to S(Top)^{op}$ defined by

$C(I) = \{X \in Top :\ X \text{ is } I\text{-indiscrete}\}$ preserves suprema.

Proof. First we prove that C preserves the order. Let $I_1 \sqsubseteq I_2$, and let $\phi \neq M \subset X \in Top$. Then, $i_1(M) \subseteq i_2(M)$ implies that if M is I_2-thin then it is also I_1-thin. Consequently if X is I_2-indiscrete then X is also I_1-indiscrete and so $C(I_2) \subseteq C(I_1)$, that means $C(I_1) \leq C(I_2)$ in $S(Top)^{op}$. Next we show that C preserves suprema. Let $\{I_k\}_{k \in K}$ be a family of interior operators. Since $I_k \sqsubseteq \bigvee I_k$ for each $k \in K$, we have that by order preservation of C, $C(\bigvee I_k) \subseteq C(I_k)$ for each $k \in K$ and so $C(\bigvee I_k) \subseteq \bigcap_{k \in K} C(I_k) = \bigvee C(I_k)$ in $S(Top)^{op}$. Now if $X \in \bigvee C(I_k) = \bigcap_{k \in K} C(I_k)$, then $X \in C(I_k)$ for all $k \in K$, and so X is I_k-indiscrete for every $k \in K$. Then for all $M \subset X$ we have $i_k(M) = \phi, \forall k \in K$. Therefore $i_{\bigvee I_k}(M) = \bigcup_{k \in K} i_k(M) = \phi$ implies that $X \in C(\bigvee I_k)$ that is, $\bigvee C(I_k) \subseteq C(\bigvee I_k)$. Now if $X \in C(\bigvee I_k)$ then we have that $\forall M \subseteq X$, $i_{\bigvee I_k}(M) = \phi$ that implies $I_k(M) = \phi, \forall k \in K$. Hence we have that $X \in \bigcap C(I_k) = \bigvee C(I_k)$ in $S(Top)^{op}$. This, together with the other containment, implies that $\bigvee C(I_k) = C(\bigvee I_k)$ that is, the function C preserves suprema. $\square$

As a consequence of the above proposition the function $C^{op} : IN(Top)^{op} \to S(Top)$ defined by $C^{op}(I) = C(I)$ preserves infima and consequently from Proposition 1.4, there exists a function $G : S(Top) \to IN(Top)^{op}$, defined for $\underline{B} \subseteq Top$ as follows: $G(\underline{B}) = \bigwedge\{I \in IN(Top)^{op} : C(I) \supseteq \underline{B}\}$ such that the diagram

$$S(Top) \underset{C}{\overset{G}{\rightleftarrows}} IN(Top)^{op}$$

is a Galois Connection.

Next we give a more practical characterization of the function G.

Proposition 4.4

For every subclass $\underline{B}$ of Top and subset M of $Y \in Top$, we have that $i_{G(B)}(M) = \bigcup\{N \subseteq M : \forall$ continuous function $f : X \to Y$ with $X \in \underline{B}$ and $f^{-1}(M) \neq X,\ f^{-1}(N) = \phi\}$.

Proof. Set $i'(M) = \bigcup\{N \subseteq M | \forall$ continuous function $f : X \to Y$ with $X \in \underline{B}$ and $f^{-1}(M) \neq X,\ f^{-1}(N) = \phi\}$. We will show that the assignment I' that to each subset M of Y associates $i'(M)$ is an interior operator on Top. Let $M \subseteq Y$, then $i'(M) \subseteq M$ by definition. Now let $M_1 \subseteq M_2 \subseteq Y$ and let $N \subseteq M_1$ occur in the construction of $i'(M_1)$. If $f : X \to Y$ is a continuous function that satisfies $X \in \underline{B}$ and $f^{-1}(M_2) \neq X$, then from $f^{-1}(M_1) \subseteq f^{-1}(M_2) \neq X$, we conclude that $f^{-1}(N) = \phi$. Thus, N also occurs in the construction of $i'(M_2)$ and so we have that $i'(M_1) \subseteq i'(M_2)$.

Next we show the continuity condition. So, let $Y \xrightarrow{g} Z$ be a continuous function and let $M \subseteq Z$. We have that $i'(g^{-1}(M)) = \bigcup\{H \subseteq g^{-1}(M) : \forall$ continuous $X \xrightarrow{h} Y$ with $X \in \underline{B}$ and $h^{-1}(g^{-1}(M)) \neq X$, $h^{-1}(H) = \phi\} \supseteq \bigcup\{g^{-1}(N) : N \subseteq M$ and $\forall$ continuous $X \xrightarrow{h} Y$ with $X \in \underline{B}$ and $h^{-1}(g^{-1}(N)) \neq X$, $h^{-1}(g^{-1}(N)) = \phi\} \supseteq \bigcup\{g^{-1}(N) : N \subseteq M$ and $\forall$ continuous $X \xrightarrow{f} Z$ with $X \in \underline{B}$ and $f^{-1}(N) \neq X$, $f^{-1}(N) = \phi\} = g^{-1}(\bigcup\{N \subseteq M$ and $\forall$ continuous $X \xrightarrow{f} Z$ with $X \in \underline{B}$ and $f^{-1}(M) \neq X$, $f^{-1}(N) = \phi\} = g^{-1}(i'(M))$. Notice that in the last containment above we have used the fact that not every function $f : X \to Z$ factors through g and h. Hence I' is an interior operator.

Next we show that $G(\underline{B})$ coincides with I'. First notice that if $X \in \underline{B}$ then for every proper subset $M \subset X$, the existence of $X \xrightarrow{id_X} X$, implies that the only subset $N \subseteq M$ that satisfies $id_X^{-1}(N) = \phi$ is $N = \phi$. Consequently $i'(M) = \phi$, that is $C(I') \supseteq \underline{B}$ and so $I' \sqsubseteq G(\underline{B})$ in $IN(Top)$. On the other hand, for every continuous function $X \xrightarrow{f} Y$ with $X \in \underline{B}$ and $M \subseteq Y$ such that $f^{-1}(M) \neq X$, we have that $f^{-1}(i_{G(B)}(M)) \subseteq i_{G(B)}(f^{-1}(M)) = \phi$. From the definition of I' follows that $i_{G(B)}(M) \subseteq i'(M)$ that is $G(\underline{B}) \sqsubseteq I'$ in $IN(Top)$. Hence, we conclude that $I' = G(\underline{B})$. $\square$

Description of the Composition

The following proposition presents a very classical result for which we omit the proof (cf. [5]).

Proposition 4.5

Let $S(Top) \xrightarrow{\Delta} S(Top)^{op}$ and $S(Top)^{op} \xrightarrow{\nabla} S(Top)$ be defined as follows:

$\Delta(\underline{B}) = \{Y \in Top | \forall$ continuous $X \xrightarrow{f} Y$, $X \in \underline{B}, f$ is constant$\}$,

$\nabla(\underline{A}) = \{X \in Top | \forall$ continuous $X \xrightarrow{g} Y$, $Y \in \underline{A}$, g is constant$\}$.

Then, $S(Top) \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} S(Top)^{op}$ is a Galois connection.

The above proposition is known in the literature as the left-right constant Galois connection. Next we present our main result.

Theorem 4.1

The following diagram of Galois connections

$$\begin{tikzcd} S(Top) \arrow[rr, "\Delta", "\nabla"'] \arrow[dr, "G", "C"'] & & S(Top)^{op} \\ & IN(Top)^{op} \arrow[ur, "D", "T"'] & \end{tikzcd}$$

commutes.

Proof. We start by showing that the function $\Delta : S(Top) \to S(Top)^{op}$ coincides with the composition of the functions D and G. Now, for $\underline{B} \in Top$, let $M \subseteq Y \in \Delta(\underline{B})$ and let $f : X \to Y$ be a continuous function with $X \in \underline{B}$ such that $f^{-1}(M) \neq X$. Then, since f is constant and $f^{-1}(M) \neq X$, we must have that $f^{-1}(M) = \phi$. Consequently, $i_{G(B)}(M) = M$ and so $Y \in D(G(\underline{B}))$. Conversely, let $Y \in D(G(\underline{B}))$ and let $f : X \to Y$ be a continuous function with $X \in \underline{B}$. First we observe that if X is empty then for every $x, y \in X$, $f(x) = f(y)$ is true by default. So, let X be nonempty and for $x \in X$, take $M = \{f(x)\} \subseteq Y$. Now, notice that if $f^{-1}(M) = X$, since $f^{-1}(M) = f^{-1}(\{f(x)\})$, we conclude that f is constant. Now suppose that $f^{-1}(M) \neq X$.

By hypothesis, $i_{G(B)}(M) = M$ and so $f^{-1}(\{f(x)\}) = f^{-1}(M) = \phi$. However, the fact $x \in f^{-1}(\{f(x)\})$ yields a contradiction. Hence, f is constant and so $Y \in \Delta(\underline{B})$. Finally we conclude that $\Delta = D \circ G$.

Now, since the composition of two Galois connection is a Galois connection as proven in Proposition 1.1 and since the functions in a Galois connection uniquely determine each other (Proposition 1.2) from $\Delta = D \circ G$ we conclude that $\nabla = C \circ T$. $\square$

Arhangel'skii and Wiegant in [2] introduced notions of connectedness and disconnectedness with respect to a class of topological spaces. Precisely, given a class $\underline{A}$ of topological spaces, a topological space X is $\underline{A}$-connected if any continuous function $X \xrightarrow{f} Y$, with $Y \in \underline{A}$ is constant. Similarly for a class $\underline{B}$ of topological spaces, X is $\underline{B}$-disconnected if every continuous function $Y \xrightarrow{f} X$, with $Y \in \underline{B}$ is constant. Many properties of these notions and related results are presented in [2]. It is important to observe that the strength of the above theorem is that it allows us to relate our notions of connectedness and disconnectedness with respect to an interior operator to Arhangel'skii and Wiegandt notions of connectedness and disconnectedness with respect to a subclass of topological spaces. More precisely, a topological space X is I-connected if and only if $X \in \nabla(D(I)) = C(T(D(I)))$. A consequence of this result is that if X is I-connected then it is $D(I)$-connected in the sense of [2]. Conversely, for a subclass A of topological spaces, if X is $\underline{A}$-connected then $X \in \nabla(\underline{A}) = \nabla(\Delta(\nabla(\underline{A}))) = \nabla(\Delta(C(T(\underline{A})))) = \nabla(D(G(C(T(\underline{A})))))$. This means that X is I-connected with $I = G(C(T(\underline{A})))$. A similar assertion can be made about the notions of I-disconnectedness and $\underline{\mathrm{B}}$-disconnectedness.

We complete this work with a few examples that illustrate the theory.

Example 4.1

Consider the interior operator K induced by the topology. Clearly, $D(K)$ coincides with the subcategory $Discr$ of discrete topological spaces. $T(D(K))$ is the clopen interior operator. In fact, we have that if $M \subseteq X \in Top$ and $f : X \to Y$ is a continuous function with Y discrete then $f^{-1}(N)$ is clopen for every $N \subseteq Y$. On the other hand, let $C \subseteq M$ be clopen and let D denote the two-point discrete topological space and $f : X \to D$ defined by $f(x) = 0$ for $x \in C$ and $f(x) = 1$ otherwise. Hence, $C = f^{-1}(\{0\})$. So from Proposition 4.2 we conclude that $T(D(K))$ is the clopen interior operator. (cf. Example 2.2).

Now, if X is an indiscrete topological space, then clearly $X \in C(K)$. So, let $X \in C(K)$ and suppose that X is not indiscrete, that is X has a non-empty proper open subset U. Then, $k(U) = U \neq \phi$ implies that $X \notin C(K)$, which is a contradiction. Hence, we conclude that $C(K) = Ind$, the subcategory of indiscrete topological spaces.

We conclude that the K-connected topological spaces are the usual connected topological spaces and the K-disconnected topological spaces are the T_0 topological spaces since Ind and Top_0 are corresponding fixed points of the Galois connection (Δ, ∇) (cf. [2]).

Example 4.2

Let Q be the clopen interior operator (cf. Example 2.2). Clearly, $D(Q) = Discr$ and consequently from Example 4.1, Q and $Discr$ are fixed points of the Galois connection (D, T). So, Q-connected means connected topological spaces. Now, we see that $C(Q) = Conn$, that is the subcategory of connected topological spaces. As a matter of fact, if X is connected then it clearly belongs to $C(Q)$ since the only proper clopen subset of X is ϕ. On the other hand, let $X \in C(Q)$ and assume X is not connected. Then, there is a non-empty proper clopen subset $U \subseteq X$. Consequently, $q(N) = N \neq \phi$, which is a contradiction. Hence, the I-disconnected topological spaces coincide with the subcategory $Tdisc$ of all totally disconnected topological space, i.e., those topological spaces whose components are single points.

Example 4.3

Let H be the closed interior operator (cf. Example 2.3) and let $X \in D(H) = \{X : \forall M \subseteq X,\ M \text{ is } H\text{-open}\}$. In particular, for $M = \{x\}$ we obtain that $\{x\}$ must be a union of closed subsets which implies that $\{x\}$ must be closed, i.e., X is a T_1 topological space. On the other hand, if X is T_1, its points are closed and since every subset M of X is the union of all its points we conclude that M is H-open. Hence, $D(H) = Top_1$. Consequently, H-connected means absolutely connected topological space (cf. [2]). Now, if X is indiscrete then it clearly belongs to $C(H)$. On the other hand, if $X \in C(H)$ and is not indiscrete, then there is a non-empty proper closed subset $N \subseteq X$. Consequently, $h_X(M) = M \neq \phi$, which is a contradiction. Hence, as in Example 4.1, the H-disconnected topological spaces are the T_0

topological spaces.

Example 4.4

Let L be the interior operator of Example 2.4, that is for $M \subseteq X$, $l_X(M)$ consists of the union of all connected components of X inside M. We are going to see that $D(L) = Tdisc$. First we observe that if X is totally disconnected then it clearly belongs to $D(L)$ since every subset M of X is a union of connected components inside M,i.e., its points. On the other hand, if $X \in D(L)$ and $x \in X$, then $\{x\}$ is L-open. So, for $x \in \{x\}$, its connected component $C_x \subseteq \{x\}$, which clearly implies that $C_x = \{x\}$, i.e., X is totally disconnected. Consequently the L-connected topological spaces are the usual connected topological spaces.

Now we see that $C(L) = Conn$. If X is connected then for any proper subset M of X, no $x \in M$ can possibly satisfy $C_x \subseteq M$ and so $l(M) = \phi$, i.e., $X \in C(L)$. On the other hand, let $\phi \neq X \in C(L)$ (the empty set belongs to both) and let $x_0 \in X$. For every $x \neq x_0$, set $M_x = X - \{x\}$. Since $l_X(M_x) = \phi$ then $C_{x_0} \nsubseteq M_x$ which implies that $x \in C_{x_0}$. Since this is true for every $x \in X$, we conclude that $C_{x_0} = X$, i.e., X is connected. Hence $C(L) = Conn$ and consequently, the L-disconnected spaces are the totally disconnected topological spaces.

Example 4.5

Let Θ be the interior operator of Example 2.5. Clearly we have that $Discr \subseteq D(\theta)$. Now, if $X \in D(\theta)$, then for every $x \in X$, $\theta_X(\{x\}) = \{x\}$. This implies that there is a neighborhood U_x of $\{x\}$ such that $x \in \overline{U_x} \subseteq \{x\}$. Hence, we conclude that $U_x = x$ and so X is discrete. As a consequence, the

Θ-connected topological spaces are the usual connected topological spaces

We call a topological space X *nowhere separated* if for any two distinct points $x, y \in X$ and neighborhoods U_x and U_y, $U_x \bigcap U_y \neq \phi$. Notice that this class of topological spaces properly contains all indiscrete topological spaces. For instance if X is an infinite set with the cofinite topology, i.e., the open sets are complements of finite subsets, then X is certainly not indiscrete but it is nowhere separated. As a matter of fact, if there would exists neighborhoods U_x, U_y in X such that $U_x \bigcap U_y = \phi$ then we would have that $X = X - \phi = X - (U_x \bigcap U_y) = (X - U_x) \bigcup (X - U_y)$ wich is a contradiction since $X - U_x$ and $X - U_y$ are finite.

Now we are going to show that $C(\theta)$ consists of all nowhere separated topological spaces. So, let $X \in C(\theta)$ and for $x \in X$ consider $M = X - \{x\}$. Since $\theta_{_X}(M) = \phi$, then for every $y \in M$ and neighborhood U_y, we have that $\overline{U_y} \nsubseteq M$. This implies that $x \in \overline{U_y}$, that is for every neighborhood U_x, $U_x \bigcap U_y \neq \phi$ and so, X is nowhere separated. On the other hand, suppose that X is nowhere separated and let M be a proper subset of X. For $x \in M$ and $y \in X - M$, we have that for every pair of neighborhoods U_x, U_y, $U_x \bigcap U_y \neq \phi$. This implies that $y \in \overline{U_x}$ and so $\overline{U_x} \nsubseteq M$, i.e., $x \notin \theta_{_X}(M)$. Hence, $\theta_{_X}(M) = \phi$, that is $X \in C(\Theta)$.

We do not have a characterization for the Θ-disconnected topological spaces.

Chapter 5

Interior operators in Algebra

Even though the definition of interior operator given in 2.1 can be translated into the category *Grp* of groups by simply replacing the word "subset" by "subgroup" and "continuous function" by "homomorphism", we could not find a way to prove Theorem 4.1 in *Grp*. The problem lies in the following fact. Although the two Galois connections (D, T) and (G, C) of Propositions 4.1 and 4.3, respectively, can be defined in *Grp*, we could not find appropriate characterization of the functions T and G that would allow us to prove Theorem 4.1. So this particular problem in *Grp* remains open. Nonetheless we include one example that shows that interior operators exist in the category of groups.

Example 5.1

Let H be a subgroup of a group $G \in Grp$. Define $i(H) = \bigvee\{K \leq H : K \trianglelefteq G\}$ that is $i_G(H)$ consist of the subgroup generated by all normal subgroups of G contained in H. Then the function that to each subgroup H associates the subgroup $i_G(H)$ is an interior operator in *Grp*.

Proof. The first two conditions are straightforward so we just need to verify the continuity condition. So, let $G \xrightarrow{\phi} G'$ be a group homomorphism. Since, the subgroup generated by the family of normal subgroups is normal, we have that $i_{G'}(N) \triangleleft G'$. This, together with the fact that the inverse image of a normal subgroup is normal, yields that $f^{-1}(i_{G'}(N))$ is a normal subgroup of G contained in $f^{-1}(N)$. Consequently, by definition of $i_G(f^{-1}(N))$ we conclude that $f^{-1}(i_{G'}(N)) \leq i_G(f^{-1}(N))$. Hence, all the conditions of interior operators are satisfied. $\square$

Conclusions and Future Work

The work done in this thesis can be summarized as follows.

- A previously introduced notion of interior operator on an arbitrary category was used in Topology to define the notions of connectedness and disconnectedness with respect to an interior operator.

- The main properties of connectedness and disconnectedness with respect to an interior operator were studied and it was shown that they have a similar behavior to classical connectedness and disconnectedness in Topology.

- The notions of discrete and indiscrete objects with respect to an interior operator were introduced and related Galois connections were constructed.

- The left-right constant Galois connection was shown to factor through the discrete and indiscrete Galois connections. This allowed us to relate our notions of connectedness and disconnectedness with respect to an interior operator to existing notions of connectedness and disconnectedness with respect to a subclass of topological spaces.

- Examples that illustrate the theory were found.

The open problems can be divided into two types. The first type consists in trying to define other classical topological notions like separation and compactness with respect to an interior operator and study their properties in the category of topological spaces. The other type consists in trying to extend the above results outside topology.

Bibliography

[1] J. Adámek, H Herrlich, G.E. Strecker, *Abstract and Concrete Categories*, Wiley, New York, 1990.

[2] A.V. Arhangel'skii, R. Wiegandt, "Connectedness and disconnectedness in topology", *Gen. Top. Appl.* 5 (1975) 9-33.

[3] G. Castellini,*Categorical Closure Operators*, Mathematics: Theory and Applications, Birkhäuser Boston, 2003.

[4] D. Dikranjan, W. Tholen, *Categorical Structure of Closure Operators, with Applications to Topology, Algebra and Discrete Mathematics*, Kluwer Academic Publishers, Dordrecht, 1995.

[5] H. Herrlich, *Topologische Reflexionen und Coreflexionen*, L.N.M. 78, Springer, Berlin, 1968.

[6] S.J.R. Voster, "Interior operators in general categories", *Quaest. Mathematicae* 23 (2000), 405-416.