INTERIOR OPERATORS AND T_1 TOPOLOGICAL SPACES

By

Henry Josue Cortez-Portillo

A thesis submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

PURE MATHEMATICS

UNIVERSITY OF PUERTO RICO
MAYAGÜEZ CAMPUS

2018

Approved by:

Reyes M. Ortiz-Albino, Ph.D — Date
Member, Graduate Committee

Juan A. Ortiz-Navarro, Ph.D — Date
Member, Graduate Committee

Gabriele Castellini, Ph.D — Date
President, Graduate Committee

Darik Cruz-Martínez, Esq. LLM — Date
Representative of Graduate Studies

Olgamary Rivera-Marrero, Ph.D — Date
Chairperson of the Department

Abstract of Dissertation Presented to the Graduate School
of the University of Puerto Rico in Partial Fulfillment of the
Requirements for the Degree of Master of Science

INTERIOR OPERATORS AND T_1 TOPOLOGICAL SPACES

By

Henrry Josue Cortez-Portillo

2018

Chair: Gabriele Castellini, Ph.D
Major Department: Mathematical Sciences

A general notion of T_1-separation with respect to an arbitrary interior operator is introduced in the category **Top** of topological spaces. This is done by means of the concept of categorical interior operator. This naturally yields a dual notion of T_1-coseparation. Each of these two notions produces a Galois connection between categorical interior operators in **Top** and subclasses of topological spaces. These two Galois connections are studied and it is shown that their composition can be described as a classical Galois connection defined in terms of the concept of constant function. This can be easily illustrated with a commutative diagram of Galois connections.

Resumen de Disertación Presentado a Escuela Graduada
de la Universidad de Puerto Rico como requisito parcial de los
Requerimientos para el grado de Maestría en Ciencias

OPERADORES DE INTERIOR Y ESPACIOS TOPOLÓGICOS T_1

Por

Henrry Josue Cortez-Portillo

2018

Consejero: Gabriele Castellini, Ph.D
Departamento: Ciencias Matemáticas

Una noción general de separación T_1 con respecto a un operador de interior arbitrario es presentada en la categoría **Top** de los espacios topológicos. Esto es hecho por medio del concepto de operador de interior categórico. Esto naturalmente implica una noción dual de coseparación T_1. Cada una de estas dos nociones produce una conexión de Galois entre los operadores de interior categóricos en **Top** y las subclases de espacios topológicos. Estas dos conexiones de Galois son estudiadas y es mostrado que su composición puede ser descrita como una conexión de Galois definida en términos del concepto de función constante. Esto puede fácilmente ser mostrado con un diagrama conmutativo de conexiones de Galois.

To my mother Teresa Portillo who motivated in me the passion of pursuing my dreams.

ACKNOWLEDGMENTS

I would like to express my appreciation to Dr. Gabriele Castellini for his guidance during the term of my Master's degree at UPRM, without his valuable assistance, this work would not have been completed.

I am also in debt with the Department of Mathematical Sciences at UPRM for the great opportunity to study here.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 1
INTRODUCTION

The notion of interior operator on an arbitrary category was introduced by Vorster in [17] and was satisfactorily used in [8] and [7] in the category of topological spaces. In [8] the notions of discrete and indiscrete objects with respect to an interior operator are introduced and it is shown that these two concepts can be used to construct two Galois connections between interior operators on the category **Top** and subclasses of topological spaces. Furthermore, Galois connections can be composed to create a third one which is characterized by using the notion of constant morphism. This gives rise to a commutative diagram of Galois connections and consequently notions of connectedness and disconnectedness with respect to an interior operator are established. The commutative diagram of Galois connections previously mentioned is used to relate these notions with the ones given by Arhangelskii and Wiegandt in [1]. That shows that at least in topology the notion of interior operator is as good as the notion of closure operator [3] to deal with the concepts of connectedness and disconnectedness.

Now, in [7] the notion of separation with respect to an interior operator through the concept of separator of two continuous functions is studied. The separator is the subset that consists of all the points on which the two functions are different and it is the counterpart in topology of the notion of equalizer of two continuous functions. Productivity and hereditary of this notion of separation are proved and like in [8] the existence of a Galois connection between subclasses of separated objects and interior operators is showed and their fixed points are characterized. Moreover, examples that illustrate the theory are given. It is important to note that as proved in [17], in a category $\mathcal{X}$ with some kind of complementation condition, a bijective correspondence exists between closure operators and interior operators on $\mathcal{X}$. However, as mentioned in that publication, it does not lead to think that in an arbitrary category with complementation such as the category **Top**, each result obtained using the theory of closure operators can be easily translated to an equivalent result for interior operators.

A detailed analysis shows that the theory of interior operators is not the symmetric counterpart of the one of closure operators. For instance, the notion of continuity with respect to a closure operator can be expressed in terms of inverse images and direct images while continuity with respect to an interior operator can be only expressed in terms of inverse images [5]. This creates a difference between closure operators and interior operators even in the basic category **Top**. A bigger implication of this fact is that the functorial property of closure operators heavily used throughout the development of the theory of closure operators simply does not hold for interior operators. And as mentioned in [7], it is this lack of symmetry between the two theories that makes worthwhile to study the theory of interior operators even though the other theory has been widely studied. The reader interested in the categorical theory of closure operators can look at the references [3] and [9].

In this work we are presenting a generalization of the well known concept of T_1 topological space (T_1 separation axiom in topology) but it is done through the framework of interior operators on the category **Top**. First, a notion of T_1-separation with respect to an interior operator I is presented and several examples of T_1 topological spaces are presented. A function T_1 between the class of all interior operators on

Top, namely $IN(\mathbf{Top})$, and the conglomerate of all subclasses of topological spaces $S(\mathbf{Top})$ ordered by inclusion is defined.

This function has the property of order preservation and allows to generate a new function H between $S(\mathbf{Top})$ and $IN(\mathbf{Top})$. In this way a Galois connection is obtained. A Galois connection is a close approximation to a bijection between two classes, so information about both its domain and codomain can be easily obtained. An explicit formula for H is achieved and some examples of interior operators using that function are shown. Moreover, theoretical properties of the notion of T_1-separation with respect to an arbitrary interior operator I are studied and some interesting equivalences of this concept are found. All the results obtained show that the notion of T_1-separation with respect to an interior operator on **Top** fits very well with the theory already known.

On the other hand, the opposite notion of T_1-separation with respect to an interior operator on **Top**, namely T_1-coseparation, is also studied and similarly to the previous work, several examples and properties are provided. Fortunately, the function C_1 that to each interior operator I associates its T_1-coseparated spaces is also order preserving and consequently a new Galois connection is found between $S(\mathbf{Top})$ and $IN(\mathbf{Top})^{op}$ where the superscript *op* means that the order is reversed. In conclusion, the notions here presented, i.e., T_1-separation and T_1-coseparation with respect to an interior operator I provide a good generalization of the T_1-separation axiom in topology. As an interesting additional result, we obtained that the composition of the two previous Galois connection gives a new factorization of the well known classical Galois connection defined in terms of constant morphisms and that also implies that the factorization is not unique.

Finally, we make a brief attempt to introduce the notion of T_1-separation with respect to an interior operator on the category **Grp**, that is, the category of all groups and homomorphisms. Examples of interior operators on **Grp** are shown and similarly to the topological case, a tentative definition of T_1-separated groups with respect to an interior operator is presented. Moreover, some examples for a few interior operators on **Grp** are provided. It is important to observe that unlike the topological case, in the category **Grp** the complements of subgroups are not subgroups and consequently, in order to define the T_1 notion in **Grp**, we use the equivalent notion of the intersection of all the I-open subgroups with respect to an interior operator being trivial. It should also be mentioned that big difficulties were encountered. For example, we have not been able to prove that the function T_1 between the class of all interior operators $IN(\mathbf{Grp})$ and the conglomerate of all subclasses of groups ordered by inclusion $S(\mathbf{Grp})$ preserves infima. Since it does not seem to be an easy task to prove or disprove this result, we have left it as an open problem for future work.

CHAPTER 2
BASIC TOPOLOGY

In the following sections we will go through fundamental concepts and properties that will be used throughout this thesis.

2.1 The interior of a set and the separation axioms

First, we recall that a topological space is a set X with a topology $\mathcal{T}$, that is a collection of subsets of X satisfying the following properties:

(1) $\emptyset$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Those elements in $\mathcal{T}$ are called *open sets*. There are many equivalent definitions about the concept of interior of an arbitrary subset M in a topological space X. For example, the classical Munkres's book [16], establishes the following statement about the interior of an arbitrary subset.

Definition 2.1.1

Given a subset M of a topological space X, the interior of M is defined as the union of all open sets contained in M. Moreover, the interior of M is denoted by *Int(M)* or $\mathring{M}$.

We use the next concept, *neighborhood*, as a shorter form to say "an open set", like Munkres uses it. For instance, U_x, with $x \in X$, (U_M, with $M \subseteq X$) means a neighborhood or an open set U that contains x (contains M). Here the concept of interior plays an important role, rather it is the key concept since there is a property that says:

$$\text{A subset } M \in X \text{ is an open set if and only if } \mathring{M} = M.$$

Now, the next definition is fundamental for any topological study.

Definition 2.1.2 [16]

Let X and Y be topological spaces. A function $f : X \longrightarrow Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Furthermore, we also remember the next classical definitions (separation axioms):

(i) A space X is called T_0 or Kolmogórov space, if for each pair of distinct points $x, y \in X$, there is a neighborhood containing x but not y or a neighborhood containing y but not x.

(ii) A space X is called T_1 or Fréchet space, if for each pair of distinct points $x, y \in X$, there is a neighborhood containing x but not y and another neighborhood containing y but not x.

(iii) A space X is called T_2 or Hausdorff space, if for every pair of distinct points $x, y \in X$, there are two disjoint open sets, one of these containing x and the other one containing y.

(iv) A space X is $T_{2\frac{1}{2}}$ or Urysohn, if for every two distinct points $x, y \in X$, there are two closed neighborhoods F_x and F_y with empty intersection.

(v) A topological space X is T_3 or regular, if it is T_1 and for each point $x \in X$ and for each closed set $F \subset X$ with $x \notin F$, there are neighborhoods U_x and U_F such that their intersection is empty.

(vi) A space X is called a $T_{3\frac{1}{2}}$ space or completely regular or Tychonoff, if it is T_1 and for each point $x \in X$ and for each closed set $F \subset X$ with $x \notin F$, there is a continuous function $f : X \longrightarrow [0,\ 1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.

(vii) A topological space is T_4 or normal, if it is T_1 and for every pair of closed sets $M, N \subset X$ with $M \cap N = \emptyset$, there are neighborhoods U_M and U_N such that their intersection is empty.

Henceforth $\mathbf{Top_i}$, with $i = 0,\ 1,\ 2,\ 2\frac{1}{2},\ 3\frac{1}{2},\ 4$, will be the collection or family of all the topological spaces satisfying the T_i property. Also, we have the following sequence of implications:

$$T_4 \Longrightarrow T_{3\frac{1}{2}} \Longrightarrow T_3 \Longrightarrow T_{2\frac{1}{2}} \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T_0$$

and consequently we have the inclusions:

$$T_4 \subseteq T_{3\frac{1}{2}} \subseteq T_3 \subseteq T_{2\frac{1}{2}} \subseteq T_2 \subseteq T_1 \subseteq T_0$$

2.2 Some important characterizations

It is important to note that our work can be done easier if we have in our hands some equivalences between the definitions and related results. For instance, the following results are very useful.

Proposition 2.2.1 [16]**.** *A topological space X is T_1 if and only if the singletons of X are closed sets.*

Proof. ($\Longrightarrow$) We shall prove that $X - \{x\}$ is an open set for every $x \in X$. Let $y \in X$ with $x \neq y$, then by T_1-definition, there is a neighborhood U_y such that $x \notin U_y$, hence $X - \{x\}$ is open.

($\Longleftarrow$) Let $x, y \in X$ with $x \neq y$. It turns out that $y \in X - \{x\}$ and this is an open set by hypothesis, similarly for x. $\square$

Now, there is another equivalent definition of T_1-space as the following proposition shows and we will use it in a later chapter.

Proposition 2.2.1 [16]**.** $X \in \mathbf{Top_1}$ *if and only if for every* $x \in X$ *the intersection of all neighborhoods of* x *is the singleton set* $\{x\}$*, in other words,* $\bigcap_{x \in X} U_x = \{x\}$*.*

Proof. ($\Longrightarrow$) Let $X \in \mathbf{Top_1}$. Suppose that $\bigcap U_x \neq \{x\}$, i.e., $\bigcap U_x$ contains other points different to x. Let $y \in \bigcap U_x$ and then $x \neq y$. Next, by definition of T_1 there is a neighborhood V_x of x such that $y \notin V_x$. But y belongs to every neighborhood of x by construction. That is a contradiction.

($\Longleftarrow$) Suppose that $X \notin \mathbf{Top_1}$. This means that there exist x, y with $x \neq y$ such that for every pair of neighborhoods U_x, U_y of x and y, respectively, we have: $y \in U_x$ or $x \in U_y$, in both cases $\{x\} \neq \{x, y\} \subset \bigcap U_x$. We have proved the statement. $\square$

CHAPTER 3
GALOIS CONNECTIONS

3.1 Pre-orders and Galois connections

The concept of Galois connection will be used in this work, whereby in this chapter we exhibit some properties that will be used later. In general, this concept is a good approximation to the notion of bijective function, moreover, this notion has some interesting properties. Thus, we start with the following statements.

We recall that a pre-order ($\leq$) is a reflexive and transitive relation, i.e., it needs satisfy the following conditions on a class $\mathcal{C}$.

(i) For every $a \in \mathcal{C}$, $a \leq a$.

(ii) For every $a, b, c \in \mathcal{C}$, $a \leq b$ and $b \leq c$ implies $a \leq c$.

If in addition for each $a, b \in \mathcal{C}$ the next holds:

(iii) $a \leq b$ and $b \leq a$ implies $a = b$, then the class $\mathcal{C}$ is called a partial order.

Now, we are ready to introduce the previously mentioned concept of Galois connection.

Definition 3.1.1 [3]

For pre-ordered classes $\mathcal{X} = \big(\mathbf{X}, \leq\big)$ and $\mathcal{Y} = \big(\mathbf{Y}, \preceq\big)$, a Galois connection $\mathcal{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{Y}$ consists of order preserving functions f and g that satisfy the following: $x \leq g\big(f(x)\big)$ for every $x \in \mathbf{X}$ and $f\big(g(y)\big) \preceq y$ for every $y \in \mathbf{Y}$.

Furthermore, if $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are such that $f(x) = y$ and $g(y) = x$, then x and y are said to be corresponding fixed points of the Galois connection $(\mathcal{X}, f, g, \mathcal{Y})$.

Now, we show two classical examples of Galois connections, other examples can be seen in [2] and [10] where there is more information about the subject.

(1) [3] Let $\mathcal{R}$ be a relation from the set X to the set Y, i.e., $R \subseteq X \times Y$. Denote by A the poset of all subsets of X ordered by inclusion and by B the poset of all subsets of Y ordered by inverse inclusion. Then, $A \underset{g}{\overset{f}{\rightleftarrows}} B$ is a Galois connection with,

$$A \xrightarrow{f} B \quad \text{defined by} \quad f(S) = \big\{y \in Y : \forall s \in S,\ \ (s, y) \in \mathcal{R}\big\}$$

$$B \xrightarrow{g} A \quad \text{defined by} \quad g(T) = \big\{x \in X : \forall t \in T,\ \ (x, t) \in \mathcal{R}\big\}$$

Proof. Firstly, let $A = \big(\mathcal{P}(X), \subseteq\big)$ and $B = \big(\mathcal{P}(Y), \overset{op}{\subseteq}\big)$. Now, let $M \subseteq N$. By definition $f(M) = \big\{y \in Y : \forall x \in M,\ (x, y) \in \mathcal{R}\big\}$ and $f(N) = \big\{y \in Y : \forall x \in N,\ (x, y) \in \mathcal{R}\big\}$. Thus, if $y \in f(N)$, then $y \in f(M)$, i.e. $f(N) \subseteq f(M)$ or equivalently $f(M) \overset{op}{\subseteq} f(N)$. Therefore f is order-preserving.

Let $U \overset{op}{\subseteq} V$. Remember, $g(U) = \big\{x \in X : \forall y \in U,\ (x, y) \in \mathcal{R}\big\}$ and $g(V) = \big\{x \in X : \forall y \in V,\ (x, y) \in \mathcal{R}\big\}$. Again like above, if $x \in g(U)$, then $x \in g(V)$. In other words, $g(U) \subseteq g(V)$. Thus, g is order-preserving also.

Let $x^* \in U$. By definition $g\big(f(U)\big) = \big\{x \in X : \forall y \in f(U),\ (x, y) \in \mathcal{R}\big\}$. First, if $f(U) = \emptyset$, we have that $g\big(f(U)\big) = X$, since there is not a pair (x, y), with $y \in f(U)$, such that $(x, y) \notin \mathcal{R}$. Consequently $U \subseteq g\big(f(U)\big)$. Now, suppose that $f(U) \neq \emptyset$ and let $y \in f(U) = \big\{y \in Y : \forall x \in U,\ (x, y) \in \mathcal{R}\big\}$. Thus, in particular for every $y \in f(U)$, $(x^*, y) \in \mathcal{R}$. Then $x^* \in g\big(f(U)\big)$ and therefore $U \subseteq g\big(f(U)\big)$.

Now, let $y^* \in V$. $f\big(g(V)\big) = \big\{y \in Y : \forall x \in g(V),\ (x, y) \in \mathcal{R}\big\}$. The empty case for $g(V)$ is similar to the $f(U)$ case. Let $x \in g(V) = \big\{x \in X : \forall y \in V,\ (x, y) \in \mathcal{R}\big\}$. Then, in particular for each $x \in g(V)$, $(x, y^*) \in \mathcal{R}$. Thus $y^* \in f\big(g(V)\big)$ and consequently, $V \subseteq f\big(g(V)\big)$, i.e. $f\big(g(V)\big) \overset{op}{\subseteq} V$. Therefore we have gotten the desired result.

(2) (Functions that induce Galois connections [3]) Let $X \xrightarrow{f} Y$ be a function between sets. Consider the functions $\mathcal{P}(X) \xrightarrow{\mathcal{P}f} \mathcal{P}(Y)$ and $\mathcal{P}(Y) \xrightarrow{\mathcal{Q}f} \mathcal{P}(X)$ between the power sets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ ordered by inclusion, defined by

$$\mathcal{P}f(A) = f(A), \text{ i.e., the direct image of } A \text{ under } f.$$

$$\mathcal{Q}f(B) = f^{-1}(B), \text{ i.e., the inverse image of } B \text{ under } f.$$

Then, $\mathcal{P}(X) \underset{\mathcal{Q}f}{\overset{\mathcal{P}f}{\rightleftarrows}} \mathcal{P}(Y)$ is a Galois connection.

Proof. By properties of direct image and inverse image we have the following:

$$A \subseteq B \text{ implies } f(A) \subseteq f(B)$$

$$U \subseteq V \text{ implies } f^{-1}(U) \subseteq f^{-1}(V)$$

The above means that f and g are order-preserving. We know also that for an arbitrary function f the following is true; for every $A \subseteq X$ and $B \subseteq Y$

$$A \subseteq f^{-1}\big(f(A)\big)$$

$$f\big(f^{-1}(B)\big) \subseteq B$$

This shows that $\mathcal{P}f$ and $\mathcal{Q}f$ both satisfy the desired conditions.

3.2 Properties of Galois connections

In later chapters we shall use some of the following important properties that will allow us to show interesting results. We only give the proof of one of them, the remaining ones can be seen in [3] with their respective proofs. For instance, in the literature the first one is often used as the definition of Galois connection.

Proposition 3.2.1 [3]. *Let $\mathcal{X}$ and $\mathcal{Y}$ be pre-ordered classes and let $\mathcal{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{Y}$ be two order preserving functions. Then, the following are equivalent.*

(a) $\mathcal{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{Y}$ *is a Galois connection.*

(b) *for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, $f(x) \leq y \Longleftrightarrow x \leq g(y)$.*

Proposition 3.2.2 [3]**.** *The composition of two Galois connections is a Galois connection, i.e., if* $\mathcal{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{Y}$ *and* $\mathcal{Y} \underset{k}{\overset{h}{\rightleftarrows}} \mathcal{Z}$ *are both Galois connections then* $\mathcal{X} \underset{g\circ k}{\overset{h\circ f}{\rightleftarrows}} \mathcal{Z}$ *is a Galois connection.*

Proposition 3.2.3 [3]**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be two pre-ordered classes and assume that infima exist in $\mathcal{Y}$. Let* $\mathcal{Y} \xrightarrow{g} \mathcal{X}$ *be a function that preserves infima. Define* $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ *as follows: for each $x \in \mathcal{X}$,*

$$f(x) = \bigwedge \{y \in \mathcal{Y} : g(y) \geq x\}.$$

Then $\mathcal{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{Y}$ *is a Galois connection.*

Proof. Let $y_1 \leq y_2$ in $\mathcal{Y}$. Since g preserves infima (for each family), we have that $g(y_1) \leq g(y_2)$. Thus, g is order preserving. Then, let $x_1 \leq x_2$ in $\mathcal{X}$. Notice that,

$$\{y \in \mathcal{Y} : g(y) \geq x_2\} \subseteq \{y \in \mathcal{Y} : g(y) \geq x_1\}$$

Then, applying the infimum we obtain,

$$f(x_1) = \bigwedge \{y \in \mathcal{Y} : g(y) \geq x_1\} \leq \bigwedge \{y \in \mathcal{Y} : g(y) \geq x_2\} = f(x_2)$$

Whence, f is order preserving. Now, let $x' \in \mathcal{X}$. Since, g preserves infima, we have

$$g\big(f(x')\big) = g\Big(\bigwedge \{y \in \mathcal{Y} : g(y) \geq x'\}\Big) = \bigwedge \{g(y) \in \mathcal{X} : g(y) \geq x'\} \geq x'$$

Moreover, let $y' \in \mathcal{Y}$. By the definition of f,

$$f\big(g(y')\big) = \bigwedge \{y \in \mathcal{Y} : g(y) \geq g(y')\} \leq y'$$

since $g(y') \leq g(y')$. $\square$

CHAPTER 4
INTERIOR OPERATORS

4.1 Interior operators on Top

In this chapter, we are going to work in the category **Top** of topological spaces, whose objects are the topological spaces and the continuous functions are its morphisms. More details about general categories can be found in [2]. The notion of a categorical interior operator was given by Vorster in [17], as we mentioned in the introductory chapter. In this section we introduce this concept on the category **Top** as in [8] and we show some of its basic properties.

Definition 4.1.1 [8]

An interior operator I on **Top** is a family of functions $\{i_X\}_{X\in\mathbf{Top}}$ on the subset lattices of **Top** with the following properties for every $X \in \mathbf{Top}$:

(i) (Contractibility) For every $M \subseteq X$ then $i_X(M) \subseteq M$.

(ii) (Monotonicity) For each pair of subsets M, N of X with $M \subseteq N$ then $i_X(M) \subseteq i_X(N)$.

(iii) (Continuity) For every continuous function $X \xrightarrow{f} Y$ and any subset $N \subseteq Y$ then $f^{-1}\big(i_Y(N)\big) \subseteq i_X\big(f^{-1}(N)\big)$.

Also, we can say that I is an *idempotent* interior operator when the following holds:

$$i_X\big(i_X(M)\big) = i_X(M) \text{ for every } M \subseteq X \text{ and } X \in \mathbf{Top}$$

The notation $IN(\mathbf{Top})$ means the class of all the interior operators on the category **Top**. Now, we are ready to show some related examples. Most of them with their respective proofs can be found in [7].

4.2 Examples of interior operators

Here, we present a list of examples of interior operators. In the next examples M represents an arbitrary subset of the topological space X.

(1) [7] Let K be the operator defined by

$$k_X(M) = \bigcup \Big\{ O \subseteq M : O \text{ is open in } X \Big\}.$$

It is called the *usual* interior operator induced by the topology of X.

(2) [7] Let H be the operator defined by

$$h_X(M) = \bigcup \Big\{ C \subseteq M : C \text{ is closed in } X \Big\} = \big\{ x \in M : \overline{\{x\}} \subseteq M \big\}.$$

(3) [7] Let B be the operator defined as follows,

$$b_X(M) = \big\{ x \in M : \exists U_x \text{ nbhd of } x \text{ such that } U_x \cap \overline{\{x\}} \subseteq M \big\}.$$

(4) [7] Let Θ be the assignment such that,

$$\theta_X(M) = \big\{ x \in M : \exists\ U_x \ \text{ nbhd of } x \text{ such that } \ \overline{U_x} \subseteq M \big\}.$$

(5) [7] Let L be the following operator:

$$l_X(M) = \big\{ x \in X : C_x \subseteq M \text{ where } C_x \text{ is the connected component of } x \big\}.$$

(6) [7] Let T be defined by

$$t_X(M) = \big\{ x \in M : x_n \longrightarrow x \text{ implies } x_n \in M \text{ eventually} \big\}.$$

(7) [7] Let Q be the assignment such that,

$$q_X(M) = \bigcup \Big\{ C \subseteq M : C \text{ is clopen in } X \Big\}.$$

The next interior operator is new and consequently we include its proof.

(8) Let W be the following assignment,

$$w_X(M) = \bigcup \{ E \subseteq M : E \text{ is open or closed in } X \}.$$

Proof. The first two conditions are easy to prove. Thus, let $X \xrightarrow{f} Y$ be a continuous function and let N be a subset of the topological space Y. Now, $f^{-1}\big(\omega_Y(N)\big) = f^{-1}\big(\bigcup E\big) = \bigcup f^{-1}(E)$, where $f^{-1}(E)$ is open or closed in $f^{-1}(N)$, because $E \subseteq N$. Consequently $f^{-1}\big(\omega_Y(N)\big) \subseteq \omega_X\big(f^{-1}(N)\big)$. □

4.3 Properties of interior operators

Here, we include some important properties of interior operators that closely resemble some properties already known in topology.

Definition 4.3.1 [8]

Let I be an interior operator on **Top** and let M be a subset of X then,

(i) M is I-open if and only if $i_X(M) = M$.

(ii) M is I-isolated if and only if $i_X(M) = \emptyset$.

(iii) X is I-discrete if and only if for every $M \subseteq X$, $i_X(M) = M$.

The following properties are easy to verify and we address the interested reader to [7] for their proofs and other interesting properties.

Proposition 4.3.2 [7]**.** *Let* $X \xrightarrow{f} Y$ *be a continuous function and let* I *be an interior operator on* ***Top****. If* $N \subseteq Y$ *is* I*-open, then so is* $f^{-1}(N)$*.*

The notation $S(X)$ means the collection of all subclasses of subsets of X.

Proposition 4.3.3 [7]. *Let I be an interior operator on **Top** and let $\{M_\alpha\}_{\alpha\in A} \subseteq S(X)$. If M_α is I-open in X for each $\alpha \in A$, then $\bigcup_{\alpha\in A} M_\alpha$ is I-open in X.*

It is important to note that, we can generate a natural order in $IN(\mathbf{Top})$ as follows:

Definition 4.3.4 [8]

Let $X \in \mathbf{Top}$ then for $I, J \in IN(\mathbf{Top})$, $I \sqsubseteq J$ if and only if $i_X(M) \subseteq j_X(M)$ for all $M \subseteq X$.

Since for every $X \in \mathbf{Top}$, $(X, \subseteq)$ is a poset, then $\sqsubseteq$ is a partial order in $IN(\mathbf{Top})$. Now, arbitrary suprema and infima exist in $IN(\mathbf{Top})$ as it is showed in the next statement and moreover their respective proofs can be found in [8].

Proposition 4.3.5 [8]. *Let $\{I_k\}_{k\in K}$ be a family of interior operators in $IN(\mathbf{Top})$. For each $M \subseteq X \in \mathbf{Top}$, define $\bigwedge_{k\in K} I_k$ by:*

$$i_{\wedge I_k}(M) = \bigcap_{k\in K} i_k(M)$$

Then $\bigwedge_{k\in K} I_k$ belongs to $IN(\mathbf{Top})$ and it is the infimum of the family $\{I_k\}_{k\in K}$.

Proposition 4.3.6 [8]. *Let $\{I_k\}_{k\in K}$ be a family of interior operators in $IN(\mathbf{Top})$. For each $M \subseteq X \in \mathbf{Top}$, define $\bigvee_{k\in K} I_k$ by:*

$$i_{\vee I_k}(M) = \bigcup_{k\in K} i_k(M)$$

Then $\bigvee_{k\in K} I_k$ belongs to $IN(\mathbf{Top})$ and it is the supremum of the family $\{I_k\}_{k\in K}$.

Finally, the next result will relate the concept of I-open with the infimum interior operator mentioned above.

Proposition 4.3.7. *Let $\{I_k\}_{k\in K} \subseteq IN(\mathbf{Top})$ and let $M \subseteq X$. Then M is $\bigwedge\limits_{k\in K} I_k$-open if and only if M is I_k-open for each $k \in K$.*

Proof. ($\Longrightarrow$) Let M be $\bigwedge_{k\in K} I_k$-open, by hypothesis we have that,

$$M = i_{\wedge I_k}(M) \subseteq i_{_k}(M)$$

for every $k \in K$, then we obtain that $i_{_k}(M) = M$ for each $k \in K$.

($\Longleftarrow$) Let M I_k-open for each $k \in K$, i.e., $i_{_k}(M) = M$. Note that

$$i_{\wedge I_k}(M) = \bigcap_{k\in K} i_k(M) = \bigcap_{k\in K} M = M$$

Thus, the desired result is obtained. $\square$

CHAPTER 5
T_1-SEPARATED TOPOLOGICAL SPACES

5.1 T_1-separation

In this chapter, we introduce the notion of T_1-separated topological space with respect to an interior operator I. Naturally, this is a generalization of the classical concept of T_1-topological space. Moreover, this theory will give us some interesting additional examples. We start with the basic definition.

Definition 5.1.1

Let I be an interior operator on the category **Top**. A space $X \in$ **Top** is T_1-separated with respect to I if and only if for every $x \in X$, $X - \{x\}$ is I-open, i.e.,

$$i_X\big(X - \{x\}\big) = X - \{x\}. \tag{5.1}$$

Now, $T_1(I)$ will denote all T_1-separated objects with respect to I. In other words:

$$T_1(I) = \Big\{X \in \mathbf{Top} : X \text{ is } T_1\text{-separated with respect to } I\Big\}. \tag{5.2}$$

At this point it is natural to ask the following question: what kind of T_1-separated objects can we obtain for a particular interior operator I? Here we will use the examples of interior operators discussed in Section 4.2 and hence a partial answer to our question is given by the following:

5.2 Examples

In the next examples M will represent a subset of the topological space X.

(1) Let K be the *usual* interior operator in **Top**, i.e,

$$k_X(M) = \bigcup \Big\{ O \subseteq M : O \text{ is open in } X \Big\}.$$

First, let $X \in T_1(K)$, then for every $x \in X$, $k_X\big(X - \{x\}\big) = X - \{x\}$. Let $y \in X - \{x\}$ then $x \neq y$. Now, there exists O_1 open in X such that $y \in O_1 \subseteq X - \{x\}$, so $x \notin O_1$. Similarly, there exists O_2 open in X such that $x \in O_2 \subseteq X - \{y\}$, so $y \notin O_2$. Therefore, we have that $X \in \mathbf{Top_1}$.
Now, let $X \in \mathbf{Top_1}$. Consider $x \in X$ and $y \in X - \{x\}$ so that $x \neq y$. Therefore, there exist O_1, O_2 open sets such that $x \in O_1$, $y \notin O_1$, $y \in O_2$ and $x \notin O_2$. Then $O_2 \subseteq X - \{x\}$. Since this is true for any $y \in X - \{x\}$, we have that $X - \{x\} \subseteq k_X\big(X - \{x\}\big)$, so that $X \in T_1(K)$. Therefore we have shown that $T_1(K) = \mathbf{Top_1}$. $\square$

(2) Let H be the interior operator defined by

$$h_X(M) = \bigcup \Big\{ C \subseteq M : C \text{ is closed in } X \Big\} = \big\{ x \in M : \overline{\{x\}} \subseteq M \big\}$$

Also in this case, we will show that $T_1(H) = \mathbf{Top_1}$. Let $X \in T_1(H)$, then for every $x \in X$ we have that $X - \{x\} \subseteq h_X\big(X - \{x\}\big)$. In fact, for each $y \in X - \{x\}$, $\overline{\{y\}} \subseteq X - \{x\}$, then $x \notin \overline{\{y\}}$. In other words, there exists O_x open set in X with $O_x \cap \{y\} = \emptyset$. Similarly, for every $x \in X - \{y\}$, $\exists\, O_y$ open set in X with $O_y \cap \{x\} = \emptyset$, hence $X \in \mathbf{Top_1}$.
On the other hand, let $X \in \mathbf{Top_1}$, $x \in X$ and $y \in X - \{x\}$. So, $x \neq y$ and moreover there exists open sets O_1 and O_2 such that,

$$x \in O_1 \;\wedge\; y \notin O_1 \tag{5.3}$$

and

$$y \in O_2 \;\wedge\; x \notin O_2 \tag{5.4}$$

Now, (5.3) implies that $x \notin \overline{\{y\}}$. Thus, $\overline{\{y\}} \subseteq X - \{x\}$ for every $y \in X - \{x\}$. Therefore, $X - \{x\} \subseteq h_X\big(X - \{x\}\big)$ and so $X \in T_1(H)$. $\square$

(3) Let B be defined by

$$b_X(M) = \big\{x \in M : \exists U_x \text{ nbhd of } x \text{ such that } U_x \cap \overline{\{x\}} \subseteq M\big\}$$

We claim that $T_1(B) = \mathbf{Top_0}$. First, let $X \notin T_1(B)$. Then, there exists $x \in X$ such that $b_X\big(X - \{x\}\big) \neq X - \{x\}$, i.e., there exists $y \in X - \{x\}$ with $y \notin b_X\big(X - \{x\}\big)$.

Therefore, for every U_y nbhd of y,

$$U_y \cap \overline{\{y\}} \nsubseteq X - \{x\}$$

This means that $x \in U_y \cap \overline{\{y\}}$, then $x \in U_y$ and $x \in \overline{\{y\}}$. Since $x \in \overline{\{y\}}$, we have the following: for every U_x, nbhd of x, $U_x \cap \{y\} \neq \emptyset$, which means that $y \in U_x$. So, every nbhd of y contains x and every nbhd of x contains y. Therefore $X \notin \mathbf{Top_0}$.

On the other hand, suppose that $X \notin \mathbf{Top_0}$. Then there exist $x, y \in X$ with $x \neq y$ such that for each U_x and U_y, nbhds of x and y respectively, one has that

$$x \in U_y \ \wedge \ y \in U_x$$

hence $x \in \overline{\{y\}}$. Now, $x \in U_y \cap \overline{\{y\}} \nsubseteq X - \{x\}$ and consequently $y \notin b_X\big(X - \{x\}\big)$. It follows that

$$b_X\big(X - \{x\}\big) \neq X - \{x\}$$

Thus $X \notin T_1(B)$. $\square$

(4) Let Θ be defined by

$$\theta_X(M) = \big\{x \in M : \exists\, U_x \text{ nbhd of } x \text{ such that } \overline{U_x} \subseteq M\big\}$$

Here, we will prove that $T_1(\Theta) = \mathbf{Top_2}$. Let $X \in T_1(\Theta)$, then for every $x \in X$ we have that $\theta_X\big(X - \{x\}\big) = X - \{x\}$. Let $y \in X - \{x\}$, so $x \neq y$. It follows that $y \in \theta_X\big(X - \{x\}\big)$. Whereby, there exists U_y such that

$$\overline{U_y} \subseteq X - \{x\}$$

Then $x \notin \overline{U_y}$, and this implies that there exists U_x where $U_x \cap U_y = \emptyset$. Therefore $X \in \mathbf{Top_2}$.

Now, let $X \in \mathbf{Top_2}$ and let $y \in X - \{x\}$, hence $x \neq y$. Then there exist U_x, U_y for which $U_x \cap U_y = \emptyset$. This implies that $x \notin \overline{U_y}$.

So, $\overline{U_y} \subseteq X - \{x\}$. Therefore $y \in \theta_X\big(X - \{x\}\big)$ and $X \in T_1(\Theta)$. $\square$

(5) Let L be the following interior operator

$$l_X(M) = \big\{x \in X : C_x \subseteq M \text{ where } C_x \text{ is the connected component of } x\big\}$$

We will show that $T_1(L) = \mathbf{TotDisc}$, where **TotDisc** is the collection of all totally disconnected topological spaces, i.e., those topological spaces whose connected components are singletons.

Let $X \in T_1(L)$. This means that for each $y \in X$, $l_X\big(X - \{y\}\big) = X - \{y\}$. Let us fix x and note that for every $y \neq x$, $C_x \subseteq X - \{y\}$, that is, for every $y \neq x$ we have $y \notin C_x$. Consequently, the connected component of x does not contain any element other than x, therefore $C_x = \{x\}$. Thus, we obtain that $X \in \mathbf{TotDisc}$.

Now, let $X \in \mathbf{TotDisc}$ and let $y \in X - \{x\}$. Then by hypothesis we have $C_y = \{y\}$, so $C_y \subseteq X - \{x\}$ whereby, $y \in l_X(X - \{x\})$. Consequently,

$l_X\big(X-\{x\}\big) = X-\{x\}$. Therefore $X \in T_1(L)$. $\square$

(6) Let T be defined by

$$t_X(M) = \big\{x \in M : x_n \longrightarrow x \text{ implies } x_n \in M \text{ eventually}\big\}$$

where the meaning of $x_n \in M$ eventually is that there is a natural number N, such that for every natural number n, with $n \geq N$, $x_n \in M$.

We are going to prove that $T_1(T) = \mathbf{Top_1}$. First, suppose that $X \notin T_1(T)$ then, there exists $y \in X-\{x\}$ such that $y \notin t_X\big(X-\{x\}\big)$. Thus, there is a sequence y_n converging to y, where y_n does not belong to $X-\{x\}$ eventually, i.e., for every $n \in \mathbb{N}$, there is $N \in \mathbb{N}$ with $N \geq n$ such that $y_N = x$. Let O_y be a nbhd of y, by definition of convergence, there exists an index k such that O_y contains all the elements of $(y_n)_{n\geq k}$, then $x \in O_y$ since there is $N \geq k$ such that $y_N = x$, hence $X \notin \mathbf{Top_1}$.

On the other hand, if $X \notin \mathbf{Top_1}$ there exist different points x, y in X such that $y \in \overline{\{x\}}$, that means the following: for each O_y nbhd of y, $O_y \cap \{x\} = \{x\}$. So, the constant sequence at x, i.e., $(x_n)_{n\geq 1}$, with $x_n = x$, converges to $y \in X-\{x\}$. Then, since $(x_n)_{n\geq 1} \nsubseteq X-\{x\}$ we have that $x_n \notin X-\{x\}$ eventually, consequently $X \notin T_1(T)$. $\square$

(7) Let Q be the interior operator defined by

$$q_X(M) = \bigcup\Big\{C \subseteq M : C \text{ is clopen in } X\Big\}$$

We shall prove that $\mathbf{TotSep} = T_1(Q) \subseteq \mathbf{TotDisc} \bigcap \mathbf{Top_{2\frac{1}{2}}}$, where $\mathbf{TotDisc}$ is the collection of all totally disconnected topological spaces and $\mathbf{TotSep}$ is the collection of all totally separated topological spaces.

In the first place, we need to show that **TotSep** $\subseteq T_1(Q)$, for which let $X \in$ **TotSep** and let $x, y \in X$ with $x \neq y$, then there is a separation U, V of X, i.e., $x \in U$ and $y \in V$ with $U \cup V = X$ and $U \cap V = \emptyset$. Note that $y \in X - \{x\}$, and moreover $x \notin V$, thus $V \subset X - \{x\}$. We remember that V is clopen in X, it follows that $y \in q_{X}\big(X - \{x\}\big)$, this is true for every $y \in X - \{x\}$. Therefore $X \in T_1(Q)$.

Now, suppose that $X \in T_1(Q)$. Then, for every $x \in X$ we have that $q_{X}\big(X - \{x\}\big) = X - \{x\}$. Let $y \in X - \{x\}$ (i.e. $x \neq y$) which implies $y \in q_{X}\big(X - \{x\}\big)$. Next, $y \in C \subseteq X - \{x\}$ with C clopen. Note that $x \notin C$ then $x \in X - C$ which is clopen too. Thus, for each $x, y \in X$ with $x \neq y$, there exists a separation U, V of X. Therefore $X \in$ **TotSep**.

To prove that **TotSep** $\subseteq$ **TotDisc**, we suppose that $X \notin$ **TotDisc**. Then, there is a connected component $D \subseteq X$ where D is not a singleton, i.e., D has at least two points, for instance $x, y \in D$ with $x \neq y$ and we can not separate these points. Whereby, there is not a separation U, V of X, with $x \in U$ and $y \in V$. Thus $X \notin$ **TotSep**.

And finally, we will show that **TotSep** $\subseteq \mathbf{Top_{2\frac{1}{2}}}$. In this case we assume that $X \in$ **TotSep**. Let $x, y \in X$, with $x \neq y$. Then, there is a separation U, V of X. Note that U and V are both clopen. Now, $U = \overline{U}$ and $V = \overline{V}$ hence $\overline{U} \cap \overline{V} = \emptyset$. Therefore $X \in \mathbf{Top_{2\frac{1}{2}}}$. $\square$

Remark. (a) We are going to show an example where **TotDisc** $\not\Rightarrow$ **TotSep**. Let X be topological subspace of $\mathbb{R}^2$ formed by $\mathbb{Q}_0 = \big\{(q,\,0)\,|\,q \in \mathbb{Q}\big\}$ and by $\mathbb{Q}_1 = \big\{(q,\,1)\,|\,q \in \mathbb{Q}\big\}$, in other words, X is formed by two copies of $\mathbb{Q}$, then we form the quotient space of X, namely X^*, by identifying each pair of nonzero rational numbers to an unique point and each copy of zero to a different point, say 0_0, 0_1. That is, X^* will be like another copy of $\mathbb{Q}$ but with an extra point. Note that X^* is totally disconnected because for each pair of different points x, y in X^*, not both of them equal to 0_0, 0_1, there exists an irrational number between them, so we can separate these two points. Now, the subset $A = \{0_0, 0_1\}$ in X^* is not connected since there exist open sets U, V in X^* that separate the subset A as subspace. For example, the subsets $U = (-1, 1)$ and $V = (-1, 0_0) \cup \{0_1\} \cup (0_0, 1)$ are both open sets satisfying the conditions for A being a disconnected subset in X^*. However, X^* it is not totally separated because the two copies of zero can not be separated by two open sets M, N in X^* with $0_0 \in M$ and $0_1 \in N$ such that $M \cup N = X^*$ and $M \cap N = \emptyset$ $\big($it always happens that $M \cap N \neq \emptyset\big)$.

(b) It is clear that if X is a discrete space then X is totally separated space. The converse is not true. Let us take $X =$ Cantor space, i.e. the Cantor set in $[0, 1]$ with the topology of subspace. Notice that X is totally separated by Example 29.6 in [15], moreover it is not discrete since every nbhd of $0 \in X$ contains the point $\dfrac{1}{3^n} \in X$ for some $n \in \mathbb{N}$.

(8) Let W be the operator defined by,

$$w_X(M) = \bigcup \big\{E \subseteq M : E \text{ is open or closed in } X\big\} \tag{5.5}$$

We claim the following statement: $T_1(W) = \mathbf{Top_1}$. First, let $X \in T_1(W)$ then for every $x \in X$, $w_X\big(X - \{x\}\big) = X - \{x\}$. Let $y \in X - \{x\}$ then $x \neq y$. Now, there is E_1 open or closed in X such that $y \in E_1 \subseteq X - \{x\}$, so $x \notin E_1$. If E_1 is open there is nothing to prove. If E_1 is closed then $X - E_1$ is open. Consequently $x \in X - E_1$ and $y \notin X - E_1$.

Similarly, there exists E_2 open or closed in X such that $x \in E_2 \subseteq X - \{y\}$, so $y \notin E_2$. If E_2 is open there is nothing to prove again. If E_2 is closed then $X - E_2$ is open where $y \in X - E_2$ and $x \notin X - E_2$. Therefore, we have obtained the desired result. $\square$

5.3 Some properties of T_1-separation

At this point it is convenient to show some closure properties of the class of T_1-separated objects, namely $T_1(I)$. Precisely, we are going to show that products and subspaces of T_1-separated objects are T_1-separated objects.

First of all, we need to recall the concept of *monosource*.

Definition 5.3.1

Given an indexed family of topological spaces $\{Y_\lambda\}_{\lambda\in\Lambda}$ and a topological space X, an indexed family of continuous functions

$$\left(X \xrightarrow{f_\lambda} Y_\lambda\right)_{\lambda\in\Lambda}$$

is called a *monosource* if and only if given $x, y \in X$, $f_\lambda(x) = f_\lambda(y)$ for every $\lambda \in \Lambda$ implies that $x = y$.

Furthermore, there is an equivalent way to express the previous definition: a family of continuous functions $(f_\lambda)_{\lambda\in\Lambda}$ is a *monosource* if and only if for every $Z \in \mathbf{Top}$ and for any pair of continuous functions

$$Z \underset{n}{\overset{m}{\rightrightarrows}} X$$

such that $f_\lambda \circ m = f_\lambda \circ n$, for every $\lambda \in \Lambda$, implies $m = n$.

Now, we are ready to prove the following result.

Proposition 5.3.2. *$T_1(I)$ is closed under monosources.*

Proof. Let $\left(X \xrightarrow{f_\lambda} Y_\lambda\right)_{\lambda\in\Lambda}$ be a monosource with $Y_\lambda \in T_1(I)$ for every $\lambda \in \Lambda$. We are going to prove that $X \in T_1(I)$.

Let $y \in X - \{x\}$, that is $x \neq y$. Now, by definition of monosource there is $\lambda_0 \in \Lambda$ such that $f_{\lambda_0}(x) \neq f_{\lambda_0}(y)$. Consequently,

$$f_{\lambda_0}(y) \in \big(Y_{\lambda_0} - \{f_{\lambda_0}(x)\}\big) = i\big(Y_{\lambda_0} - \{f_{\lambda_0}(x)\}\big)$$

since Y_{λ_0} is T_1-separated. Then,

$$y \in {f_{\lambda_0}}^{-1}\Big(i\big(Y_{\lambda_0} - \{f_{\lambda_0}(x)\}\big)\Big) \subseteq i\Big({f_{\lambda_0}}^{-1}\big(Y_{\lambda_0} - \{f_{\lambda_0}(x)\}\big)\Big).$$

by the continuity property of interior operators. Notice that

$${f_{\lambda_0}}^{-1}\big(Y_{\lambda_0} - \{f_{\lambda_0}(x)\}\big) \subseteq X - \{x\}$$

thus by the order preservation property of interior operators,

$$i\Big({f_{\lambda_0}}^{-1}\big(Y_{\lambda_0} - \{f_{\lambda_0}(x)\}\big)\Big) \subseteq i(X - \{x\})$$

and so $y \in i\big(X - \{x\}\big)$. Therefore $i\big(X - \{x\}\big) = X - \{x\}$ and so $X \in T_1(I)$. $\square$

Corollary 5.3.3. *For any interior operator I, $T_1(I)$ is closed under subspaces and products.*

Proof. It is a direct consequence of the previous proposition, since inclusions of subspaces and the projections of a product of topological spaces are monosources. $\square$

Remark. Just a clarification, the above corollary states that if $X \in T_1(I)$ and M is a subspace of X then also $M \in T_1(I)$. Furthermore, if $\{X_i\}_{i\in I}$ is a family of topological spaces that all belong to $T_1(I)$ then also $\prod_{i\in I} X_i \in T_1(I)$.

Let $S(\mathbf{Top})$ denote all subclasses of objects of $\mathbf{Top}$, that is subclasses of topological spaces ordered by inclusion. We are going to show that there is an interesting relationship between $IN(\mathbf{Top})$ and $S(\mathbf{Top})$.

We start by presenting the following important result.

Proposition 5.3.4. *The function* $IN(\mathbf{Top}) \xrightarrow{T_1} S(\mathbf{Top})$ *defined by*

$$T_1(I) = \big\{X \in \mathbf{Top} : X \textit{ is } T_1\textit{-separated with respect to } I\,\big\}$$

preserves infima.

Proof. Let $\{I_k\}_{k\in K}$ be a family of interior operators in $\mathbf{Top}$. We are going to prove that

$$T_1\Big(\bigwedge_{k\in K} I_k\Big) = \bigcap_{k\in K} T_1(I_k)$$

By definition of infimum, for every $k \in K$, $\bigwedge\limits_{k\in K} I_k \sqsubseteq I_k$. If we show that T_1 is order preserving then, for each $k \in K$,

$$T_1\Big(\bigwedge_{k\in K} I_k\Big) \subseteq T_1(I_k)$$

and hence,

$$T_1\Big(\bigwedge_{k\in K} I_k\Big) \subseteq \bigcap_{k\in K} T_1(I_k)$$

Let $I, J \in IN(\mathbf{Top})$, with $I \sqsubseteq J$ and let $X \in T_1(I)$. Then for all $x \in X$, $i_X\big(X-\{x\}\big) = X - \{x\}$. It follows that,

$$X - \{x\} = i_X\big(X - \{x\}\big) \subseteq j_X\big(X - \{x\}\big)$$

Now, by definition of interior operator, $j_X\big(X - \{x\}\big) \subseteq X - \{x\}$. Consequently $j_X\big(X - \{x\}\big) = X - \{x\}$ for every $x \in X$ and so $X \in T_1(J)$. Thus $T_1(I) \subseteq T_1(J)$ and thence T_1 is order preserving.

Let $X \in \bigcap\limits_{k\in K} T_1(I_k)$, then for every $k \in K$ we have $X \in T_1(I_k)$, that is, for every $x \in X$ and each $k \in K$

$$i_k\big(X - \{x\}\big) = X - \{x\}.$$

By definition of $\bigwedge_{k\in K} I_k$ we have that

$$i_{\bigwedge I_k}(M) = \bigcap_{k\in K} i_k(M), \tag{5.6}$$

which implies that, $i_{\bigwedge I_k}\big(X - \{x\}\big) = \bigcap_{k\in K} i_k\big(X - \{x\}\big) = X - \{x\}$. Therefore $X \in T_1\Big(\bigwedge_{k\in K} I_k\Big)$. We have proved that,

$$T_1\Big(\bigwedge_{k\in K} I_k\Big) = \bigcap_{k\in K} T_1(I_k)$$

in other words, T_1 preserves infima. $\square$

The previous result and Proposition 3.2.3 yield the following Galois connection

$$S(\mathbf{Top}) \underset{T_1}{\overset{H}{\rightleftarrows}} IN(\mathbf{Top})$$

where the function H is defined by: for $\mathscr{A} \in S(\mathbf{Top})$,

$$H(\mathscr{A}) = \bigwedge \big\{I \in IN(\mathbf{Top}) : \mathscr{A} \subseteq T_1(I)\big\} \tag{5.7}$$

In the next result we are going to relate the concept studied in [7], namely, separation with respect to an interior operator I, $Sep(I)$ to $T_1(I)$.

Proposition 5.3.5. *Let I be an arbitrary interior operator, then the following statement is true,*

$$Sep(I) \subseteq T_1(I).$$

Proof. First, we need to recall the following definition: A space Y is called I-separated if and only if for all $X \in \mathbf{Top}$ and for each pair of continuous functions,

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

we have that $Sep(f, g)$ is *I-open*, i.e., $i\big(Sep(f, g)\big) = Sep(f, g)$.

Let X be *I-separated*, we will prove that if $X \in Sep(I)$ then $X \in T_1(I)$. For some $x \in X$ consider the diagram below,

$$X - \{x\} \overset{i}{\underset{c}{\rightrightarrows}} X$$

with i being the inclusion function and c the constant function in x. Both functions are continuous and it is true that for each $y \in X - \{x\}$, $i(y) \neq c(y)$.
Accordingly, $Sep(i, c) = X - \{x\}$ and thus for every $x \in X$, $X - \{x\}$ is I-open, that is $X \in T_1(I)$. $\square$

Now, it is necessary to raise the following question: is there any explicit representation of (5.7)? Surprisingly and fortunately the answer is affirmative and for this purpose we have the next proposition.

Note. Henceforth, unless otherwise stated, $X \xrightarrow{f} Y$ means that f is a continuous function between the topological spaces X and Y.

Proposition 5.3.6. *Let $\mathscr{A} \in S(\mathbf{Top}), X \in \mathbf{Top}$ and $M \subseteq X$. We define $J_{\mathscr{A}}$ by*

$$(j_{\mathscr{A}})_X(M) = \bigcup \Big\{ f^{-1}\big(Y - \{y\}\big) \subseteq M : \ X \xrightarrow{f} Y, \ \ Y \in \mathscr{A}, \ y \in Y \Big\} \tag{5.8}$$

Then, $J_{\mathscr{A}}$ is an idempotent interior operator on **Top**.

Proof. It is easy to see that $J_{\mathscr{A}}$ satisfies the contractiveness property of an interior operator, since if $X \in \mathbf{Top}$ and $M \subseteq X$, by definition $(j_{\mathscr{A}})_X(M) \subseteq M$. Moreover, if $M_1 \subseteq M_2$, then

$$\Big\{ f^{-1}\big(Y - \{y\}\big) \subseteq M_1 : \ X \xrightarrow{f} Y, \ \ Y \in \mathscr{A}, \ \ y \in Y \Big\} \subseteq$$
$$\Big\{ f^{-1}\big(Y - \{y\}\big) \subseteq M_2 : \ X \xrightarrow{f} Y, \ \ Y \in \mathscr{A}, \ \ y \in Y \Big\}$$

Finally, we will prove the continuity property. Let X, $Y \in \mathbf{Top}$, $N \subseteq Y$ and moreover $X \xrightarrow{f} Y$. Hence

$$\begin{aligned}
f^{-1}\Big((j_{\mathscr{A}})_Y(N)\Big) &= f^{-1}\Bigg(\bigcup\Big\{g^{-1}\big(Z-\{z\}\big) \subseteq N : Y \xrightarrow{g} Z,\ Z \in \mathscr{A},\ z \in Z\Big\}\Bigg)\\
&= \bigcup\Big\{f^{-1}\Big(g^{-1}\big(Z-\{z\}\big)\Big) \subseteq f^{-1}(N) : Y \xrightarrow{g} Z,\ Z \in \mathscr{A},\ z \in Z\Big\}\\
&= \bigcup\Big\{(g\circ f)^{-1}\big(Z-\{z\}\big) \subseteq f^{-1}(N) : Y \xrightarrow{g} Z,\ Z \in \mathscr{A},\ z \in Z\Big\}\\
&\subseteq \bigcup\Big\{h^{-1}\big(Z-\{z\}\big) \subseteq f^{-1}(N) : X \xrightarrow{h} Z,\ Z \in \mathscr{A},\ z \in Z\Big\}\\
&= (j_{\mathscr{A}})_X\big(f^{-1}(N)\big).
\end{aligned}$$

We observe that the above inclusion is true since $g\circ f$ is a continuous function from X to Z and there are additional continuous functions $h : X \longrightarrow Z$ that do not factor as $g \circ f$. Therefore $J_{\mathscr{A}} \in IN(\mathbf{Top})$.

To prove idempotency we have that if $X \xrightarrow{f} Y$ with $Y \in \mathscr{A}$ and $y \in Y$ satisfies $f^{-1}\big(Y-\{y\}\big) \subseteq M$, then $f^{-1}\big(Y-\{y\}\big) \subseteq (j_{\mathscr{A}})_X(M)$ and consequently $f^{-1}\big(Y-\{y\}\big) \subseteq (j_{\mathscr{A}})_X\big((j_{\mathscr{A}})_X(M)\big)$. Whereby, $J_{\mathscr{A}}$ is idempotent. □

The next lemma is crucial for our purposes, because it will allows us to relate the family $\mathscr{A}$ and the separated objects with respect to the interior operator $J_{\mathscr{A}}$.

Lemma 5.3.7. *If $X \in \mathscr{A}$, then $X \in T_1(J_{\mathscr{A}})$.*

Proof. We shall prove that for every $x \in X$, $(j_{\mathscr{A}})_X\big(X-\{x\}\big) = X-\{x\}$. We recall that $(j_{\mathscr{A}})_X\big(X-\{x\}\big) = \bigcup\Big\{f^{-1}\big(Y-\{y\}\big) \subseteq X-\{x\} :\ X \xrightarrow{f} Y,\ Y \in \mathscr{A},\ y \in Y\Big\}$. Let $x \in X$ and consider $X-\{x\}$. Now, since $X \in \mathscr{A}$ and moreover

$$X \xrightarrow{id} X$$

is a continuous function, then $X-\{x\}=id^{-1}\big(X-\{x\}\big)$, so that $X-\{x\}$ occur in the construction of $(j_{\mathscr{A}})_X\big(X-\{x\}\big)$, that is $X-\{x\}\subseteq (j_{\mathscr{A}})_X\big(X-\{x\}\big)$. Hence $(j_{\mathscr{A}})_X\big(X-\{x\}\big)=X-\{x\}$ and therefore $X\in T_1(J_{\mathscr{A}})$. $\square$

At this point we are ready to prove the equality between (5.7) and (5.8). This is an important result that shows that there exists an explicit expression for the operator J.

Proposition 5.3.8. *For every $M\subseteq X\in\mathbf{Top}$*

$$H(\mathscr{A})(M)=J_{\mathscr{A}}(M)$$

Proof. It is clear by the lemma shown above, that $J_{\mathscr{A}}\in\big\{I\in IN(\mathbf{Top}):\mathscr{A}\subseteq T_1(I)\big\}$, and therefore $H(\mathscr{A})\sqsubseteq J_{\mathscr{A}}$.

Next, we will prove that $J_{\mathscr{A}}\sqsubseteq H(\mathscr{A})$. Since,

$$S(\mathbf{Top}) \underset{T_1}{\overset{H}{\rightleftarrows}} IN(\mathbf{Top})$$

is a Galois connection, $\mathscr{A}\subseteq T_1(H(\mathscr{A}))$, so that for every $X\in\mathscr{A}$, X is T_1-separated with respect to the interior operator $H(\mathscr{A})$.

Now, let $Y\in\mathscr{A}$ and let $M\subseteq X\in\mathbf{Top}$. Consider $X\xrightarrow{f}Y$ such that $f^{-1}\big(Y-\{y\}\big)\subseteq M$. Because $Y\in T_1\big(H(\mathscr{A})\big)$ implies $Y-\{y\}=h(\mathscr{A})_Y\big(Y-\{y\}\big)$ then,

$$f^{-1}\big(Y-\{y\}\big)=f^{-1}\Big(h(\mathscr{A})_Y\big(Y-\{y\}\big)\Big)$$

by continuity of $H(\mathscr{A})$,

$$\begin{aligned}&\subseteq h(\mathscr{A})_X\Big(f^{-1}\big(Y-\{y\}\big)\Big)\\&\subseteq h(\mathscr{A})_X(M).\end{aligned}$$

Hence, by definition of $(j_{\mathscr{A}})_X(M)$, we conclude that $(j_{\mathscr{A}})_X(M)\subseteq h(\mathscr{A})_X(M)$ accordingly, $J_{\mathscr{A}}\sqsubseteq H(\mathscr{A})$. This, together with the other inequality yields $J_{\mathscr{A}}=H(\mathscr{A})$. $\square$

Corollary 5.3.9. *Let $I_{\mathscr{A}}$ be the interior operator as established in [7], in other words, $(i_{\mathscr{A}})_X(M) = \bigcup \left\{ sep\ (f,g) \subseteq M : \ X \overset{f}{\underset{g}{\rightrightarrows}} Y,\ Y \in \mathscr{A} \right\}$ for $M \subseteq X$ and let $J_{\mathscr{A}}$ be as defined in (5.8). Then, we have that $J_{\mathscr{A}} \sqsubseteq I_{\mathscr{A}}$.*

Proof. Let $y^* \in Y$, we recall that $Y - \{y^*\} = sep\ (id,\ c)$, where id is the identity function and c is the constant function, $c(y) = y^*$, both defined from Y into Y. Now let $f^{-1}\big(Y - \{y^*\}\big) \subseteq M$, then

$$f^{-1}\big(Y - \{y^*\}\big) = f^{-1}\big(sep\ (id, c)\big) = sep\ \big(id \circ f,\ c \circ f\big) = sep\ (f, c)$$

Therefore $J_{\mathscr{A}} \sqsubseteq I_{\mathscr{A}}$. □

Remark. The converse of the above result is not true. In the definition of $(j_{\mathscr{A}})_X(M) = h(\mathscr{A})_X(M)$ consider $\mathscr{A} = \mathbf{Top_1}$ and let $f^{-1}\big(Y - \{y\}\big)$ be a subset of M. Since $Y \in \mathbf{Top_1}$, the one point sets are closed and consequently $Y - \{y\}$ is an open set. Thus we have that $f^{-1}\big(Y - \{y\}\big)$ is an open set contained in M, then $f^{-1}\big(Y - \{y\}\big) \subseteq \mathring{M}$ which implies that $h(\mathbf{Top_1})_X(M) \subseteq \mathring{M}$. Now, in Example 3.13 of [7] the authors found the following: $(i_{\mathscr{A}})_X(M) = M$. This gives us the proper inclusion below,

$$h(\mathbf{Top_1})_X(M) \subseteq \mathring{M} \subsetneq M = i(\mathbf{Top_1})_X(M) \tag{5.9}$$

when M is not open. For instance, for $X = \mathbb{R}$ and $M = [0, 1]$, $\mathring{M} \subset M$ and therefore $J_{\mathscr{A}} \sqsubset I_{\mathscr{A}}$.

Now, we will show some explicit examples using the assignment H defined above. Our first example shows that the family of topological spaces with the coarsest topology yields the discrete interior operator.

Example 5.1: Let $\mathscr{A} = \mathbf{Ind}$, where $\mathbf{Ind}$ is the collection of all indiscrete topological spaces. Consider the interior operator $H_{\mathbf{Ind}} = \big(h(\mathbf{Ind})_X\big)_{X \in \mathbf{Top}}$ that to $M \subseteq X$

assigns:

$$h(\mathbf{Ind})_X(M) = \bigcup \Big\{ f^{-1}\big(Y - \{y\}\big) \subseteq M : \ X \xrightarrow{f} Y, \quad Y \in \mathscr{A}, \ \ y \in Y \Big\}$$

Consider $Y = \{y_1, y_2\}$ with the indiscrete topology. Then, every $f : X \longrightarrow Y$ is continuous and so is in particular the function f defined by:

$$f(x) = \begin{cases} y_1 & \text{if} \quad x \in M \\ y_2 & \text{if} \quad x \notin M \end{cases}$$

Thus, $f^{-1}\big(Y - \{y_2\}\big) = M$ and consequently $h(\mathbf{Ind})_X(M) = M$.

This means that each $M \subseteq X$ is an $H_{\mathbf{Ind}}$-open set and therefore $H_{\mathbf{Ind}}$ is the discrete interior operator.

Example 5.2: Let $\mathscr{A} = \mathcal{S}$ consist of only the *Sierpinski* topological space Y, that is $Y = \{0, 1\}$ with topology $\mathcal{T} = \big\{\emptyset, \{0\}, Y\big\}$. Let $X \in \mathbf{Top}$ and let $M \subseteq X$. Now, for $U \subseteq M$ open in X. We define the function $X \xrightarrow{f} Y$ by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in U \\ 1 & \text{if} \quad x \notin U \end{cases}$$

We note that $f^{-1}\big(Y - \{1\}\big) = f^{-1}(0) = U$ and $f^{-1}(1) = X - U$. Clearly f is continuous and,

$$\overset{\circ}{M} \subseteq \ h\big(\mathcal{S}\big)_X(M)$$

since $U \subseteq h(\mathscr{A})(M)$ for every U open in X and contained in M.

Similarly if $F \subseteq M$ is closed, we define $X \xrightarrow{g} Y$ in the form

$$g(x) = \begin{cases} 0 & \text{if} \quad x \notin F \\ 1 & \text{if} \quad x \in F \end{cases}$$

Therefore g is continuous too. This, tells us that the union of all open or closed sets contained in M belong to $h(\mathcal{S})_X(M)$. Whereby,

$$w_X(M) \subseteq h\big(\mathcal{S}\big)_X(M)$$

where W is the interior operator discussed in equation (5.5). On the other hand, the other inclusion can be obtain easily since $f^{-1}\big(Y - \{a\}\big)$ and $f^{-1}\big(Y - \{b\}\big)$ either are open or closed sets contained in M, then

$$h(\mathcal{S})_X(M) \subseteq w_X(M)$$

and so,

$$w_X(M) = h\big(\mathcal{S}\big)_X(M)$$

Remark. The union of all open sets and closed sets contained in M does not necessarily equal M. That depends on the topology on X. For example, let $X = \mathbb{R}$ with the topology $\mathcal{T}$ consisting of $\emptyset$, $\mathbb{R}$ and all intervals of the form $(-\infty,\ p)$ for $p \in \mathbb{R}$. Clearly $X \in \mathbf{Top_0}$.
If we take $M = (-\infty,\ p]$, the union of all open and closed sets contained in M is equal to $(-\infty,\ p) \neq M$.

Example 5.3: Let $\mathscr{A} = \mathbf{Top_4}$ (normal spaces) and let $Y \in \mathbf{Top_4}$. Notice that the singletons of Y are closed sets and if $X \xrightarrow{f} Y$ is continuous, then

$$\begin{aligned} f^{-1}\big(Y - \{y\}\big) \subseteq M &\Longrightarrow f^{-1}\big(Y - \{y\}\big) \subseteq \mathring{M} \\ &\Longrightarrow h(\mathbf{Top_4})_X(M) \subseteq \mathring{M} \end{aligned}$$

Now, is it true that $\mathring{M} \subseteq h(\mathbf{Top_4})_X(M)$? We have not been able to answer this question but fortunately the next example comes a bit to the rescue.

Example 5.4: If $\mathscr{A} = \mathbf{Top_4}$ and $X \in \mathbf{Top}$ is a completely regular topological space (i.e, $X \in \mathbf{Top_{3\frac{1}{2}}}$), then $\mathring{M} \subseteq h(\mathbf{Top_4})_X(M)$.

Proof. We recall from [16] that $X \in \mathbf{Top}$ is completely regular if one-point sets are closed in X and if for each point x_0 and each closed set A not containing x_0, there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

Now let $U \subseteq M \subseteq X$ be open, then $X - U$ is closed. If $x \in U$ we have by hypothesis that there exists continuous function $f_x : X \longrightarrow [0,1]$ such that $f_x(x) = 1$ and $f_x\big(X - U\big) = \{0\}$.

Notice that by repeating this process for every $x \in U$ we have found a family of continuous functions $\{f_x\}_{x\in U}$ meeting the above conditions.
We note that $\bigcup\limits_{x\in U} f_x^{-1}\big((0,1]\big) = U \subseteq M$. So, as $\mathring{M}$ is an open set contained in M it can be represented as $\bigcup\limits_{x\in \mathring{M}} f_x^{-1}\big((0,1]\big)$ and by definition of the interior operator $H(\mathscr{A})$ we have that $\mathring{M} \subseteq h(\mathbf{Top_4})_X(M)$.

In this particular case we have obtained that

$$h(\mathbf{Top_4})_X(M) = \mathring{M}$$

as we wished. □

So far, we have the following: the interior operator $H(\mathscr{A}) = J_{\mathscr{A}}$ is order preserving then $h(\mathbf{Top_4})_X(M) \subseteq ... \subseteq h(\mathbf{Top_1})_X(M)$. By equation (5.9) we have $h(\mathbf{Top_1})_X(M) \subseteq \mathring{M}$ and by example 5.2, $w_X(M) = h(\mathcal{S})_X(M)$; thus, putting together these results, we obtain:

$$\boxed{h(\mathbf{Top_4})(M) \subseteq ... \subseteq h(\mathbf{Top_1})(M) \subseteq \mathring{M} \subseteq w(M) = h(\mathcal{S})(M) \subseteq M.}$$

where the last inclusion is by definition of interior operator.

Now, the following properties are important since they give us equivalent definitions of the concept of T_1-separated topological space with respect to an interior operator and furthermore they are also generalizations of the usual definition of T_1 topological space.

Proposition 5.3.10. *Let I be an interior operator in the category* **Top**. *$X \in T_1(I)$ if and only if for every $x, y \in X$ with $x \neq y$ there exists O_x, O_y I-open sets containing x and y, respectively, such that $y \notin O_x$ and $x \notin O_y$.*

Proof. ($\Longrightarrow$) We note that, $y \in X - \{x\} = i_X\big(X - \{x\}\big) = O_y$. Hence, we have $x \notin O_y$. Also, if $x \in X - \{y\} = i_X\big(X - \{y\}\big) = O_x$, and then $y \notin O_x$.
($\Longleftarrow$) Let $y \in X - \{x\}$ then $y \neq x$. By hypothesis there are O_x, O_y I-open sets such that $y \notin O_x$ and $x \notin O_y$. Then $O_y \subseteq X - \{x\}$, thus

$$O_y = i_X(O_y) \subseteq i_X\big(X - \{x\}\big)$$

Therefore $y \in i_X\big(X - \{x\}\big)$, that is $X - \{x\}$ is I-open and so $X \in T_1(I)$. □

Proposition 5.3.11. *Let I be an interior operator on the category* **Top**. *Then $X \in$* **Top** *is T_1-separated with respect to I if and only if for every $x \in X$, $\bigcap O_x = \{x\}$, where O_x is an I-open set containing x.*

Proof. ($\Longrightarrow$) Let $X \in T_1(I)$, i.e. X is T_1-separated with respect to I. Suppose that $\bigcap O_x \neq \{x\}$. In other words, there is $y \in \bigcap O_x$ such that $x \neq y$. We note that $x \in X - \{y\} = i_X\big(X - \{y\}\big)$. So, $X - \{y\}$ is a I-open set containing x but not y. Nevertheless by construction y belongs to every I-open set containing x and that yields a contradiction.
($\Longleftarrow$) Now, assume that $\bigcap O_x = \{x\}$ for each $x \in X$ and let $y \in X - \{x\}$. Then $\{y\} = \bigcap O_y \subseteq X - \{x\}$, that is, $x \notin \bigcap O_y$. Therefore, there is O_y, a nbhd of y, such that $x \notin O_y$.

Hence, by the properties of interior operators and the definition of I-open set we have,

$$O_y = i_X(O_y) \subseteq i_X\big(X - \{x\}\big)$$

thus $y \in i_X\big(X - \{x\}\big)$. We have proved that $X - \{x\} \subseteq i_X\big(X - \{x\}\big)$. Therefore $X \in T_1(I)$. $\square$

CHAPTER 6
T_1-COSEPARATED TOPOLOGICAL SPACES

6.1 T_1-coseparation

In this chapter we will study a notion that can be considered as the opposite of the concept of T_1-separated topological space with respect to an interior operator I. This notion will be very important later in order to define a Galois connection between the conglomerate of all subclasses of topological spaces $S(\mathbf{Top})$ ordered by inclusion and $IN(\mathbf{Top})^{op}$, i.e, the dual (or opposite) category of $IN(\mathbf{Top})$ with the order reversed. Hence for $I, J \in IN(\mathbf{Top})^{op}$, $I \sqsubseteq J$ if and only if for every $N \subseteq X \in \mathbf{Top}$, $i_X(M) \supseteq j_X(M)$. Thus, we start with the following definition:

Definition 6.1.1

Let I be an interior operator on the category **Top**. A space $X \in \mathbf{Top}$ is T_1-*coseparated* with respect to I if and only if for every $x \in X$,

$$i_X\big(X - \{x\}\big) = \emptyset. \tag{6.1}$$

$C_1(I)$ will denote all T_1-*coseparated* objects in **Top** with respect to I, i.e.,

$$C_1(I) = \big\{X \in \mathbf{Top} : X \text{ is } T_1\text{-coseparated with respect to } I\big\}.$$

Next we apply the previous definition to the examples of interior operators presented in Chapter 4. We observe that even though the concept of T_1-coseparation produces a more limited variety of classes of topological spaces compared to T_1-separation, this is essential for the theory developed in Chapter 7.

6.2 Examples

Here M will represent a subset of the topological space X.

(1) Let K be the usual interior operator in **Top**, i.e,

$$k_X(M) = \bigcup \Big\{ O \subseteq M : O \text{ is open in } X \Big\}$$

In this case we are going to see that $C_1(K) = \mathbf{Ind}$, where **Ind** consist of all indiscrete topological spaces, i.e., topological spaces with only two open sets, the empty set and the space itself.

First, if $X \in C_1(K)$ then for each $x \in X$,

$$k_X\big(X - \{x\}\big) = \bigcup \Big\{ O \subseteq X - \{x\} : O \text{ is an open set in } X \Big\} = \emptyset. \tag{6.2}$$

Let $x \in X$ and let O be an arbitrary nonempty open set in X. By equality (6.2) we have that $x \in O$. The previous sentence is true for every $x \in X$, hence $O = X$. Thus, the only open sets in X are $\emptyset$ and X itself. Therefore $X \in \mathbf{Ind}$.

Now, let $X \in \mathbf{Ind}$ and $x \in X$. We note that $\emptyset$ is the only open set in X contained in $X - \{x\}$, then $k_X\big(X - \{x\}\big) = \emptyset$. Therefore $X \in C_1(K)$. $\square$

(2) Let H be the interior operator defined by

$$h_X(M) = \bigcup \Big\{ C \subseteq M : C \text{ is closed in } X \Big\} = \big\{ x \in M : \overline{\{x\}} \subseteq M \big\}$$

We shall see that $C_1(H) = \mathbf{Ind}$. Consider $X \in C_1(H)$ then for every $x \in X$,

$$k_X\big(X - \{x\}\big) = \bigcup \Big\{ C \subseteq X - \{x\} : C \text{ is a closed set in } X \Big\} = \emptyset.$$

This means that if C is a nonempty closed set in X in a similar fashion to Example (a) it can be proved that C must contain every $x \in X$, and so $C = X$. Consequently $X \in \mathbf{Ind}$. On the other hand if $X \in \mathbf{Ind}$ then $X \in C_1(H)$, because the only closed set contained in $X - \{x\}$ is $\emptyset$. $\square$

(3) Let B be defined by

$$b_X(M) = \big\{x \in M : \exists\, U_x \text{ nbhd of } x \text{ such that } U_x \cap \overline{\{x\}} \subseteq M\big\}$$

In this case we also have that $C_1(B) = \mathbf{Ind}$.

Suppose that $X \in C_1(B)$. Then, for every $x \in X$, $b_X\big(X - \{x\}\big) = \emptyset$. Let $y \in X - \{x\}$, then for each U_y, nbhd of y, $U_y \cap \overline{\{y\}}$ must contain x and this is true for every $x \in X - \{y\}$, consequently $X \subseteq U_y \cap \overline{\{y\}} \subseteq U_y$. So, the only nonempty open set containing y is X itself. Since this is true for every $y \in X$ then $X \in \mathbf{Ind}$.

On the other hand if $X \in \mathbf{Ind}$ we have that $U_y \cap \overline{\{y\}} = X$ and then $b_X\big(X - \{x\}\big) = \emptyset$. $\square$

(4) Let Θ be defined by

$$\theta_X(M) = \big\{x \in M : \exists\, U_x \text{ nbhd of } x \text{ such that } \overline{U_x} \subseteq M\big\}$$

First, we recall that a topological space X is *nowhere separated* if for any two distinct points $x, y \in X$, if U_x and U_y are nbhds of x and y, respectively, then $U_x \cap U_y \neq \emptyset$. Let **NowhSep** be the class of all nowhere separated topological spaces. Notice that if $X \in \mathbf{Ind}$ then $X \in \mathbf{NowhSep}$. But $\mathbf{Ind} \neq \mathbf{NowhSep}$ for instance, $\mathbb{R}$ with the topology given by subsets of the form $(a, +\infty)$, is nowhere separated but not indiscrete. Now, in this case we have $C_1(\Theta) = \mathbf{NowhSep}$. Suppose that $X \in C_1(\Theta)$, then for each $x \in X$, $\theta_X(X - \{x\}) = \big\{y \in X - \{x\} : \exists U_y, \text{ nbhd of } y, \text{ such that } \overline{U_y} \subseteq X - \{x\}\big\} = \emptyset$, this means that for every $y \in X - \{x\}$ and every U_y nbhd of y we have $x \in \overline{U_y}$, that is, $U_x \cap U_y \neq \emptyset$ for every U_x, nbhd of x. Therefore $X \in \mathbf{NowhSep}$.

Now, let $X \in$ **NowhSep** and let $y \in X - \{x\}$, we have that for every U_x and U_y nbhds of x and y, respectively, it happens that $U_x \cap U_y \neq \emptyset$, hence $x \in \overline{U_y}$ and then $\overline{U_y} \nsubseteq X - \{x\}$. Whereby $\theta_X\big(X - \{x\}\big) = \emptyset$, that is $X \in C_1(\Theta)$. $\square$

(5) Let L be the following interior operator

$$l_X(M) = \big\{x \in X : C_x \subseteq M \text{ where } C_x \text{ is the connected component of } x\big\}$$

Here, we have that $C_1(L) =$ **Conn**, where **Conn** denotes all the connected topological spaces.

Let $X \in C_1(L)$, then for every $x \in X$

$$l_X\big(X - \{x\}\big) = \emptyset$$

in other words, for every $y \in X$ such that $y \neq x,\ x \in C_y$. Hence $X = C_y$ and consequently $X \in$ **Conn**.

If $X \in$ **Conn**, then for each $y \in X,\ C_y = X$. Thus $l_X\big(X - \{x\}\big) = \emptyset$, and we have $X \in C_1(L)$. $\square$

(6) Let T be defined by

$$t_X(M) = \big\{x \in M : x_n \longrightarrow x \text{ implies } x_n \in M \text{ eventually}\big\}$$

We will prove that $C_1(T) =$ **Ind**.

Suppose that $X \in C_1(T)$. Now, for every x in X, $t_X(X - \{x\}) = \big\{y \in X - \{x\} : y_n \longrightarrow y \text{ implies } y_n \in X - \{x\} \text{ eventually}\big\} = \emptyset$. This means that, if $y \in X - \{x\}$ and $y_n \longrightarrow y$ then for each $n \in \mathbb{N}$, there is $m > n$ such that $y_m = x$. Now, by definition of convergence: for every U_y nbhd of y, $x \in U_y$. This is true for every $x \in X$ with $x \neq y$, thus $X = U_y$. So, the only open sets are $\emptyset$ and X itself. Thence $X \in$ **Ind**.

On the other hand, let $X \in \mathbf{Ind}$ and let $x \neq y$, i.e., $y \in X - \{x\}$. We will prove that there exists a sequence converging at y but is not eventually in $X - \{x\}$. Consider $(y_n)_{n\geq 1}$ with $y_n = x$ for $n \geq 1$, then $y_n \longrightarrow y$ (because X indiscrete). We have that $(y_n)_{n\geq 1} \nsubseteq X - \{x\}$, hence $y \notin t_X\big(X - \{x\}\big)$. Consequently, for every $x \in X$, $t_X\big(X - \{x\}\big) = \emptyset$. Then $X \in C_1(T)$. □

(7) Let Q be the interior operator defined by

$$q_X(M) = \bigcup \Big\{C \subseteq M : C \text{ is a clopen set in } X\Big\}$$

We shall see that $C_1(Q) = \mathbf{Conn}$.

If $X \in C_1(Q)$ then, for every $x \in X$,

$$q_X(X - \{x\}) = \bigcup \Big\{C \subseteq X - \{x\} : C \text{ is clopen in } X\Big\} = \emptyset.$$

Now, if $C \neq \emptyset$ is clopen in X, we have $x \in C$. It follows that $X = C$. So, the only open and closed sets are $\emptyset$ and X. Hence X is connected.

It is easy to see that if X is connected then for every $x \in X$, $q_X\big(X - \{x\}\big) = \emptyset$. Therefore $X \in C_1(Q)$. □

(8) Let W be the operator defined by:

$$\omega_X(M) = \bigcup \big\{E \subseteq M : E \text{ is open or closed in } X\big\}$$

We shall show that $C_1(W) = \mathbf{Ind}$.

First, let $X \in C_1(W)$. This means that for each $x \in X$,

$$w_X\big(X - \{x\}\big) = \bigcup \big\{E \subseteq X - \{x\} : E \text{ is an open or closed set in } X\big\} = \emptyset$$

Thus, if E is an open or closed set in X, then E should be either $\emptyset$ or X; hence $X \in \mathbf{Ind}$.

On the other hand, if $X \in \mathbf{Ind}$ then $\emptyset$ and X are the only open or closed sets in X. Consequently, $w_X\big(X - \{x\}\big) = \emptyset$, i.e., $\mathbf{Ind} \subseteq C_1(W)$. Hence $C_1(W) = \mathbf{Ind}$ $\square$

In the following table we present a list of all the available examples of T_1-separation and T_1-coseparation with respect to some interior operators.

Table 6–1: Review of T_1-separated and T_1-coseparated objects

Example	Interior Operator (I)	T_1-separated objects $\big(T_1(I)\big)$	T_1-coseparated objects $\big(C_1(I)\big)$
(1)	K	$\mathbf{Top}_1$	$\mathbf{Ind}$
(2)	H	$\mathbf{Top}_1$	$\mathbf{Ind}$
(3)	B	$\mathbf{Top}_0$	$\mathbf{Ind}$
(4)	Θ	$\mathbf{Top}_2$	$\mathbf{NowSep}$
(5)	L	$\mathbf{TotDisc}$	$\mathbf{Conn}$
(6)	T	$\mathbf{Top}_1$	$\mathbf{Ind}$
(7)	Q	$\mathbf{TotSep}$	$\mathbf{Conn}$
(8)	W	$\mathbf{Top}_1$	$\mathbf{Ind}$

6.3 Properties of the T_1-coseparated objects

In this section we will present some properties related to the coseparated topological spaces. These results will help us to create a Galois connection between $S(\mathbf{Top})$ and $IN(\mathbf{Top})^{op}$. This connection will be very important later, because it will allow us to factor a classical Galois connection via interior operators. We start with the following result.

Proposition 6.3.1. *The function* $IN(\mathbf{Top})^{op} \xrightarrow{C_1} S(\mathbf{Top})$ *defined by*

$$C_1(I) = \big\{X \in \mathbf{Top} : X \textit{ is } T_1\textit{-coseparated with respect to } I\big\}$$

preserves infima.

Proof. Let $\{I_k\}_{k\in K}$ be a family of interior operators on **Top**. We will prove that

$$C_1\Big(\bigwedge_{k\in K}^{op} I_k\Big) = \bigcap_{k\in K} C_1(I_k)$$

We need to remember that in $IN\big(\mathbf{Top}\big)^{op}$, the interior operator $\bigwedge\limits_{k\in K}^{op} I_k$ represents the infima, in other words, $\bigwedge\limits_{k\in K}^{op} I_k \overset{op}{\sqsubseteq} I_k$ for every $k \in K$.

First, we will prove that C_1 is order preserving. Let $I \overset{op}{\sqsubseteq} J$, i.e., $j_{_X}(M) \subseteq i_{_X}(M)$. Let $X \in C_1(I)$, then for each $x \in X$, $i_{_X}\big(X - \{x\}\big) = \emptyset$. By assumption $j_{_X}\big(X - \{x\}\big) \subseteq i_{_X}\big(X - \{x\}\big)$. Consequently, for every $x \in X$, $j_{_X}\big(X - \{x\}\big) = \emptyset$ and so $X \in C_1(J)$ and we have obtained our desired result.

Order preservation of C_1 implies that for every $k \in K$,

$$C_1\Big(\bigwedge_{k\in K}^{op} I_k\Big) \subseteq C\big(I_k\big)$$

and therefore,

$$C_1\Big(\bigwedge_{k\in K}^{op} I_k\Big) \subseteq \bigcap_{k\in K} C_1(I_k)$$

On the other hand, let $X \in \bigcap\limits_{k\in K} C_1(I_k)$, then $X \in C_1(I_k)$ for each $k \in K$. So, that for every $x \in X$ and $k \in K$,

$$(i_{_k})_{_X}\big(X - \{x\}\big) = \emptyset \tag{6.3}$$

Furthermore, from Proposition 4.3.5 we have the following:

$$i_{\wedge^{op}_{I_k}}(M) = i_{\vee_{I_k}}(M) = \bigcup_{k\in K} i_k(M) \tag{6.4}$$

thus from (6.3) and (6.4) we have

$$i_{\wedge^{op}_{I_k}}\big(X - \{x\}\big) = \bigcup_{k\in K} i_k\big(X - \{x\}\big) = \emptyset$$

Consequently, $X \in C_1\Big(\bigwedge_{k\in K}^{op} I_k\Big)$. Therefore C_1 preserves infima. $\square$

Proposition 6.3.1 and Proposition 3.2.3 [3] imply the existence of the following Galois connection.

$$S(\mathbf{Top}) \underset{C_1}{\overset{D}{\rightleftarrows}} IN(\mathbf{Top})^{op}$$

where D is defined by,

$$D(\mathscr{A}) = \bigwedge \left\{ I \in IN\big(\mathbf{Top}\big)^{op} : \mathscr{A} \subseteq C_1(I) \right\} \tag{6.5}$$

Now, we are going to show a more explicit form of the function D.

Proposition 6.3.2. *Let $\mathscr{A} \in S(\mathbf{Top})$ and let $M \subseteq Y \in \mathbf{Top}$. We define $I_{\mathscr{A}}$ by*

$$(i_{\mathscr{A}})_Y(M) = \bigcup \Big\{ N \subseteq M : \forall\ X \xrightarrow{f} Y \ \text{ and } X \in \mathscr{A} \text{ with } f^{-1}(M) \neq X,\ f^{-1}(N) = \emptyset \Big\}$$

Then, $I_{\mathscr{A}}$ is an interior operator on **Top**.

Proof. First, it is clear that $(i_{\mathscr{A}})_Y(M) \subseteq M$. Then, if $M \subseteq M'$ and $N \subseteq M$ satisfies the conditions in the definition of $(i_{\mathscr{A}})_Y(M)$, then we have that N also satisfies the conditions in the definition of $(i_{\mathscr{A}})_Y(M')$, hence $(i_{\mathscr{A}})_Y(M) \subseteq (i_{\mathscr{A}})_Y(M')$.

Finally, let $Y \xrightarrow{f} Z$ be a continuous function and let $M \subseteq Z$.

$$\begin{tikzcd} Y \arrow[rr, "f"] & & Z \\ & X \arrow[ul, "g"] \arrow[ur, "f\circ g"'] & \end{tikzcd}$$

Then by definition we have:

$(i_{\mathscr{A}})_Y\big(f^{-1}(M)\big) =$

$= \bigcup \Big\{ K \subseteq f^{-1}(M) : \forall X \xrightarrow{g} Y,\ X \in \mathscr{A} \text{ with } g^{-1}\big(f^{-1}(M)\big) \neq X,\ g^{-1}(K) = \emptyset \Big\}$

$\supseteq \bigcup \Big\{ f^{-1}(N) \subseteq f^{-1}(M) : \forall X \xrightarrow{g} Y,\ X \in \mathscr{A} \text{ with } g^{-1}\big(f^{-1}(M)\big) \neq X,\ g^{-1}(f^{-1}(N)) = \emptyset \Big\}$

$\supseteq \bigcup \Big\{ f^{-1}(N) \subseteq f^{-1}(M) : \forall\ X \xrightarrow{h} Z,\ X \in \mathscr{A} \text{ with } h^{-1}(M) \neq X,\ h^{-1}(N) = \emptyset \Big\}$

$$\supseteq f^{-1}\Big(\bigcup\Big\{N\subseteq M : \forall\ X \xrightarrow{h} Z \text{ and } X\in\mathscr{A} \text{ with } h^{-1}(M)\neq X,\ h^{-1}(N)=\emptyset\Big\}\Big)$$
$$= f^{-1}\big((i_{\mathscr{A}})_Z(M)\big)$$

Notice that the second " $\supseteq$ " is true since not all continuous functions from X into Z factor as $f\circ g$. Thus, $I_{\mathscr{A}}$ is an interior operator. □

Additionally, we have the following consequence.

Lemma 6.3.3. *If $Y\in\mathscr{A}$ then $Y\in C_1(I_{\mathscr{A}})$, i.e., $\mathscr{A}\subseteq C_1(I_{\mathscr{A}})$.*

Proof. The existence of the identity function from Y into Y, implies the only $N\subseteq Y-\{y\}$ with $f^{-1}(N)=\emptyset$ is $N=\emptyset$. Hence, $(i_{\mathscr{A}})_Y\big(Y-\{y\}\big)=\emptyset$. Therefore $Y\in C_1(I_{\mathscr{A}})$. □

Now, we are ready to prove that $I_{\mathscr{A}}$ is the explicit representation of (6.5).

Proposition 6.3.4. *We have that $D(\mathscr{A})=I_{\mathscr{A}}$, where D is defined by (6.5)*

Proof. First of all, we shall prove that $I_{\mathscr{A}}\sqsubseteq D(\mathscr{A})$. Notice that by the previous lemma,

$$I_{\mathscr{A}}\in\Big\{I\in IN\big(\mathbf{Top}\big)^{op} : \mathscr{A}\subseteq C_1(I)\Big\}=\mathbf{C}_{\mathscr{A}}$$

then from (6.5) we have that $D(\mathscr{A})\sqsubseteq I_{\mathscr{A}}$ in $IN\big(\mathbf{Top}\big)^{op}$, i.e., $D(\mathscr{A})\overset{op}{\sqsubseteq} I_{\mathscr{A}}$.

On the other hand, let $f : X\longrightarrow Y$ be a continuous function with $X\in\mathscr{A}$, and let $M\subseteq Y$ such that $f^{-1}(M)\neq X$. Notice that for every $I\in\mathbf{C}_{\mathscr{A}}$ if $X\in\mathscr{A}$ then $X\in C_1(I)$, that means for every $x\in X$, $i_X\big(X-\{x\}\big)=\emptyset$. Now, by hypothesis $f^{-1}(M)\subseteq X-\{x\}$ for some $x\in X$ and hence $i_X\big(f^{-1}(M)\big)\subseteq i_X\big(X-\{x\}\big)=\emptyset$. So, $i_X\big(f^{-1}(M)\big)=\emptyset$ for each $I\in\mathbf{C}_{\mathscr{A}}$. Since,

$$(d_{\mathscr{A}})_Y(M)=\Big(\bigwedge_{I\in\mathbf{C}_{\mathscr{A}}}^{op} I\Big)(M)=\bigcap_{I\in\mathbf{C}_{\mathscr{A}}}^{op} i(M)=\bigcup_{I\in\mathbf{C}_{\mathscr{A}}} i(M),$$

We conclude that $(d_{\mathscr{A}})_X\big(f^{-1}(M)\big) = \emptyset$. Then, by the continuity property of interior operators,

$$f^{-1}\Big((d_{\mathscr{A}})_Y(M)\Big) = \emptyset.$$

Since $(d_{\mathscr{A}})_Y(M) \subseteq M$, then $(d_{\mathscr{A}})_Y(M)$ is one of the N's that occur in the definition of $(i_{\mathscr{A}})_Y(M)$ and consequently for every $M \subseteq Y$, $(d_{\mathscr{A}})_Y(M) \subseteq (i_{\mathscr{A}})_Y(M)$. Hence, $I_{\mathscr{A}} \sqsubseteq D(\mathscr{A})$ in $IN\big(\mathbf{Top}\big)^{op}$, i.e., $I_{\mathscr{A}} \overset{op}{\sqsubseteq} D(\mathscr{A})$.

Finally we conclude that $I_{\mathscr{A}} = D(\mathscr{A})$. □

Next we are going to show that any quotient topological space of a T_1-coseparated space X is a T_1-coseparated space.

Proposition 6.3.5. *Consider an interior operator I on* ***Top****. Let $X \in C_1(I)$ and let $Y \in$* ***Top*** *be the quotient space of X, under some equivalence relation $\sim$, with $X \xrightarrow{f} Y$ being the quotient function (natural projection), then $Y \in C_1(I)$.*

Proof. We need to prove that for each $y \in Y$, $i_Y\big(Y - \{y\}\big) = \emptyset$. Let $y \in Y$ and by the properties of interior operators we have,

$$f^{-1}\Big(i_Y\big(Y - \{y\}\big)\Big) \subseteq i_X\Big(f^{-1}\big(Y - \{y\}\big)\Big)$$

and since f is onto,

$$f^{-1}\big(Y - \{y\}\big) \subseteq X - \{x\},$$

for some $x \in X$. Consequently,

$$i_X\Big(f^{-1}\big(Y - \{y\}\big)\Big) \subseteq i_X\big(X - \{x\}\big) = \emptyset$$

that is,

$$f^{-1}\Big(i_Y\big(Y - \{y\}\big)\Big) = \emptyset,$$

and as f is onto this implies,

$$i_Y\big(Y - \{y\}\big) = \emptyset.$$

Hence $Y \in T_1(I)$, as we wanted to show. $\square$

6.4 Composition of Galois connections

In this section we are going to show that a classical Galois connection, that first appeared in an early work by Herrlich [12] can be factorized by the previous two Galois connections given by the concepts of T_1-separation and T_1-coseparation. In other words, the composition of these Galois connections can be represented by this new Galois connection. First, we need to define the following:

Definition 6.4.1

Let $S\big(\mathbf{Top}\big) \xrightarrow{\Delta} S\big(\mathbf{Top}\big)^{op}$ be the function defined by,

$$\Delta(\mathscr{A}) = \big\{Y \in \mathbf{Top} : \ \forall\, X \xrightarrow{f} Y, \ \ X \in \mathscr{A}, \ f \text{ is constant}\big\},$$

and let $S\big(\mathbf{Top}\big)^{op} \xrightarrow{\nabla} S\big(\mathbf{Top}\big)$ be the function defined by,

$$\nabla(\mathscr{B}) = \big\{X \in \mathbf{Top} : \ \forall\, X \xrightarrow{f} Y, \ \ Y \in \mathscr{B}, \ f \text{ is constant}\big\}.$$

As it was stablished in [12], $S(\mathbf{Top}) \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} S\big(\mathbf{Top}\big)^{op}$ forms a Galois connection. It is very important to mention that this was used by [1] to define the concepts of topological connectedness and disconnectedness with respect to a subclass of topological spaces and afterwards in [8] it was related to new notions of these last concepts but means of interior operators. Now, we present the mentioned result.

Proposition 6.4.2. *In the next diagram we have that:*

$$\begin{tikzcd}
 & IN\big(\mathbf{Top}\big)^{op} \arrow[dl, shift left, "C_1"'] \arrow[dr, shift left, "T_1"] & \\
S\big(\mathbf{Top}\big) \arrow[ur, shift left, "D"] \arrow[rr, shift left, "\Delta"] & & S\big(\mathbf{Top}\big)^{op} \arrow[ul, shift left, "H"'] \arrow[ll, shift left, "\nabla"]
\end{tikzcd}$$

(a) $C_1 \circ H = \nabla$

(b) $T_1 \circ D = \Delta$

Proof. **(a)** Let $\mathscr{B} \in S\big(\mathbf{Top}\big)^{op}$, we suppose that $X \notin \nabla(\mathscr{B})$. Then, there is $X \xrightarrow{f} Y$, with $Y \in \mathscr{B}$ such that f is not constant, i.e, the image of f has at least two different points y_1, y_2 in Y.

Now, let x_1 be a point in X such that $f(x_1) = y_1$. Notice that $f^{-1}\big(Y - \{y_1\}\big) \neq \emptyset$ and moreover $f^{-1}\big(Y - \{y_1\}\big) \subseteq X - \{x_1\}$, so there exists $x_1 \in X$ such that $(h_{\mathscr{B}})_X\big(X - \{x_1\}\big) \neq \emptyset$. Consequently $X \notin C_1\big(H(\mathscr{B})\big)$. We have proved that $\big(C_1 \circ H\big)(\mathscr{B}) \subseteq \nabla(\mathscr{B})$.

On the other hand, let $X \in \nabla(\mathscr{B})$. If $f : X \longrightarrow Y$ is a continuous function with $Y \in \mathscr{B}$ then $f(X) = \{y^*\}$. It is easy to see that,

$$f^{-1}\big(Y - \{y\}\big) = \begin{cases} \emptyset & \text{if} \quad y = y^* \\ X & \text{if} \quad y \neq y^* \end{cases}$$

Then, we have that for every $x \in X$,

$$(h_{\mathscr{B}})_X\big(X - \{x\}\big) = \bigcup \Big\{ f^{-1}\big(Y - \{y\}\big) \subseteq X - \{x\} : \; X \xrightarrow{f} Y, \; Y \in \mathscr{B}, \; y \in Y \Big\} = \emptyset,$$

and therefore $X \in C_1\big(H(\mathscr{B})\big)$. So, we have obtained that $\nabla(\mathscr{B}) \subseteq \big(C_1 \circ H\big)(\mathscr{B})$, that is $C_1 \circ H = \nabla$.

(b) Let $\mathscr{A} \in S\big(\mathbf{Top}\big)$, we suppose that $Y \notin \Delta(\mathscr{A})$. So, there is $X \xrightarrow{f_0} Y$ and $X \in \mathscr{A}$ such that f_0 is not constant, i.e, the image of f_0 has at least two different points y_0, y_1 in Y. Let $x_0 \in X$ with $f_0(x_0) = y_0$. Notice that $f_0^{-1}\big(\{y_1\}\big) \neq X$ since $x_0 \notin f_0^{-1}\big(\{y_1\}\big)$ and also, $f_0^{-1}\big(\{y_1\}\big) \neq \emptyset$ since there is $x_1 \in X$ such that $f_0(x_1) = y_1 \in \{y_1\}$.

Hence, $\{y_1\}$ is not one of the N's in the definition of the interior operator $D(\mathscr{A})$ in Proposition 6.3.2. Moreover $\{y_1\}$ cannot be included in any of those N's, since $\emptyset \neq f_0^{-1}\big(\{y_1\}\big) \subseteq f_0^{-1}\big(N\big)$, would yield a contradiction. So, we have that $Y - \{y_0\} \nsubseteq (d_{\mathscr{A}})_Y\big(Y - \{y_0\}\big)$. Therefore $Y \notin T_1\big(D(\mathscr{A})\big)$, in other words, $\big(T_1 \circ D\big)(\mathscr{A}) \subseteq \Delta(\mathscr{A})$.

Now, let $Y \in \Delta(\mathscr{A})$ and let $f : X \to Y$ be a continuous function with $X \in \mathscr{A}$ suppose that for $y \in Y$, $f^{-1}\big(Y - \{y\}\big) \neq X$ is true. Then, since f is constant by hypothesis, $f(X) = \{y^*\}$ and consequently we have that,

$$f^{-1}\big(Y - \{y\}\big) = \begin{cases} \emptyset & \text{if} \quad y = y^* \\ X & \text{if} \quad y \neq y^*. \end{cases}$$

However, by assumption $f^{-1}\big(Y - \{y\}\big) \neq X$ and so, the only possible option is $f^{-1}\big(Y - \{y\}\big) = \emptyset$. Hence, $Y - \{y\} \subseteq (d_{\mathscr{A}})_Y\big(Y - \{y\}\big)$. Thus $\Delta(\mathscr{A}) \subseteq \big(T_1 \circ D\big)(\mathscr{A})$ and therefore we conclude that $\Delta = T_1 \circ D$. □

In [8] the concept of *I-indiscrete* topological space was defined. It is interesting to see that for any interior operator I there is an equivalence between topological spaces that are T_1-*coseparated* with respect to I and I-*indiscrete* topological spaces, as it is shown in the next result.

Proposition 6.4.3. *Let I be an interior operator on **Top**. $X \in$ **Top** is T_1-coseparated if and only if X is I-indiscrete, in other words*

$$\text{For every } x \in X,\ i_X\big(X - \{x\}\big) = \emptyset \Longleftrightarrow \text{for every } M \subset X,\ i_X(M) = \emptyset.$$

Proof. ($\Longrightarrow$) Let $M \subset X$, then there is $x \in X$ such that $x \notin M$ then $M \subseteq X - \{x\}$. By interior operators properties, $i_X(M) \subseteq i_X\big(X - \{x\}\big) = \emptyset$ and by hypothesis $i_X(M) = \emptyset$.

($\Longleftarrow$) This part is trivial because $X - \{x\}$ is a proper subset of X. □

Remark. The previous proposition has an unexpected consequence, that is, the interior operator $I_{\mathscr{A}}$ in Proposition 6.3.2 matches with the interior operator defined in Proposition 3.22 in [8]. But, this leads to another unexpected result, that is the factorization of a Galois connection may not be unique even when one of the factors is fixed. In other words, two different factorizations that have a common factor may exist.

CHAPTER 7
OTHER RESULTS

We are interested in introducing and studying a notion of T_1-separation on the category **Grp** of groups and homomorphisms like we did in **Top** by using a similar approach. In [4], Castellini defined T_1-separation in an arbitrary category, in particular on groups, but in that case he used the concept of closure operator. It is important to remember that in **Grp** the sub-objects are the subgroups and in **Top** they are the subspaces. In Proposition 2.2.2, we established an interesting equivalence between the notion of T_1-separation and the intersection of all the neighborhoods of the points being singletons. We are going to use this result in order to define a similar notion on the category **Grp**, since in this category we cannot talk about the complements of the points as subobjects, because these are not subgroups. So, the idea is to try to recreate the work done in **Top**, but using a different approach in the category **Grp**. First, we are going to give a few examples of interior operators in **Grp**.

7.1 Interior operators on Grp

The following examples of interior operators together with their respective proofs can be found in [14]. For groups, the definition of interior operator is similar to the given in Chapter 4 on the category **Top**. Now, we need to work on the subgroup lattices of **Grp**. Here M is an arbitrary subgroup of the group X.

(1) [14] Let D be defined by $d_X(M) = M$ for every $M \leq X,\ X \in \mathbf{Grp}$. Thus D is called the *discrete* interior operator.

(2) [14] The assignment T defined by $T_X(M) = \{e_X\}$ for every $M \leq X,\ X \in \mathbf{Grp}$ is the *trivial* interior operator.

(3) [14] For every $M \leq X,\ X \in \mathbf{Grp}$ we consider P with,

$$p_X(M) = \bigvee \left\{K \leq M : K \trianglelefteq X\right\}$$

This is the *normal* interior operator.

(4) [14] We consider the function $I_{\mathbf{Ab}}$ given by

$$i_{\mathbf{Ab}}(M) = \bigvee \left\{K \leq M : K \trianglelefteq X,\ X/K \in \mathbf{Ab}\right\}.$$

Then, $I_{\mathbf{Ab}}$ is an interior operator.

Next, we present two different ways to construct interior operators from a given class of groups satisfying certain conditions:

(5) [14] Let $\mathscr{C}$ be any subclass of groups that is closed under subgroups and quotients. That is, if $M \leq X$, then $X \in \mathscr{C}$; and if $K \trianglelefteq X$, then $X/K \in \mathscr{C}$. Thus, for each $M \subseteq X,\ X \in \mathbf{Grp}$ the following function,

$$i_{\mathscr{C}}(M) = \bigvee \left\{K \leq M : K \trianglelefteq X,\ X/K \in \mathscr{C}\right\} \tag{7.1}$$

defines an interior operator.

(6) [14] Let $\mathscr{H}$ be a class of subgroups closed under suprema and inversa images under homomorphisms. For each $M \leq X$ and $X \in \mathbf{Grp}$ consider $I^{\mathscr{H}}$ the interior operator defined by,

$$i^{\mathscr{H}}(M) = \bigvee \left\{K \leq M : K \in \mathscr{H}\right\} \tag{7.2}$$

Moreover, $I^{\mathscr{H}}$ is idempotent.

Definition 7.1.1

Let I be an interior operator on **Grp**. Then $X \in$ **Grp** is T_1-*separated* with respect to I if and only if the intersection of all I-open subgroups M of X, with $M \neq \{e_X\}$, is equal to the identity, i.e.,

$$\bigcap_{\substack{M \leq X \\ M \neq \{e_X\} \\ i_X(M)=M}} M = \{e_X\}. \tag{7.3}$$

Furthermore, we denote by $T_1^G(I)$ the class of all T_1-separated groups with respect to the interior operator I.

We recall that a group X is called a torsion group if and only if every element $x \in X$ has finite order. Let **Tor** denote all torsion groups. The following lemma is useful to characterize some classes of groups that will arise later.

Lemma 7.1.2 [13]**.** *Let X be a group such that the intersection of all its subgroups different from $\{e_X\}$ is a subgroup different from $\{e_X\}$. Then, every element of X is of finite order, that is, X is a torsion group ($X \in$ **Tor**).*

Proof. Suppose not. So, there should exists an element $x \neq e$ such that $|x| = \infty$, that is, $x^k \neq e$ for every $k \geq 1$. This implies that all the elements in $\langle x \rangle = \{e, x^1, x^2, x^3, ...\}$ are different. Now, let H be the intersection of all the subgroups in $\langle x \rangle$ different from $\{e\}$ and suppose that there exists an element $g \neq e$. If $g = x^m$, then $g \in \bigcap_{n \geq m} \langle x^n \rangle$. But this is not true, because as we previously mentioned, all the elements in $\langle x \rangle$ are different. Hence, we have a contradiction. □

Now, we present some examples:

Example 1: Let D be the *discrete* interior operator in **Grp**, i.e., $d_X(M) = M$ for each $M \leq X$. Let $X \in T_1^G(D)$ then,

$$\{e_X\} = \bigcap_{\substack{M \leq X \\ M \neq \{e_X\} \\ d_X(M)=M}} M = \bigcap_{\substack{M \leq X \\ M \neq \{e_X\}}} M.$$

We have that $\mathbf{Grp} - \mathbf{Tor} \subseteq T_1^G(D)$. Suppose not, then, there would exist a group X with at least an element of infinite order that satisfies,

$$\bigcap_{\substack{M \leq X \\ M \neq \{e_X\}}} M \neq \{e\}$$

and this is a contradiction to Lemma 7.1.2. However, $T_1^G(D) \nsubseteq \mathbf{Grp} - \mathbf{Tor}$, since $S_3 \in T_1^G(D) \bigcap \mathbf{Tor}$.

Example 2: Let T be the *trivial* interior operator in $\mathbf{Grp}$, i.e., $t_X(M) = \{e_X\}$ for every $M \leq X$. We note that if $X \in \mathbf{Grp}$, then the only subgroup M in X such that $t_X(M) = M$ is $\{e_X\}$, but by hypothesis $M \neq \{e_X\}$, that is, there are no T-open subgroups different from the identity.

Now, we recall the definition of empty intersection precisely, if $\{M_i\}$ is an empty family of subgroups of a given group X then $\bigcap_{i \in \emptyset} M_i = X$. So, vacuously the only $X \in \mathbf{Grp}$ satisfying that $\bigcap_{i \in \emptyset} M_i = \{e_X\}$ is $X = \{e\}$, i.e, the identity group. Thus, we have showed that $T_1^G(T) = \{e\}$.

Example 3: Let P be the *normal* interior operator definined by $p_X(M) = \bigvee\{K \leq M : K \trianglelefteq X\}$. We first note that, $p_X(M) = M \iff M \trianglelefteq X$ [14]. Let $X \in T_1^G(P)$, then

$$\{e_X\} = \bigcap_{\substack{M \leq X \\ M \neq \{e_X\} \\ p_X(M)=M}} M = \bigcap_{\substack{M \trianglelefteq X \\ M \neq \{e_X\}}} M.$$

As in Example 1, it can be proved that $\mathbf{Ab} - \mathbf{AbTor} \subseteq T_1^G(P)$, where $\mathbf{AbTor}$ denotes all abelian torsion groups, that is, abelian groups in which every element has finite order. However, $T_1^G(P) \nsubseteq \mathbf{Ab} - \mathbf{AbTor}$, since $Z_6 \in T_1^G(P) \cap \mathbf{AbTor}$.

Example 4: Consider the interior operator $I_{\mathbf{Ab}}$, defined by

$$i_{\mathbf{Ab}}(M) = \bigvee \{K \leq M : K \trianglelefteq X \text{ and } X/K \in \mathbf{Ab}\}$$

for every $M \leq X$ and every $X \in \mathbf{Grp}$.

In this case is easy to see that, $i_{\mathbf{Ab}}(M) = M \iff M \trianglelefteq X$ and $X/M \in \mathbf{Ab}$ [14]. Now, let $X \in T_1^G(I_{\mathbf{Ab}})$ and hence by applying the definition of T_1-separated group we obtain,

$$\{e_X\} = \bigcap_{\substack{M \leq X \\ M \neq \{e_X\} \\ i_{\mathbf{Ab}}(M)=M}} M = \bigcap_{\substack{M \neq \{e_X\} \\ M \trianglelefteq X,\ X/M \in \mathbf{Ab}}} M$$

We claim that $T_1^G(I_{\mathbf{Ab}}) \subseteq \mathbf{Ab}$.

Again, let $X \in T_1^G(I_{\mathbf{Ab}})$ and let $\{M_i\}_{i\in I}$ be a family of normal subgroups in X such that $X/M_i \in \mathbf{Ab}$, for every $i \in I$. Consider $\left\{X \xrightarrow{q_i} X/M_i\right\}_{i\in I}$ where q_i is the quotient homomorphism.

Now, let $X \xrightarrow{\langle q_i \rangle} \prod X/M_i$ be the induced homomorphism into the product. We have that $\ker\langle q_i\rangle = \bigcap_{i\in I} \ker q_i = \bigcap M_i$. Since X satisfies $\bigcap M_i = \{e_X\}$, then $\ker\langle q_i\rangle = \{e_X\}$, that is, $\langle q_i\rangle$ is injective and so,

$$X \simeq \langle q_i\rangle(X) \leq \prod X/M_i \in \mathbf{Ab}.$$

Hence $X \in \mathbf{Ab}$, that is $T_1^G(I_{\mathbf{Ab}}) \subseteq \mathbf{Ab}$.

On the other hand, let $X \in \mathbf{Ab} - \mathbf{AbTor}$, that is, X is an abelian group where not all its elements are of finite order. Also, every subgroup of X satisfies $M \trianglelefteq X$ and $X/M \in \mathbf{Ab}$.

As an easy consequence of Lemma 7.1.2 [13] we conclude that $X \in T_1^G(I_{\mathbf{Ab}})$:

$$\mathbf{Ab} - \mathbf{AbTor} \subseteq T_1^G(I_{\mathbf{Ab}}) \subseteq \mathbf{Ab}$$

In a more general way we look at the following interior operator generated by a subclass $\mathscr{C}$ of groups that is closed under subgroups, quotients and products.

Example 5: Let $\mathscr{C}$ be a subclass of groups closed under subgroups, quotients and products and let $I_{\mathscr{C}}$ be the interior operator defined by,

$$i_{\mathscr{C}}(M) = \bigvee \left\{ K \leq M : K \trianglelefteq X \text{ and } X/K \in \mathscr{C} \right\}$$

Then it turns out that $i_{\mathscr{C}}(M) = M \iff M \trianglelefteq X$ and $X/M \in \mathscr{C}$ [14]. Furthermore, if $X \in T_1^G(I_{\mathscr{C}})$, then

$$\{e_X\} = \bigcap_{\substack{M \leq X \\ M \neq \{e_X\} \\ i_{\mathscr{C}}(M) = M}} M = \bigcap_{\substack{M \neq \{e_X\} \\ M \trianglelefteq X,\ X/M \in \mathscr{C}}} M.$$

Similarly to Example 4 we have that $T_1^G(I_{\mathscr{C}}) \subseteq \mathscr{C}$.

Next we present an equivalence that will allow us to prove an interesting property later.

Proposition 7.1.5. *Let I be an interior operator on **Grp** and let $X \in$ **Grp**. Then the following are equivalent.*

(1) *X is T_1-separated with respect to I.*

(2) *For every $x \in X$, $x \neq e_X$, there is an I-open subgroup that does not contain x.*

Proof. We have the following equivalences:

$$\bigcap_{\substack{M \leq X \\ M \neq \{e_X\} \\ i_X(M) = M}} M = \{e_X\} \iff \forall x \in X - \{e_X\},\ x \notin \bigcap_{\substack{M \leq X \\ M \neq \{e_X\} \\ i_X(M) = M}} M$$

$$\iff \forall x \in X - \{e_X\}, \exists\, M_0, I\text{-open subgroup such that } x \notin M_0.$$

Thus, we have proved the desired result. $\square$

Analogously to the topological case we have the following.

Lemma 7.1.6. *The inverse image under a homomorphism of an I-open subgroup is an I-open subgroup.*

Proof. Let $M \leq Y$ an I-open subgroup, then $i_Y(M) = M$. Let $f : X \longrightarrow Y$ be a homomorphism. Consequently by the continuity condition of interior operators we have that

$$i_X\big(f^{-1}(M)\big) \geq f^{-1}\big(i_Y(M)\big) = f^{-1}(M)$$

whence $f^{-1}(M)$ is an I-open subgroup. $\square$

Proposition 7.1.7. *Let I be an interior operator on the category **Grp**. Then $T_1^G(I)$ is closed under monosources.*

Proof. Let $\left(X \xrightarrow{f_k} Y_k\right)_{k\in K}$ be a monosource. Let $x \in X$ with $x \neq e_X$, then by definition of monosource there is f_{k_0} such that $f_{k_0}(x) \neq f_{k_0}(e_X) = e_Y$. Now $Y_{k_0} \in T_1^G(I)$, then by Proposition 7.1.5 exists N_0, I-open subgroup in Y_{k_0}, such that $f_{k_0}(x) \notin N_0$. Therefore $x \notin f_{k_0}^{-1}(N_0)$ and by the previous lemma, $f_{k_0}^{-1}(N_0)$ is I-open in X. Thus X is T_1-separated. $\square$

Since the inclusion of a subgroup M into a group G is a monomorphism (and so a singleton monosource) and the projections $\prod G_i \xrightarrow{\pi_i} G_i$ of a product of groups into its factors is also a monosource, then as a consequence of the above proposition we have the following.

Corollary 7.1.8. *For any interior operator I on **Grp**, $T_1^G(I)$ is closed under subgroups and products.*

As in the topological case, we can establish a natural relation between the class of all the interior operators on the category **Grp**, namely $IN(\mathbf{Grp})$, and the conglomerate of all the classes of groups ordered by inclusion $S(\mathbf{Grp})$. This is done by the following lemma.

Lemma 7.1.9. *The function* $IN(\boldsymbol{Grp}) \xrightarrow{T_1^G} S(\boldsymbol{Grp})$ *defined by,*

$$T_1^G(I) = \big\{X \in \boldsymbol{Grp} : X \text{ is } T_1\text{-separated with respect to } I\,\big\}$$

is order preserving.

Proof. Consider I_k, $I_h \in IN(\mathbf{Grp})$ such that $I_k \sqsubseteq I_h$, that is, $i_k(M) \subseteq i_h(M)$ for every $M \subseteq X$. Let $X \in T_1^G(I_k)$ so,

$$\bigcap_{\substack{M \leq X \\ M \neq \{e_X\} \\ i_k(M)=M}} M = \{e_X\}.$$

If M is I_k-open, then by hypothesis it is also I_h-open, since $i_k(M) \subseteq i_h(M)$, consequently,

$$\bigcap_{\substack{M \leq X \\ M \neq \{e_X\} \\ i_h(M)=M}} M = \{e_X\}.$$

This means that $X \in T_1^G(I_h)$ and thus we have obtained our desired result. $\square$

As a consequence of the previous lemma, we have that, for a given indexed family of interior operators $\{I_k\}_{k\in K}$,

$$T_1^G\Big(\bigwedge_{k\in K} I_k\Big) \subseteq \bigcap_{k\in K} T_1^G(I_k).$$

Since by definition of infimum, for every $k \in K$, $\bigwedge\limits_{k\in K} I_k \sqsubseteq I_k$. As the function T_1^G is order preserving, for each $k \in K$,

$$T_1^G\Big(\bigwedge_{k\in K} I_k\Big) \subseteq T_1^G(I_k)$$

and hence,

$$T_1^G\Big(\bigwedge_{k\in K} I_k\Big) \subseteq \bigcap_{k\in K} T_1^G(I_k).$$

At this point, similarly to the topological case, a question can naturally arise.

Is it true that

$$\bigcap_{k\in K} T_1^G(I_k) \subseteq T_1^G\Big(\bigwedge_{k\in K} I_k\Big)\,?$$

First, consider $X \in \bigcap_{k\in K} T_1^G(I_k)$, that is, for every $k \in K$, $X \in T_1^G(I_k)$. Then by definition of T_1-separated group we have that for each $k \in K$,

$$\bigcap_{\substack{M\leq X\\ M\neq\{e_X\}\\ i_k(M)=M}} M = \{e_X\}.$$

but certainly the class of all $\bigwedge I_k$-open subgroups is contained in every class of I_k-open subgroups for each $k \in K$. But we do not have any information about if the converse is true for at least some interior operator I_k. Since for every index $k \in K$ there may exist subgroups M being I_k-open, but not necessarily I_h-open, for $h \neq k$.

Now, let us try to assume that $X \notin T_1^G\Big(\bigwedge_{k\in K} I_k\Big)$, that is,

$$\bigcap_{\substack{M\leq X\\ M\neq\{e_X\}\\ i_{\wedge}(M)=M}} M \neq \{e_X\}.$$

Hence, there exists $y \in X$ such that $y \in M$ for every $M = i_{\wedge}(M) = \bigcap i_k(M)$. This does not mean that y belongs to all the I_k-open subgroups M for some given index $k \in K$, thus nothing can be concluded.

After a careful examination of the situation we realized that the problem is quite complex and we have left it as future work.

CHAPTER 8
CONCLUSION AND FUTURE WORK

A summary of the done work in this thesis is as follows:

- Using a general notion of interior operator I defined on the category **Top** of topological spaces notions of T_1-separation and T_1-coseparation with respect to I on the category **Top** were introduced.
- Examples of T_1-separated topological spaces and T_1-coseparated topological spaces with respect to some specific interior operators I were presented.
- Properties of T_1-separation and T_1-coseparation with respect to an interior operator were studied and in particular the behavior of T_1-separation with respect to I matches very well with the one of the classical notion of T_1 separation in topology. For instance, T_1-separation with respect to I is closed under subspaces and products.
- A new factorization of a classical Galois connection was found by means of two Galois connections induced by the notions of T_1-separation and T_1-coseparation with respect to I.
- Finally, a tentative of introducing a notion of T_1-separation with respect to an interior operator on the category **Grp** was made. Several partial examples of this notion in the category of groups were presented.

The following ideas are left as future work.

- Find explicit examples of interior operators of the form $D(\mathscr{A})$ for $\mathscr{A} \in$ **Top**.

- On the category **Grp** prove or disprove that

$$\bigcap_{k\in K} T_1^G(I_k) \subseteq T_1^G\Big(\bigwedge_{k\in K} I_k\Big).$$

  In the affirmative case, this would yield a Galois connection and further development of the theory would be possible.

- Find a better classification of T_1-separated groups with respect to the given interior operators in the category of groups.

REFERENCES

[1] A. Arhangel'skii, R. Wiegandt, "Connectedness and disconnectedness in topology", *General Topology and its Applications*, **5** (1975), 9-33.

[2] J. Adámek, H. Herrlich, G. E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*, open access file in http://katmat.math.uni-bremen.de/acc

[3] G. Castellini, *Categorical Closure Operators*, Mathematics: Theory and Applications, Birkhauser, Boston, 2003.

[4] G. Castellini, "T_1-separation in a category", *Quaestiones Mathematicae*, **29** (2006), 151-170.

[5] G. Castellini, "Interior operators in a category: idempotency and heredity", *Topology and its Applications*, **158** (2011), 2332-2339.

[6] G. Castellini, D. Hajek, "Closure operators and connectedness", *Topology and its Applications*, **5** (1994), 29-45.

[7] G. Castellini, E. Murcia, "Interior operators and topological separation", *Topology and its Applications*, **160** (2013), 1476-1485.

[8] G. Castellini, J. Ramos, "Interior operators and topological connectedness", *Quaestiones Mathematicae*, **33** (2010), 1-15.

[9] D. Dikranjan, W. Tholen, *Categorical Structure of Closure Operators with Applications to Topology, Algebra and Discrete Mathematics*, Kluwer Academic Publishers, Dordrecht, 1995.

[10] M. Erne, J. Koslowski, A. Melton, G. Strecker, “A primer on Galois connections”, Proceedings of the 1991 summer conference on general topology and applications in honor of Mary Ellen Rudin and her work, *Annals of the New York Academy of Sciences*, **704** (1993), 103-125.

[11] P. R. Franzosa, C. Adams, *Introduction to Topology: Pure and Applied, First Edition*, Pearson Education, Inc, 2008.

[12] H. Herrlich, *Topologische Reflexionen und Coreflexione*, L.N.M. **78**, Springer, Berlin, 1968.

[13] I. N. Herstein, *Topics in Algebra, second edition*, John Wiley & Sons, Inc., New York, 1975.

[14] E. Medina, “Interior operators in the category of groups”, Master Degree Thesis, University of Puerto Rico, Mayaguez, 2015.

[15] L. A. Stenn, J. A. Seebach, Jr. *Counterexamples in Topology*, Dover Publications, Mineola, N.Y. 1995.

[16] J. R. Munkres, *Topology, second edition*, Prentice Hall, Inc., Upper Saddle River, N.J., 2000.

[17] S. R. Vorster, “Interior operators in general categories”, *Quaestiones Mathematicae*, **23**(4) (2000), 405-416.

INTERIOR OPERATORS AND T_1 TOPOLOGICAL SPACES

Henrry Josue Cortez-Portillo
henrry.cortez@upr.edu
Department of Mathematical Sciences
Chair: Gabriele Castellini, Ph.D
Degree: Master of Science
Graduation Date: 2018

A general notion of T_1-separation with respect to an arbitrary interior operator is introduced in the category **Top** of topological spaces. This is done by means of the concept of categorical interior operator. This naturally yields a dual notion of T_1-coseparation. Each of these two notions produces a Galois connection between categorical interior operators in **Top** and subclasses of topological spaces. These two Galois connections are studied and it is shown that their composition can be described as a classical Galois connection defined in terms of the concept of constant function. This can be easily illustrated with a commutative diagram of Galois connections.