ALEXANDER POLYNOMIAL FOR TORUS KNOTS VIA BURAU MATRICES FOR PERIODIC BRAIDS

By

Raúl Alfonso Beltrán Hoyos

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Approved by:

Gabriele Castellini, Ph.D Member, Graduate Committee	Date
Juan Romero, Ph.D Member, Graduate Committee	Date
Juan A. Ortiz-Navarro, Ph.D President, Graduate Committee	Date
Betsy Morales Caro, Ph.D Representative of Graduate Studies	Date
Olgamary Rivera-Marrero, Ph.D Chairperson of the Department	Date

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Chair: Juan A. Ortiz-Navarro
Department: Mathematical Sciences

This thesis provides a characterization of the reduced Burau matrices for braids of the form $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$, with $gcd(n,d) = 1$, $n,d \geq 2$, and exposes its relationship with the Alexander polynomial for (n,d)-torus knot by using Markov functions theory. In addition, a similar characterization for a particular case of periodic braids is provided, whose closures is the mirror of a (n,d)-torus knot.

Resumen de Tesis Presentado a Escuela Graduada
de la Universidad de Puerto Rico como requisito parcial de los
Requerimientos para el grado de Maestría en Ciencias

POLINOMIO DE ALEXANDER PARA NUDOS TOROIDALES VIA MATRICES DE BURAU PARA TRENZAS PERIODICAS

Por

Raúl Alfonso Beltrán Hoyos

2017

Consejero: Juan A. Ortiz-Navarro
Departamento: Ciencias Matemáticas

Esta tesis provee una caraterización de las matrices reducidas de Burau para trenzas de la forma $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$, con $mcd(n,d) = 1$, $n, d \geq 2$, y expone su relación con el polinomio de Alexander para nudos toroidales, usando la teoria de funciones de Markov. En adición, proporcionamos una caracterización similar para un caso particular de trenzas periodicas cuya clausura es el espejo de un (n,d)-nudo toroidal.

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Raúl Alfonso Beltrán Hoyos

To my wife, my parents and my sisters.

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I would like to express my gratitude to my advisor Juan A. Ortiz Navarro for his continuous support, patience, and motivation.

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I am also thankful to my wife for her unconditional help and love of things that are important to me.

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CHAPTER 1
INTRODUCTION

1.1 Justification

Braid theory has been studied by many mathematicians throughout its development. The idea of this mathematical object was introduced by Emil Artin [3] to formalize topological objects that represent the intertwining of several strings in $\mathbb{R}^3$. Its attractiveness lies from its close relations to other objects in low dimensional topology, such as knots, links, surfaces, and configuration spaces.

Braid groups have been extensively studied and their relation with representation theory has led to numerous important results such as the linearity and the orderability of the nth braid group B_n. One remarkable presentation of the braid group is the Burau representation. This is a homological representation of the braid groups obtained by classes of self-homeomorphisms acting on the topological spaces obtained from the puntured disks, from which we can obtain powerful knot invariants such as the Jones polynomial and the Alexander polynomial.

To construct the last polynomial mentioned above for knots(links) it is first necessary to study the reducibility of the Burau representation. Then, using the theory of Markov functions, one can construct rational functions with values in the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$. All these functions form a Markov Funtion, and the associated knot(link) invariant coincides with the Alexander polynomial.

Knots are classified into three families: satellite, hyperbolic, and torus knots. An important result is that each knot or link represents the closure of a braid, a theorem stated and proved by Emil Artin [3]. In particular, torus knots have a unique periodic braid representation.

My interest is to analyze what is happening in the reduced Burau matrices for the periodic braids mentioned above; the rational functions derived from these matrices, and the Alexander polynomial.

1.2 Previous publications

In 1925, Emil Artin introduced the braid group B_n . In his paper [3], he provided an algebraic definition in terms of a group presentation by generators and relationship and he proved that each knot(link) is isotopic to the closure of a braid.

In 1935 Burau [9] provided a nontrivial linear representation called Burau representation, he also gave a close relationship between this representation and the Alexander polynomial.

An interesting type of braid are periodic braids, that is, braids that have the form $w = b^k$, where b is word and k is the period of w. In recent work, Seong Ju Kim, Ryan Stees and Laura Taalman [8] gave an important result that relates a big knot class and periodic braids. This result provides an easier demonstration that every torus knot has a unique periodic braid representation.

1.3 Objectives

1.3.1 Main objective

- The main objective of this thesis is to find a pattern in the reduced Burau matrices for braids of the form $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$, and $\left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)^d$ where $gcd(n,d)=1$ and $n,d\geq 2$.

1.3.2 Secondary objectives

- To analyze the reduced Burau representation for braids in B_n.
- To characterize the reduced Burau matrices for the braids mentioned in the main objetive.
- To expose a general formula for the Alexander Polynomial for the knot associated to the closure of the braids $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$, and $\left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)^d$.

CHAPTER 2
PRELIMINARIES

This chapter provides basic definitions and results of knot theory and braid theories.

2.1 Knots

Definition 2.1. *A link is an embedding of a disjoint union of n circles into $\mathbb{R}^3$ or S^3. A link of one component is a knot.*

A way to visualize and manipulate knots is to project the knot in $\mathbb{R}^2$. If the projection is injective everywhere, except at a finite number of points called **crossings,** and there is not triple intersections, tangencies or cusps, the projection is called **regular**

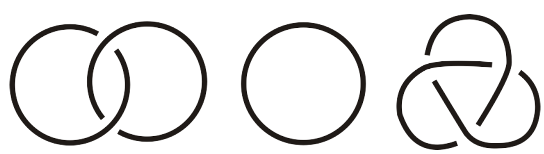


Figure 2–1: Hopf link, unknot and trefoil

Figure 2–2: Triple intersections, tangencies, and cusps

An **oriented knot (link)** is a knot (link) with an orientation defined. The orientation of a knot is usually represented by placing an arrow on its diagram in a chosen direction.

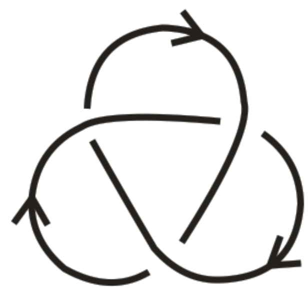


Figure 2–3: An oriented knot

Definition 2.2. *Let $f, g : X \to Y$ be continuous functions. A continuous function $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$ is called an isotopy if $F|_{X \times \{t\}}$ is homeomorphism for all $t \in [0,1]$.*

Definition 2.3. *Let $f, g : Y \to X$ be embeddings of Y into X. f and g are ambient isotopic if there is an isotopy $F : X \times [0,1] \to Y$ such that $F(x,0) = x$ for all $x \in X$ and $F(f(y),1) = g(y)$ for all $y \in Y$.*

Definition 2.4. *Two knots K_1,K_2 are equivalent if they are ambient isotopic.*

Definition 2.5. *A planar isotopy of a knot projection is a continuous deformation of the projection.*

Then, K_1,K_2 are equivalent if it is possible to deform one to the other by ambient or planar isotopy.

Knots are classified into three large families: **satellite, hyperbolic, and torus knots**. This thesis provides an important result for torus knots, making use of braid theory, which will be discussed in section 2.4

Definition 2.6. *A torus knot is a knot that lies on an unknotted (standard) torus in $\mathbb{R}^3$; without crossing over or under itself as it lies on the torus.*

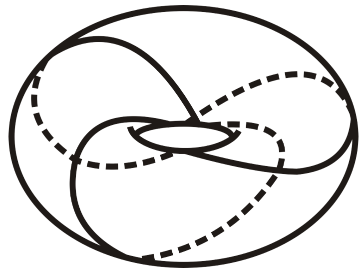


Figure 2–4: A trefoil on a torus

We can draw a torus knot, traveling p times vertically and q times horizontally around the torus, where $p, q \geq 2$ and p and q are relatively prime. A $(p, q)-$ *torus knot* is denoted by $T(p, q)$. For example, the trefoil knot is denoted by $T(3, 2)$. It goes three times vertically around the torus and twice horizontally.

Theorem 1. *$T(p, q)$ is equivalent to $T(q, p)$. [1]*

Theorem 2. *The least number of crossings that occurs in any projection, for a $T(p, q)$ is exactly the minimum of $p(q-1)$ y $q(p-1)$. [5]*

2.2 Reidemeister Moves

In 1927, Kurt Reidemeister defined a series of move known as **Reidemeister Moves**.

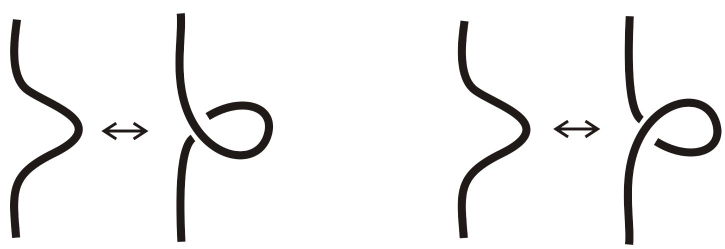


Figure 2–5: Type I Reidemeister move

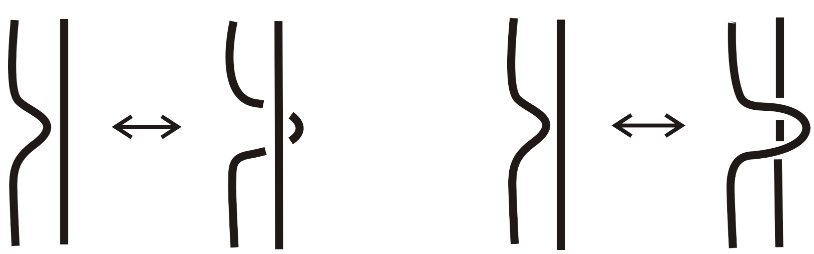


Figure 2–6: Type II Reidemeister move

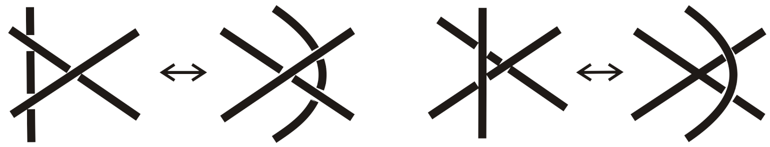


Figure 2–7: Type III Reidemeister move

Two projections of a knot(link) are isotopic if and only if one can be transformed into the other by a finite sequence of Reidemeister moves [7]

2.3 Knot invariants

Definition 2.7. *A knot invariant is a property of a knot that does not change under ambient isotopy.*

Two projections of a knot have the same knot invariant, but two projections with the same knot invariant need not be the same knot.

2.3.1 Alexander polynomial

The Alexander polynomial is an invariant of oriented Knot(Link) in $\mathbb{R}^3$, discovered in 1923 by J. W. Alexander (Alexander, 1928). There are generally difficult routes to compute the Alexander polynomial, but in 1969, John Conway provided an axiomatic form to compute it.

Let L be an oriented knot projection, let $+1$ be a right handed (positive) and -1 a left handed (negative) crossing, respectively, as in the figure (2–8)

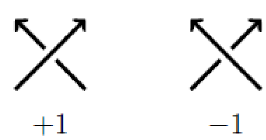


Figure 2–8: Positive and negative crossings

Each crossing can be smoothed in two diferent ways, either by $0-$ *smoothing* or $1-$ *smoothing* according to the figure (2–9)

Figure 2–9: Smoothings for a crossing

Define L_+, L_- and L_0 by isolating and changing one crossing of L as shown in figure (2–10)

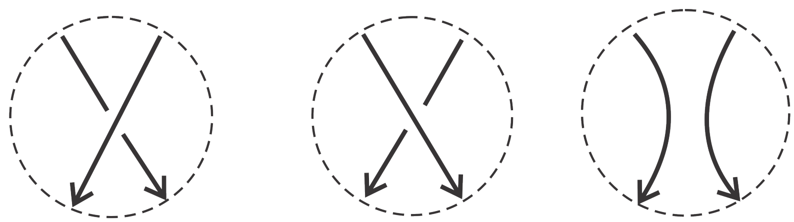


Figure 2–10: A Conway triple

L_+, L_- and L_0 are three oriented links in $\mathbb{R}^3$, and form a *Conway triple*.

The Alexander polynomial of links is a mapping Δ assigning every oriented link $L \subset \mathbb{R}^3$ a Laurent polynomial $\Delta(L) \in \mathbb{Z}[t, t^{-1}]$ satisfying the following three axioms:

1. $\Delta(L)$ is invariant under isotopy of L;
2. if L is a trivial knot, then $\Delta(L) = 1$;
3. for any *Conway triple* L_+, L_-, $L_0 \subset \mathbb{R}^3$,

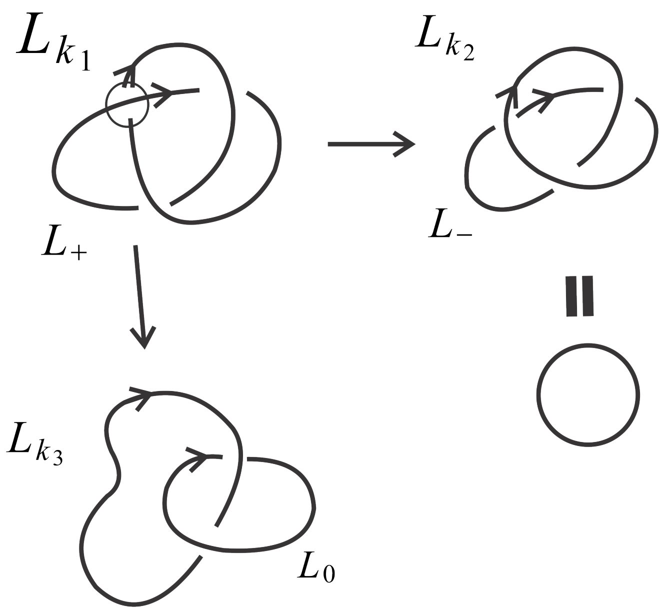
$$\Delta(L_+) - \Delta(L_-) = \left(t^{-1/2} - t^{1/2}\right) \Delta(L_0)$$

The latter equality is known as the Alexander *skein relation*. [2]

Definition 2.8. *A splittable link is a link that can be separated by a 2-sphere embedded in S^3.*

Proposition 2.1. *If L is splittable with at least two components, then $\Delta(L) = 0$. ([4] and [6])*

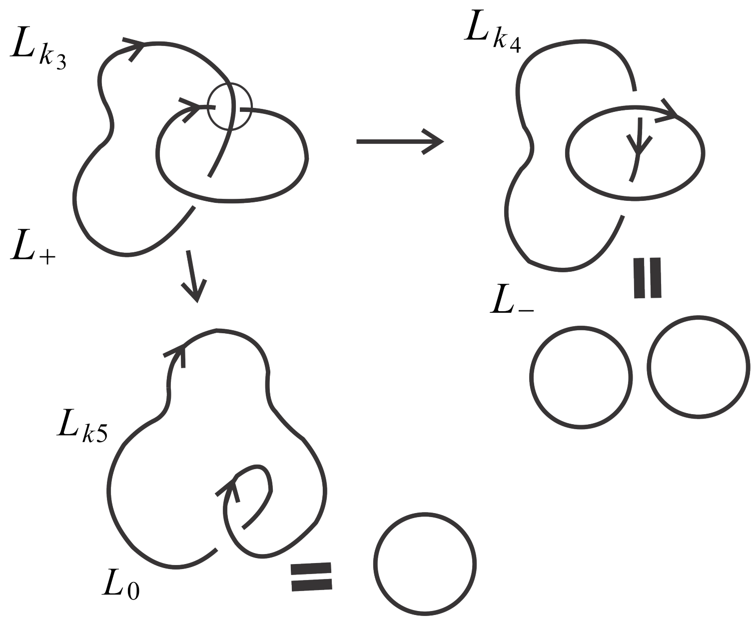
Example: Let's compute the Alexander polynomial for the trefoil knot, choosing crossings sequentially, and then applying the skein relation until trivial links are left.



then

$$\begin{aligned}\Delta\left(L_{k_1}\right) &= \Delta\left(L_{k_2}\right) + \left(t^{-1/2} - t^{1/2}\right)\Delta\left(L_{k_3}\right)\\ &= 1 + \left(t^{-1/2} - t^{1/2}\right)\Delta\left(L_{k_3}\right).\end{aligned}$$

for $\Delta\left(L_{k_3}\right)$,



then

$$\begin{aligned}\Delta(L_{k_3}) &= \Delta(L_{k_4}) + \left(t^{-1/2} - t^{1/2}\right)\Delta(L_{k_5}) \\ &= 0 + \left(t^{-1/2} - t^{1/2}\right)(1) \\ &= t^{-1/2} - t^{1/2}\end{aligned}$$

therefore

$$\begin{aligned}\Delta(L_{k_1}) &= 1 + \left(t^{-1/2} - t^{1/2}\right)\left(t^{-1/2} - t^{1/2}\right) \\ &= t^{-1} + t - 1\end{aligned}$$

which is the Alexander polynomial for the trefoil knot.

2.4 Braid groups

Basic relations between the braid group, the symmetry groups S_n, links in $\mathbb{R}^3$ and representation theory are presented.

Definition 2.9. *Consider two parallel planes A and A' in $\mathbb{R}^3$, each containing n distinct Collinear points $\{p_i\}$ and $\{p'_i\}$ respectively. Let $\mathscr{A}_i : [0,1] \to \mathbb{R}^3$ be embedded arcs in $\mathbb{R}^3$. A geometric braid (or braid with n strand) is said to be the set $\mathscr{A} = \{\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n\}$, where $\mathscr{A}_i$ connects the point $p_i \in A$ to the point $p'_{\tau(i)} \in A'$, $\tau \in S_n$ and*

1. *Each $\mathscr{A}_i$ intersects each intermediate parallel plane between A and A' exactly once.*
2. *$\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ intersect each intermediate parallel plane between A and A' in exactly n different points.*

The permutation τ is called the associated permutation to the braid, and $\mathscr{A}_i$ is called the ith string (or strand) in the braid.

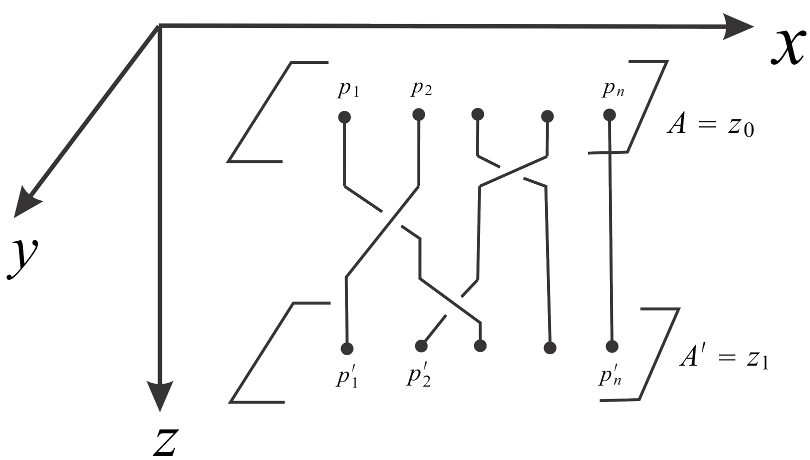


Figure 2–11: A geometric braid

Definition 2.10. *Two $n-$braids $\mathscr{A}$ and $\mathscr{A}'$ with the same permutation τ are equivalent if there is a homotopy with permutation τ from $\mathscr{A}$ to $\mathscr{A}'$. In other words, if n continuous maps exist*

$$F_i : [0,1] \times [0,1] \to \mathbb{R}^3, \quad 1 \le i \le n$$

such that

1. $F_i(t,0) = \mathscr{A}_i(t),\ F_i(t,1) = \mathscr{A}'_i(t),\ 1 \le i \le n,\ 0 \le t \le 1$
2. $F_i(0,k) = p_i,\ F_i(1,k) = p'_{\tau(i)},\ 1 \le i \le n,\ 0 \le k \le 1$

and defining $\mathscr{A}_i^k : [0,1] \to \mathbb{R}^3$ *by* $\mathscr{A}_i^k(t) = F_i(t,s)$, *then* $\mathscr{A}^k$ *is a* $n-$*braid for each* $k \in [0,1]$.

An oriented braid is a braid with an orientation defined. The orientation of a braid is usually downwards.

According to the definition of braid only type II and type III Reidemeister moves can be made. In the same way of knot theory, any two geometric braids $\mathscr{A}$, $\mathscr{A}'$ are equivalent if $\mathscr{A}$ can be transformed into $\mathscr{A}'$ by a finite sequence of isotopies and Reidemeister moves.

2.4.1 The Artin braid groups

The Artin braid group was introduced by Emil Artin [3] in 1925. He provided an algebraic definition of the braid group denoted by B_n in terms of a group presentation by generators and relations.

Definition 2.11. *The Artin braid group* B_n *is the group generated by* $n-1$ *generators* $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ *and the braid relations*

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for all $i, j = 1, 2, ..., n-1$ *with* $|i-j| \geq 2$, *and*

$$\sigma_i \sigma_{i+1} \sigma_i \quad = \quad \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for $i, j = 1, 2, ..., n-1$

By definition $B_1 = \{1\}$ is a trivial group and B_2 is generated by σ_1 without relations.

For $n > 2$, if $b \in B_n$, then

$$b = \sigma_{i_1}^{n_1} \sigma_{i_2}^{n_2} ... \sigma_{i_r}^{n_r}$$

where $i_j \in \{1, 2, ..., n-1\}$ for all j, $i_j \neq i_k$, $i \neq k$ and $n_j \in \mathbb{Z}$ for all j. b is known as a word.

Proposition 2.2. *If $s_1, ..., s_{n-1}$ are elements of a group G satisfying the braid relations, then there is a unique group homomorphism $f : B_n \to G$ such that $s_i = f(\sigma_i)$ for all $i = 1, 2, ..., n-1$. See [2]*

Lemma 2.1. *The group B_n with $n \geq 3$ is nonabelian. See [2]*

Denote by $\mathscr{B}_n$ the set of geometric braids with multiplication. The basic blocks to build any braid in $\mathscr{B}_n$ are shown in the figure(2–12)

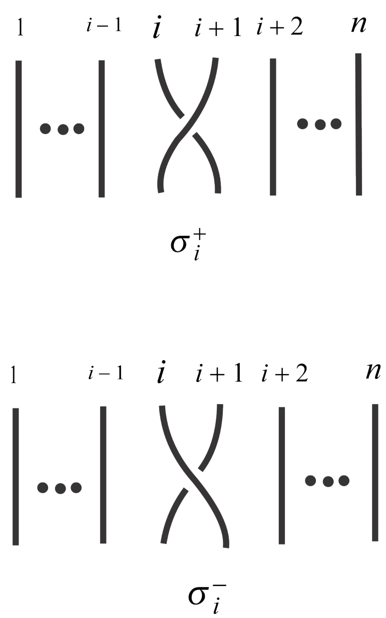
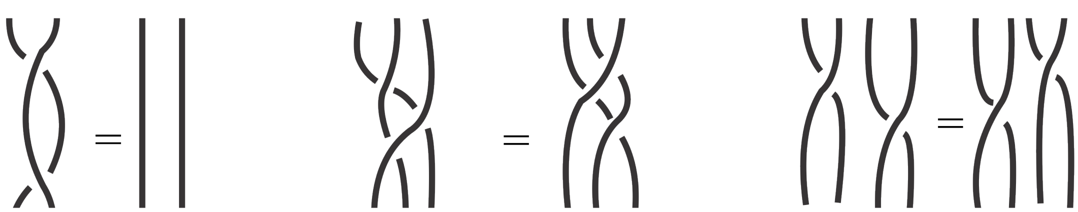


Figure 2–12: The elementary braids

Geometrically, the first and second relations represent equivalent braids by isotopies.



$\sigma_1\sigma_1^{-1} = 1$ $\qquad$ $\sigma_1\sigma_2\sigma_3 = \sigma_2\sigma_1\sigma_2$ $\qquad$ $\sigma_1\sigma_3 = \sigma_3\sigma_1$

The elements σ_i^+, $i = 1, 2, ..., n-1$ satisfy the braid relations and each $\beta \in \mathscr{B}_n$ has an inverse β^{-1}, giving $\mathscr{B}_n$ a group structure, then, there is a unique homomorphism $\varphi_\pm : B_n \to \mathscr{B}_n$ such that $\varphi_\pm(\sigma_i) = \sigma_i^\pm$ for all $i = i = 1, 2, ..., n-1$. The homomorphism $\varphi_\pm$ is an isomorphism [2]. $\varphi_\pm$ allows us to identy B_n and $\mathscr{B}_n$, thus, the elements in B_n are called braids on n strings.

2.4.2 Markov funtions

In order to show the connection between knots and braids, concepts such as Markov functions and Markov moves will appear in this section as a complement to Alexander's theorem

Theorem 3. *(Alexander's theorem) Any oriented Knot (link) in* $\mathbb{R}^3$ *is isotopic to a closed braid. See [5]*

The process of passing from a knot to a braid is called *braiding*, and it is described in the proof of the theorem 3

Example

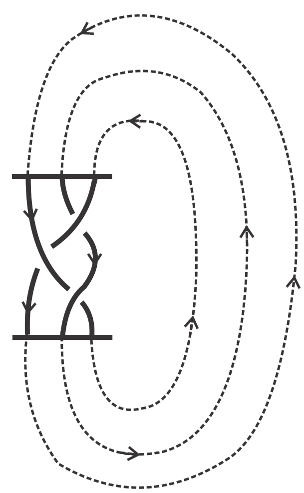


Figure 2–13: Braid closure

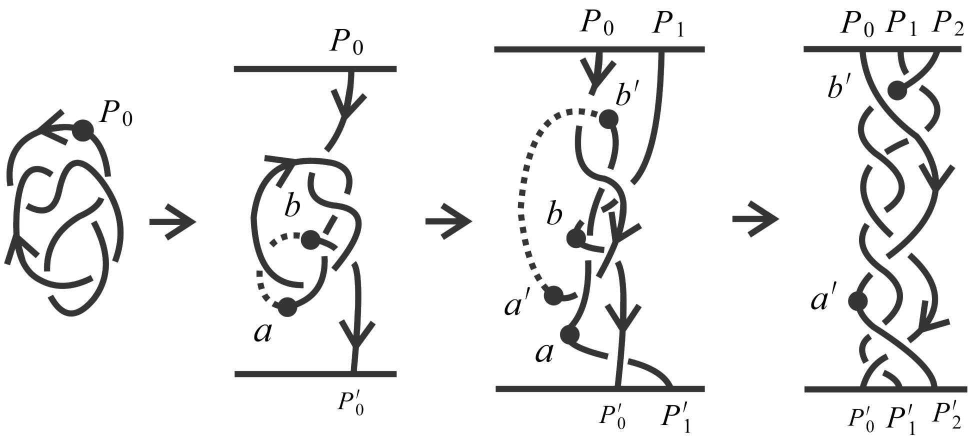


Figure 2–14: Braiding

Definition 2.12. *Let B_∞ be the union of the groups $B_1,\ B_2, ..., B_n, ...$ i.e $B_\infty := \bigcup_{k\geq 1} B_k$. The following two operations in B_∞; are called Markov moves:*

1. *M_1 : If $\beta \in B_n$, then $\beta \mapsto \gamma\beta\gamma^{-1}$ for some $\gamma \in B_n$. The element $\gamma\beta\gamma^{-1} \in B_n$ is a conjugate of β.*
2. *M_2 : If $\beta \in B_n$, then $\beta \mapsto \beta\sigma_n^{\pm 1}$, where σ_n is a generator of B_{n+1}.*

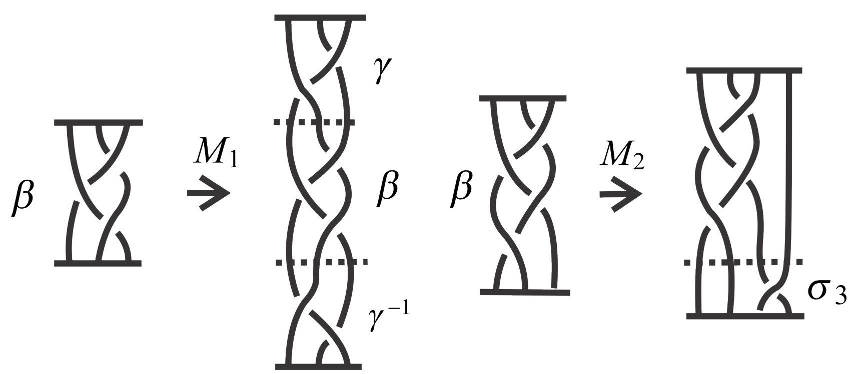


Figure 2–15: Markov moves

Definition 2.13. *Let $\alpha, \beta \in B_n$. If β can be obtained from α by a finite number of Markov moves, then α is said to be Markov equivalent ($M-$equivalent) to β and is denoted by $\alpha \sim_M \beta$.*

Theorem 4. *(A. Markov) Two braids have isotopic closures in Euclidean space $\mathbb{R}^3$ if and only if these braids are $M-$equivalent. See [2]*

Definition 2.14. *A periodic braid in B_n is a braid with n-strands of the form $w = b^k$, where $b = \sigma_{i_1}^{n_1}\sigma_{i_2}^{n_2}...\sigma_{i_r}^{n_r}$, $i_j \in \{1, 2, ..., n-1\}$ for all j, $i_j \neq i_k$, $i \neq k$, $n_j \in \mathbb{Z}$ for all j, and k is the number of repetitions of b or the period of w.*

For any integers $n \geq 2$ and $k \geq 1$. A spiral link of type (n, k) is a link that admits an n-strand braid word of the form $w = b^k$ with word lenth $n - 1$.

Example $\left(\sigma_1\sigma_2\sigma_4^{-1}\sigma_3^{-1}\right)^2$ is the braid representation of a spiral link of type $(5, 2)$.

It is proved that every spiral knot of type (n, k) must have $\gcd(n, k) = 1$ [8], and we say that a spiral link with one component is a spiral knot.

Theorem 5. *Let $n \geq 2$ and $k \geq 1$. Every spiral knot of type (n, k) admits a braid word of the form $\left(\sigma_1^{\epsilon_1}\sigma_2^{\epsilon_2}...\sigma_{n-1}^{\epsilon_{n-1}}\right)^k$, where $\epsilon_i = \pm 1$. [8]*

Spiral knots are denoted by $S(n, k, \epsilon)$, where $\epsilon = (\epsilon_1, \epsilon_1, ..., \epsilon_{n-1})$. Note that, a torus knot $T(p, q) = S(p, q, \epsilon)$ where $\epsilon = (1, 1, ..., 1)$, thus $T(p, q)$ admits a braid word of the form $(\sigma_1\sigma_2...\sigma_{p-1})^q$.

2.4.3 The Burau representation

The Burau representation is a homological representation of the braid groups obtained by classes of self-homeomorphisms acting on the homology of topological spaces obtained from puntured disks.

Fix $n \geq 2$. For $i = 1, ..., n-1$, consider the $n \times n$ matrix over the ring of Laurent polynomials $\Lambda = \mathbb{Z}[t, t^{-1}]$

$$U_i = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}$$

where I_k is the unit $k \times k$ matrix. Each matrix U_i has a block-diagonal form. These blocks can be the unit matrix or the 2×2 matrix

$$U = \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}.$$

By the Cayley-Hamilton theorem, $U^2 - (1-t)\,U - tI_2 = 0$. Since I_k satisfy this equation,

$$U_i^2 - (1-t)\,U_i - tI_n = 0, \quad \forall i$$

then, $U_i U_i^{-1} = I_n,$. where $U_i^{-1} = t^{-1}\left(U_i - (1-t)\,I_n\right)$. Hence U_i is invertible over Λ and its inverse is given by:

$$U_i^{-1} = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t^{-1} & 1-t^{-1} & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}$$

The block form of the matrices $U_1, ..., U_{n-1}$ implies that

$$\begin{aligned} U_i U_j &= U_j U_i, \quad |i-j| \geq 2; \\ U_i U_{i+1} U_i &= U_{i+1} U_i U_{i+1}, \quad |i-j| \geq 1. \end{aligned}$$

Then $U_1, ..., U_{n-1}$ satisfy the braid relations. By proposition 2.2 there is a homomorphism $\Psi_n : B_n \to GL_n(\Lambda)$ Such that $\Psi_n(\sigma_i) = U_i$, $n \geq 2$, $i = 1, ..., n-1$, where $GL_n(\Lambda)$ is the group of invertible $n \times n$ matrices over Λ. Ψ_n is called the *Burau representaion* of B_n.

- By convention, Ψ_1 for the group B_1 is the trivial homomorphism $B_1 \to GL_1(\Lambda)$
- For $n = 2$, Ψ_2 for the group B_2 is the homomorphism $B_2 \to GL_2(\Lambda)$ such that $\Psi_2(\sigma_1) = U$.

Since $\det\left(U_i\right) = -t\ \forall i$, then $\det \Psi_n\left(b\right) = \left(-t\right)^{\langle b\rangle}$, $\forall \beta \in B_n$, where $\langle b\rangle : B_n \to \mathbb{Z}$ such that $\langle b\rangle\left(\sigma_{i_1,\ldots,}^{v_i}\sigma_{i_k}^{v_k}\right) = \sum_{j=1}^{k} v_j$ is a homomorphism, which tells us the number of generators involved in the word.

Example The generators for B_4 are σ_1,σ_2, and σ_3, then

$$\Psi\left(\sigma_1\right) = \begin{bmatrix} 1-t & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Psi\left(\sigma_2\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Psi\left(\sigma_3\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-t & t \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

consider $b \in B_4$ such that $b = \sigma_1\sigma_2\sigma_3$, hence $\Psi_4\left(b\right) = \Psi_4\left(\sigma_1\sigma_2\sigma_3\right) = \Psi_4\left(\sigma_1\right)\Psi_4\left(\sigma_2\right)\Psi_4\left(\sigma_3\right)$, then

$$\begin{aligned} \Psi_4\left(\sigma_1\sigma_2\sigma_3\right) &= \begin{bmatrix} 1-t & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-t & t \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1-t & -t\left(t-1\right) & -t^2\left(t-1\right) & t^3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

and $\det \Psi_4\left(b\right) = -t^3$.

A linear representation is said to be faithful if its kernel is trivial. The homomorphism Ψ_n is not faithful for all $n \geq 5$ i.e $ker\Psi_n \neq \{1\}$ for all $n \geq 5$ [2]; and an easy way to study the kernel is studying first the reducibility of the representation.

The reduced Burau representation Ψ_n

Fix $n \geq 3$. consider the $(n-1) \times (n-1)$ matrices $V_1, V_2, ...V_{n-1}$ over the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$ given by:

$$V_1 = \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}, \qquad V_{n-1} = \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix}$$

and for $1 < i < n-1$

$$V_i = \begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{pmatrix}.$$

Consider the $n \times n$ matrix C,

$$C = \begin{pmatrix} 1 & 1 & 1 & .. & 1 \\ 0 & 1 & 1 & .. & 1 \\ 0 & & 1 & .. & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & .. & 1 \end{pmatrix}$$

and

$$V_i' = \begin{pmatrix} V_i & 0 \\ *_i & 1 \end{pmatrix}$$

where $*_i$ is the row of length $n-1$ equal to $(0, ...0, 0)$ if $i < n-1$ and to $(0, ...0, 1)$ if $i = n-1$.

Note that, fixing i, for all $k = 1, ..., n$, the $kth-$column of U_iC is the sum of the first k columns of U_i. A direct calculation shows that, U_iC is obtained from C replacing the $(i,i)th-$entry for $1-t$ and replacing the $(i+1,i)\,th-$entry by 1.

Similarly, for all $l = 1, ..., n$, the $lth-$row of CV_i' is the sum of the last l rows of V_i'. Then, CV_i' is obtained from C by the same substitution above . Hence,

$$C^{-1}U_iC = V_i'$$

then, the matrices $V_1', V_2', ..., V_{n-1}'$ satisfy the braid relations, hence the matrices $V_1, V_2, ...V_{n-1}$ also satisfy the braid relations. By proposition 2.2 there is a homomorphism $\Psi_n^r : B_n \to GL_{n-1}(\Lambda)$ such that $\Psi_n^r(\sigma_i) = V_i$, $n \geq 3$, $i = 1, ..., n-1$, where $GL_{n-1}(\Lambda)$ is the group of invertible $(n-1) \times (n-1)$ matrices over Λ. Ψ_n^r is called the *reduced Burau representaion* of B_n.

- For $n = 2$, Ψ_2^r for the group B_2 is the homomorphism $B_2 \to GL(\Lambda)$ such that $\Psi_2^r(\sigma_1) = (-t)_{1\times 1}$.

Example The generator for B_3 are σ_1, and σ_2, then

$$\Psi_3^r(\sigma_1) = \begin{pmatrix} -t & 0 \\ 1 & 1 \end{pmatrix} \quad \Psi_3^r(\sigma_2) = \begin{pmatrix} 1 & t \\ 0 & -t \end{pmatrix}$$

consider $b \in B_3$ such that $b = \sigma_1\sigma_2\sigma_1$, hence $\Psi_3^r(b) = \Psi_3^r(\sigma_1\sigma_2\sigma_1) = \Psi_3^r(\sigma_1)\Psi_3^r(\sigma_2)\Psi_3^r(\sigma_1)$, then

$$\Psi_3^r(b) = \begin{pmatrix} -t & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & -t \end{pmatrix} \begin{pmatrix} -t & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -t^2 \\ -t & 0 \end{pmatrix}.$$

In order to see an application of this representation, it is important to construct a Markov function with values in $\mathbb{Z}[s, s^{-1}]$. The link invariant associated to this function is the Alexander polynomial.

Definition 2.15. *A Markov function with values in a set E is a sequence of set-theoretic maps $\{f_n : B_n \to E\}_{n\geq 1}$, satisfying the following conditions:*

1. $\forall n \geq 1$ *y* $\forall \alpha, \beta \in B_n$, $f_n(\alpha\beta) = f_n(\beta\alpha)$;
2. $\forall n \geq 1$ *y* $\forall \beta \in B_n$, $f_n(\beta) = f_{n+1}(\sigma_n\beta)$ *y* $f_n(\beta) = f_{n+1}(\sigma_n^{-1}\beta)$.

A Markov function allows us to identify isotopy invariants of oriented link in $\mathbb{R}^3$.

Consider the ring homomorphism $g : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[s, s^{-1}]$ Such that $t \longmapsto s^2$. Let $\beta \in B_n$, $n \geq 2$, then the rational function in s given by

$$f_n(\beta) = (-1)^{n+1} \frac{s^{-\langle \beta \rangle}(s - s^{-1})}{(s^n - s^{-n})} g\left(\det\left(\Psi_n^r(\beta) - I_{n-1}\right)\right), \tag{2.1}$$

is the Alexander polynomial, where $\langle b \rangle : B_n \to \mathbb{Z}$ such that $\langle b \rangle \left(\sigma_{i_1,\ldots,}^{v_i} \sigma_{i_k}^{v_k}\right) = \sum_{j=1}^{k} v_j$.

Lemma 2.2. *The mapping* $\{f_n : B_n \to \mathbb{Z}[s, s^{-1}]\}_{n \geq 1}$ *forms a Markov funtion. See [2].*

Set $\widehat{f}(K) = f_n(\beta)$, for an oriented knot(link) $K \in \mathbb{R}^3$ and an arbitrary $\beta \in B_n$ whose closure is isotopic to K. By the previous lemma $\widehat{f}(K)$ is an isotopy invariant of K that does not depend on the choice of β.

CHAPTER 3
REDUCED BURAU MATRICES FOR BRAIDS OF THE FORM $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$ WITH $gcd(n,d)=1$

This chapter studies the reduced Burau matrices for braids in B_n of the form $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$, where $gcd(n,d) = 1$ and $n, d \geq 2$. These types of braids are a subclass of periodic braids, where all their crossings are positive *i.e.* the base words of the braid involve the first strand passing over all other strands in order.

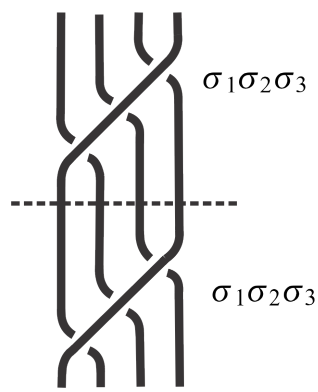


Figure 3–1: Projection of the braid $(\sigma_1\sigma_2\sigma_3)^2$

By theorem 5, a torus knot $T(n,d)$ admits a braid word of the form $(\sigma_1\sigma_2...\sigma_{n-1})^d$ *i.e* the closure of $(\sigma_1\sigma_2...\sigma_{n-1})^d$ is a knot K isotopic to $T(n,d)$. Now, by theorem 1, $T(n,d)$ is equivalent to $T(d,n)$; observe that $T(d,n)$ has a standard braid projection $(\sigma_1\sigma_2...\sigma_{d-1})^n$, then $(\sigma_1\sigma_2...\sigma_{n-1})^d$ and $(\sigma_1\sigma_2...\sigma_{d-1})^n$ are braids whose closures are equivalent knots, but this does not mean that $(\sigma_1\sigma_2...\sigma_{n-1})^d$ and $(\sigma_1\sigma_2...\sigma_{d-1})^n$ are equivalent. When $gcd(n,d) \neq 1$ the closure of $(\sigma_1\sigma_2...\sigma_{n-1})^d$ is a torus link and $\Psi_n^r\left((\sigma_1\sigma_2...\sigma_{n-1})^d\right) = t^n I_{n-1}$ for all $n \geq 2$, this can be proven using the homological interpretation of Ψ_n^r. In order to characterize the reduced Burau matrices for the braids mentioned above, show the following cases:

- For $n = 2$, let $\beta \in B_2$, such that $\beta = (\sigma_1)^d$ .with $gcd(2, d) = 1$. Then $d \geq 3$ and $d = 1 + 2k$, $k \in \mathbb{N}$. Therefore,

$$\Psi_2^r(\beta) = \Psi_2^r\left((\sigma_1)^d\right) = \left[\Psi_2^r(\sigma_1)\right]^d = (-t)^d = -t^d$$

  where Ψ_2^r is the reduced Burau representation for B_2 mentioned in the section (2.4.3).

- For $n = 3$, let $\beta \in B_3$, such that $\beta = (\sigma_1\sigma_2)^d$ with $gcd(3, d) = 1$. Then $d \geq 2$ and $d = 2 + 3k$, $k \in \mathbb{N} \cup 0'$ or $d = 1 + 3k$, $k \in \mathbb{N}$. Therefore,

$$\Psi_3^r(\beta) = \left[\Psi_3^r(\sigma_1\sigma_2)\right]^d = \begin{pmatrix} -t & -t^2 \\ 1 & 0 \end{pmatrix}^d$$

  hence,

$$\Psi_3^r(\beta) = \begin{cases} \begin{pmatrix} 0 & t^{3+3k} \\ -t^{1+3k} & -t^{2+3k} \end{pmatrix} & , \quad d = 2 + 3k,\ k \in \mathbb{N} \cup 0'. \\ \\ \begin{pmatrix} -t^{1+3k} & -t^{2+3k} \\ t^{3k} & 0 \end{pmatrix} & , \quad d = 1 + 3k,\ k \in \mathbb{N}. \end{cases}.$$

  Proving this by induction on k for $d = 2 + 3k$, $k \in \mathbb{N} \cup 0'$.

  If $k = 0$,

$$\Psi_3^r(\beta) = \begin{pmatrix} -t & -t^2 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & t^3 \\ -t^1 & -t^2 \end{pmatrix} = \begin{pmatrix} 0 & t^{3+3(0)} \\ -t^{1+3(0)} & -t^{2+3(0)} \end{pmatrix}.$$

Suppose that the result is valid for k_0, then,

$$\Psi_3^r(\beta) = [\Psi_3^r(\sigma_1\sigma_2)]^{2+3k_0} = \begin{pmatrix} 0 & t^{3+3k_0} \\ -t^{1+3k_0} & -t^{2+3k_0} \end{pmatrix}.$$

Now, for k_0+1, then $d = 2+3k_0+3 = 5+3k_0$, and note that $5 \equiv 2mod(3)$, thus $\Psi_3^r\left((\sigma_1\sigma_2)^{2+3k_0+3}\right)$ is well defined, therefore,

$$\begin{aligned}
\Psi_3^r\left((\sigma_1\sigma_2)^{2+3k_0+3}\right) &= \Psi_3^r\left((\sigma_1\sigma_2)^{2+3k_0}(\sigma_1\sigma_2)^3\right) \\
&= [\Psi_3^r(\sigma_1\sigma_2)]^{2+3k_0}[\Psi_3^r(\sigma_1\sigma_2)]^3 \\
&= \begin{pmatrix} 0 & t^{3+3k_0} \\ -t^{1+3k_0} & -t^{2+3k_0} \end{pmatrix}\begin{pmatrix} t^3 & 0 \\ 0 & t^3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & t^{3+3k_0+3} \\ -t^{1+3k_0+3} & -t^{2+3k_0+3} \end{pmatrix} \\
& \begin{pmatrix} 0 & t^{3+3(k_0+1)} \\ -t^{1+3(k_0+1)} & -t^{2+3(k_0+1)} \end{pmatrix}.
\end{aligned}$$

The result is valid for all $k \in \mathbb{N} \cup 0'$. The proof for the case $d = 1+3k$, $k \in \mathbb{N}$ is similar.

In addition, the matrices $\Psi_3^r(\beta)$ seen in terms of d would be,

$$\Psi_3^r(\beta) = \begin{cases} \begin{pmatrix} 0 & t^{d+1} \\ -t^{d-1} & -t^d \end{pmatrix} & , \quad d = 2+3k,\ k \in \mathbb{N} \cup 0'. \\ \\ \begin{pmatrix} -t^d & -t^{d+1} \\ t^{d-1} & 0 \end{pmatrix} & , \quad d = 1+3k,\ k \in \mathbb{N}. \end{cases}$$

Now, by reasoning in the same way for n equal to 4 and 5, the following matrices are obtained:

$$\Psi_4^r(\beta) = \begin{pmatrix} -t & -t^2 & -t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^d = \begin{cases} \begin{pmatrix} 0 & t^{d+1} & 0 \\ 0 & 0 & t^{d+1} \\ -t^{d-2} & -t^{d-1} & -t^d \end{pmatrix} & d = 3+4k,\ k \in \mathbb{N} \cup 0'. \\ \\ \begin{pmatrix} -t^d & -t^{d+1} & -t^{d+2} \\ t^{d-1} & 0 & 0 \\ 0 & t^{d-1} & 0 \end{pmatrix} & d = 1+4k,\ k \in \mathbb{N}. \end{cases}$$

and

$$\Psi_5^r = \begin{pmatrix} -t & -t^2 & -t^3 & -t^4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^d$$

$$= \overbrace{\underbrace{\begin{pmatrix} 0 & 0 & 0 & t^{d+3} \\ -t^{d-1} & -t^d & -t^{d+1} & -t^{d+2} \\ t^{d-2} & & 0 & 0 \\ 0 & t^{d-2} & & 0 \end{pmatrix}}_{d=2+5k,k\in\mathbb{N}\cup 0'.}, \underbrace{\begin{pmatrix} 0 & 0 & t^{d+2} & 0 \\ 0 & 0 & 0 & t^{d+2} \\ -t^{d-2} & -t^{d-1} & -t^d & -t^{d+1} \\ t^{d-3} & 0 & 0 & 0 \end{pmatrix}}_{d=3+5k,k\in\mathbb{N}\cup 0'.}}$$

$$\underbrace{\begin{pmatrix} 0 & t^{d+1} & 0 & 0 \\ 0 & 0 & t^{d+1} & 0 \\ 0 & 0 & 0 & t^{d+1} \\ -t^{d-3} & -t^{d-2} & -t^{d-1} & -t^d \end{pmatrix}}_{d=4+5k,k\in\mathbb{N}\cup 0'.}, \underbrace{\begin{pmatrix} -t^d & -t^{d+1} & -t^{d+2} & -t^{d+3} \\ t^{d-1} & 0 & 0 & 0 \\ 0 & t^{d-1} & 0 & 0 \\ 0 & 0 & t^{d-1} & 0 \end{pmatrix}}_{d=1+5k,k\in\mathbb{N}.}$$

Generalizing, consider $\beta \in B_n$ such that $\beta = (\sigma_1\sigma_2\cdots\sigma_{n-1})^d$ with $gcd(n,d) = 1$, $n, d \geq 2$ and define the Burau matrix for de braid $\sigma_1\sigma_2\cdots\sigma_{n-1}$ as:

$$\Psi_n^r(\sigma_1\sigma_2\cdots\sigma_{n-1}) := \begin{pmatrix} -t & -t^2 & -t^3 & \cdot\cdot & -t^{n-3} & -t^{n-2} & -t^{n-1} \\ 1 & 0 & 0 & \cdot\cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdot\cdot & 0 & 0 & 0 \\ . & . & . & & . & . & . \\ . & . & . & & . & . & . \\ . & . & . & & . & . & . \\ 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot\cdot & 0 & 1 & 0 \end{pmatrix}.$$

Since Ψ_n^r is a homomorphism, then $\Psi_n^r(\beta) = \Psi_n^r\left((\sigma_1\sigma_2\cdots\sigma_{n-1})^d\right) = [\Psi_n^r(\sigma_1\sigma_2\cdots\sigma_{n-1})]^d$. Now, let $d = p + nk$, $k \in \mathbb{N} \cup 0'$ or $k \in \mathbb{N}$ with $\gcd(n,p) = 1$, where p indicates the $(p,p)\,th$ position where the term $-t^d$ appears in $\Psi_n^r(\beta)$, and the position of the row equal to $\left(-t^{d-p+1}, -t^{d-p+2}, ..., -t^{d-1}, -t^d, -t^{d+1}, ..., -t^{n-p-2}, -t^{n-p-1}\right)$ of length $n-1$. Therefore $\Psi_n^r(\beta)$ is given by:

- If $d = 1 + nk$, $k \in \mathbb{N}$, then $\Psi_n^r(\beta)$ is

$$\Psi_n^r(\beta) = \begin{pmatrix} -t^d & -t^{d+1} & -t^{d+2} & \cdots & -t^{d+n-3} & -t^{d+n-2} \\ t^{d-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & t^{d-1} & 0 & \cdots & 0 & 0 \\ . & . & . & & . & . \\ . & . & & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & \cdots & t^{d-1} & 0 \end{pmatrix}_{n-1}, \tag{3.1}$$

The term t^{d-1} is in the $(n-1, n-2)\,th$ position.

- If $d = p + nk$, $k \in \mathbb{N} \cup 0'$, $1 < p < n$, then $\Psi_n^r(\beta)$ is

$$\left(\begin{array}{ccccccccc}
0 & \cdots & t^{d+n-p} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
\cdot & & 0 & 0 & & & & \cdot & \cdot \\
\cdot & & & & & & & \cdot & \cdot \\
\cdot & & & & & & & 0 & 0 \\
0 & 0 & \cdot\cdot & 0 & 0 & 0 & \cdot\cdot & t^{d+n-p} & 0 \\
0 & 0 & \cdot\cdot & 0 & 0 & 0 & \cdot\cdot & 0 & t^{d+n-p} \\
-t^{d-p+1} & -t^{d-p+2} & \cdot\cdot & -t^{d-1} & -t^{d} & -t^{d+1} & \cdot\cdot & -t^{d+n-p-2} & -t^{d+n-p-1} \\
t^{d-p} & 0 & \cdot\cdot & 0 & 0 & 0 & \cdot\cdot & 0 & 0 \\
0 & t^{d-p} & \cdot\cdot & 0 & 0 & 0 & \cdot\cdot & 0 & 0 \\
0 & 0 & & & & & & & \\
\cdot & & & & & & & & \\
\cdot & & & & & 0 & 0 & & \\
0 & 0 & \cdot & \cdot & \cdot & 0 & t^{d-p} & \cdots & 0
\end{array}\right)_{n-1}, \tag{3.2}$$

The terms t^{d+n-p} and t^{d-p} are in the $(1, n-p+1)\,th$ position and the $(n-1, n-p-1)\,th$ position respectively.

Since the number of classes of relative primes to n can be obtained by the function $\varphi - euler$, then the number of classes of Burau matrices is obtained by the same function, and observe that, when $p > n$, then $p \equiv Q \bmod(n)$ for some $1 \leq Q < n$ with $\gcd(n, Q) = 1$.

Let us prove (3.1) by induction on k.

If $k = 1$, then $d = 1 + n$, thus

$$\Psi_n^r(\beta) = \begin{pmatrix} -t & -t^2 & -t^3 & \cdot\cdot & -t^{n-3} & -t^{n-2} & -t^{n-1} \\ 1 & 0 & 0 & \cdot\cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdot\cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot\cdot & 0 & 1 & 0 \end{pmatrix}^{1+n}$$

$$= \begin{pmatrix} -t & -t^2 & -t^3 & \cdot\cdot & -t^{n-3} & -t^{n-2} & -t^{n-1} \\ 1 & 0 & 0 & \cdot\cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdot\cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot\cdot & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -t & -t^2 & -t^3 & \cdot\cdot & -t^{n-3} & -t^{n-2} & -t^{n-1} \\ 1 & 0 & 0 & \cdot\cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdot\cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot\cdot & 0 & 1 & 0 \end{pmatrix}^{n}$$

$$= \begin{pmatrix} -t & -t^2 & -t^3 & \cdot\cdot & -t^{n-3} & -t^{n-2} & -t^{n-1} \\ 1 & 0 & 0 & \cdot\cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdot\cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot\cdot & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} t^n & 0 & 0 & \cdots & 0 & 0 \\ 0 & t^n & 0 & \cdots & 0 & 0 \\ 0 & 0 & & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & & 0 & 0 \\ 0 & 0 & 0 & \cdots & t^n & 0 \\ 0 & 0 & 0 & \cdots & 0 & t^n \end{pmatrix}$$

$$= \begin{pmatrix} -t^{1+n} & -t^{1+(1+n)} & -t^{2+(1+n)} & \cdot\cdot & -t^{(1+n)+n-4} & -t^{(1+n)+n-3} & -t^{(1+n)+n-2} \\ t^{(1+n)-1} & 0 & 0 & \cdot\cdot & 0 & 0 & 0 \\ 0 & t^{(1+n)-1} & 0 & \cdot\cdot & 0 & 0 & 0 \\ . & . & . & & . & . & . \\ . & . & . & & . & . & . \\ . & . & . & & . & . & . \\ 0 & 0 & 0 & & t^{(1+n)-1} & 0 & 0 \\ 0 & 0 & 0 & \cdot\cdot & 0 & t^{(1+n)-1} & 0 \end{pmatrix}$$

Suppose that the result in(3.1) is valid for k_0, then $d = 1 + nk_0$, therefore

$$\Psi_n^r(\beta) = \begin{pmatrix} -t & -t^2 & -t^3 & \cdot\cdot & -t^{n-3} & -t^{n-2} & -t^{n-1} \\ 1 & 0 & 0 & \cdot\cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdot\cdot & 0 & 0 & 0 \\ . & . & . & & . & . & . \\ . & . & . & & . & . & . \\ . & . & . & & . & . & . \\ 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot\cdot & 0 & 1 & 0 \end{pmatrix}^{1+nk_0}$$

$$= \begin{pmatrix} -t^{1+nk_0} & -t^{1+nk_0+1} & -t^{1+nk_0+2} & \cdots & -t^{1+nk_0+n-3} & -t^{1+nk_0+n-2} \\ t^{1+nk_0-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & t^{1+nk_0-1} & 0 & \cdots & 0 & 0 \\ . & . & . & & . & . \\ . & . & & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & \cdots & t^{1+nk_0-1} & 0 \end{pmatrix}$$

Now, for $k_0 + 1$, then $d = 1 + nk_0 + n = (n+1) + nk_0$, and note that $(n+1) \equiv 1 mod(n)$, so $\Psi_n^r(\beta)$ is well defined. Therefore,

$$\begin{aligned}\Psi_n^r\left((\sigma_1\sigma_2\cdots\sigma_{n-1})^{1+nk_0+n}\right) &= \Psi_n^r\left((\sigma_1\sigma_2\cdots\sigma_{n-1})^{1+nk_0}(\sigma_1\sigma_2\cdots\sigma_{n-1})^{n}\right)\\ &= \left[\Psi_n^r(\sigma_1\sigma_2\cdots\sigma_{n-1})\right]^{1+nk_0}\left[\Psi_n^r(\sigma_1\sigma_2\cdots\sigma_{n-1})\right]^{n}\end{aligned}$$

$$= \begin{pmatrix} -t^{1+nk_0} & -t^{1+nk_0+1} & -t^{1+nk_0+2} & \cdot\cdot & -t^{1+nk_0+n-3} & -t^{1+nk_0+n-2}\\ t^{1+nk_0-1} & 0 & 0 & \cdot\cdot & 0 & 0\\ 0 & t^{1+nk_0-1} & 0 & \cdot\cdot & 0 & 0\\ . & . & . & & . & .\\ . & . & & & . & .\\ . & . & . & & . & .\\ 0 & 0 & 0 & \cdot\cdot & t^{1+nk_0-1} & 0 \end{pmatrix} \begin{pmatrix} t^n & 0 & \cdot\cdot & 0 & 0\\ 0 & t^n & \cdot\cdot & 0 & 0\\ 0 & 0 & & . & .\\ . & . & & . & .\\ . & . & & 0 & 0\\ 0 & 0 & \cdot\cdot & t^n & 0\\ 0 & 0 & \cdot\cdot & 0 & t^n \end{pmatrix}$$

$$= \begin{pmatrix} -t^{1+nk_0+n} & -t^{1+nk_0+1+n} & -t^{1+nk_0+2+n} & \cdot\cdot & -t^{1+nk_0+n-3+n} & -t^{1+nk_0+n-2+n}\\ t^{1+nk_0-1+n} & 0 & 0 & \cdot\cdot & 0 & 0\\ 0 & t^{1+nk_0-1+n} & 0 & \cdot\cdot & 0 & 0\\ . & . & . & & . & .\\ . & . & & & . & .\\ . & . & . & & . & .\\ 0 & 0 & 0 & \cdot\cdot & t^{1+nk_0-1+n} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -t^{1+n(k_0+1)} & -t^{1+n(k_0+1)+1} & -t^{1+n(k_0+1)+2} & \cdot\cdot & -t^{1+n(k_0+1)+n-3} & -t^{1+n(k_0+1)+n-2}\\ t^{1+n(k_0+1)-1} & 0 & 0 & \cdot\cdot & 0 & 0\\ 0 & t^{1+n(k_0+1)-1} & 0 & \cdot\cdot & 0 & 0\\ . & . & . & & . & .\\ . & . & & & . & .\\ . & . & . & & . & .\\ 0 & 0 & 0 & \cdot\cdot & t^{1+n(k_0+1)-1} & 0 \end{pmatrix}$$

The result is valid for all $k \in \mathbb{N}$. The proof for the matrix defined in (3.2) is similar.

Therefore, the form of the reduced Burau matrices for braids of the form $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$ has been explicitly provided.

Now, let I_{n-1} be the unit $(n-1)\times(n-1)$ matrix, and consider de matrix defined in (3.1), then

$$\Psi_n^r(\beta)-I_{n-1}=\begin{pmatrix} -t^d-1 & -t^{d+1} & -t^{d+2} & \cdots & -t^{d+n-3} & -t^{d+n-2} \\ t^{d-1} & -1 & 0 & \cdots & 0 & 0 \\ 0 & t^{d-1} & -1 & \cdots & 0 & 0 \\ . & . & . & & . & . \\ . & . & & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & \cdots & t^{d-1} & -1 \end{pmatrix}_{n-1}$$

therefore,

$$\det\left(\Psi_n^r(\beta)-I_{n-1}\right)=\begin{vmatrix} -t^d-1 & -t^{d+1} & -t^{d+2} & \cdots & -t^{d+n-3} & -t^{d+n-2} \\ t^{d-1} & -1 & 0 & \cdots & 0 & 0 \\ 0 & t^{d-1} & -1 & \cdots & 0 & 0 \\ . & . & . & & . & . \\ . & . & & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & \cdots & t^{d-1} & -1 \end{vmatrix}_{n-1}$$

$$= (-1)^3 \left(t^d + 1\right) \begin{vmatrix} -1 & 0 & 0 & \cdot\cdot & 0 & 0 \\ t^{d-1} & -1 & 0 & \cdot\cdot & 0 & 0 \\ 0 & t^{d-1} & -1 & \cdot\cdot & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & -1 & \cdot \\ 0 & 0 & 0 & \cdot\cdot & t^{d-1} & -1 \end{vmatrix} + (-1)^4 \, t^{d+1} \begin{vmatrix} t^{d-1} & 0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & -1 & 0 & \cdot\cdot & 0 & 0 \\ 0 & t^{d-1} & -1 & \cdot\cdot & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & -1 & \cdot \\ 0 & 0 & 0 & \cdot\cdot & t^{d-1} & -1 \end{vmatrix} +$$

$$(-1)^5 \, t^{d+2} \begin{vmatrix} t^{d-1} & -1 & 0 & \cdot\cdot & 0 & 0 \\ 0 & t^{d-1} & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & -1 & \cdot\cdot & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & -1 & \cdot \\ 0 & 0 & 0 & \cdot\cdot & t^{d-1} & -1 \end{vmatrix} + (-1)^6 \, t^{d+3} \begin{vmatrix} t^{d-1} & -1 & 0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & t^{d-1} & -1 & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & t^{d-1} & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdot\cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & -1 & \cdot \\ 0 & 0 & 0 & 0 & \cdot\cdot & t^{d-1} & -1 \end{vmatrix} +$$

$\bullet$

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$\bullet$

$$+ (-1)^n \, t^{d+n-3} \begin{vmatrix} t^{d-1} & -1 & 0 & \cdot\cdot & 0 & 0 \\ 0 & t^{d-1} & -1 & \cdot\cdot & 0 & 0 \\ 0 & 0 & t^{d-1} & \cdot\cdot & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & t^{d-1} & \cdot \\ 0 & 0 & 0 & \cdot\cdot & 0 & -1 \end{vmatrix} + (-1)^{n+1} \, t^{d+n-2} \begin{vmatrix} t^{d-1} & -1 & 0 & \cdot\cdot & 0 & 0 \\ 0 & t^{d-1} & -1 & \cdot\cdot & 0 & 0 \\ 0 & 0 & t^{d-1} & \cdot\cdot & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & t^{d-1} & -1 \\ 0 & 0 & 0 & \cdot\cdot & 0 & t^{d-1} \end{vmatrix}$$

$$\begin{aligned} &= (-1)^{n+1}\left(t^d+1\right)+(-1)^{n+1}t^{2d}+(-1)^{n+1}t^{3d}+...+(-1)^{n+1}t^{d(n-2)}+(-1)^{n+1}t^{d(n-1)} \\ &= (-1)^{n+1}\left(1+t^d+t^{2d}+t^{3d}+...+t^{d(n-2)}+t^{d(n-1)}\right) \end{aligned}$$

hence,

$$\det\left(\Psi_n^r(\beta)-I_{n-1}\right)=(-1)^{n+1}\left(1+t^d+t^{2d}+t^{3d}+...+t^{d(n-2)}+t^{d(n-1)}\right). \tag{3.3}$$

The determinants previously estimated are $(n-2)\times(n-2)$ determinants.

Since β has the form $(\sigma_1\sigma_2...\sigma_{n-1})^d$, then $\langle\beta\rangle = (n-1)\,d$. Replacing the determinant calculated in (3.3) in the rational function defined in (2.1), we obtain:

$$\begin{aligned} f_n(\beta) &= (-1)^{n+1}\frac{s^{-(n-1)d}\left(s-s^{-1}\right)}{\left(s^n-s^{-n}\right)}g\left((-1)^{n+1}\left(1+t^d+t^{2d}+t^{3d}+...+t^{d(n-2)}+t^{d(n-1)}\right)\right) \\ &= \frac{s^{-(n-1)d}\left(s-s^{-1}\right)}{\left(s^n-s^{-n}\right)}\left(1+s^{2d}+s^{4d}+s^{6d}+...+s^{d(2n-4)}+s^{d(2n-2)}\right) \\ &= \frac{\left(s^2-1\right)\left(1+s^{2d}+s^{4d}+s^{6d}+...+s^{d(2n-4)}+s^{d(2n-2)}\right)}{s^{(d-1)(n-1)}\left(s^{2n}-1\right)} \\ &= \frac{\left(1+s^{2d}+s^{4d}+s^{6d}+...+s^{d(2n-4)}+s^{d(2n-2)}\right)}{s^{(d-1)(n-1)}\left(1+s^2+s^4+s^6+...+s^{2n-4}+s^{2n-2}\right)} \end{aligned}$$

hence,

$$f_n(\beta)=\frac{\left(1+s^{2d}+s^{4d}+s^{6d}+...+s^{d(2n-4)}+s^{d(2n-2)}\right)}{s^{(d-1)(n-1)}\left(1+s^2+s^4+s^6+...+s^{2n-4}+s^{2n-2}\right)}. \tag{3.4}$$

Recall that the knot invariant associated to f_n is the Alexander polynomial, and is obtained under the transformation $s\to\sqrt{1/t}$. Since the clousure of β is a knot K isotopic to a torus knot $T(n,d)$, the formula (3.4) provides a relatively easy way to calculate its Alexander polynomial. In addition, since the size of the Burau matrices changes depending on the value of n, the formula (3.4) also changes for every $T(n,d)$.

Knot	*Braid representation*	$\det\left(\Psi_n^r(\beta) - I_{n-1}\right)$	*Alexander polynomial*
$T(2,d)$	$(\sigma_1)^d$	$-1-t^d$	$\dfrac{1+s^{2d}}{s^{d-1}\left(1+s^2\right)}$
$T(3,d)$	$(\sigma_1\sigma_2)^d$	$1+t^d+t^{2d}$	$\dfrac{1+s^{2d}+s^{4d}}{s^{2d-2}\left(1+s^2+s^4\right)}$
$T(4,d)$	$(\sigma_1\sigma_2\sigma_3)^d$	$-1-t^d-t^{2d}-t^{3d}$	$\dfrac{1+s^{2d}+s^{4d}+s^{6d}}{s^{3d-3}\left(1+s^2+s^4+s^6\right)}$
$T(5,d)$	$(\sigma_1\sigma_2\sigma_3\sigma_4)^d$	$1+t^d+t^{2d}+t^{3d}+t^{4d}$	$\dfrac{1+s^{2d}+s^{4d}+s^{6d}+s^{8d}}{s^{4d-4}\left(1+s^2+s^4+s^6+s^8\right)}$

The equation (3.4) can be algebraically reduced as:

$$f_n(\beta) = \frac{\left(s^{2nd}-1\right)\left(s^2-1\right)}{s^{(d-1)(n-1)}\left(s^{2d}-1\right)\left(s^{2n}-1\right)}.$$

A posteriori result

Considering $\alpha \in B_n$ such that $\alpha = \left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)^d$, where $gcd(n,d) = 1$ and $n, d \geq 2$, the type of braid is the mirror of the the braid $\left(\sigma_1\sigma_2\cdots\sigma_{n-1}\right)^d$ *i.e.* α is a periodic braid where the base word involve the first strand passing under all other strands in order. Since the clousure of $\left(\sigma_1\sigma_2\cdots\sigma_{n-1}\right)^d$ is a knot K isotopy to $T(n,d)$, then the clousure of $\left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)^d$ is a knot K' isotopy to the mirror of $T(n,d)$.

Defining the Burau matrix for the braid $\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$ as:

$$\Psi_n^r\left(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1\right) := \begin{pmatrix} 0 & t & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & t & 0 \\ 0 & 0 & & 0 & t \\ -t & -t & \cdots & -t & -t \end{pmatrix}_{n-1}.$$

Observe that, $\Psi_n^r\left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)=\Psi_n^r\left[(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)^{-1}\right]=\left[\Psi_n^r\left(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1\right)\right]^{-1}$, then

$$\Psi_n^r\left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)=\begin{pmatrix}-t^{-1} & -t^{-1} & \cdots & -t^{-1} & t^{-1}\\ t^{-1} & 0 & \cdots & 0 & 0\\ 0 & t^{-1} & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & t^{-1} & 0\end{pmatrix}.$$

Since $\Psi_n^r\left((\sigma_1\sigma_2...\sigma_{n-1})^n\right)=t^nI_{n-1}$ for all $n\geq 2$, then

$$\begin{aligned}\Psi_n^r\left(\left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)^n\right) &= \Psi_n^r\left[\left((\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)^{-1}\right)^n\right]\\ &= \left[\Psi_n^r\left((\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)^n\right)\right]^{-1}\\ &= \left(t^nI_{n-1}\right)^{-1}\\ &= t^{-n}I_{n-1}\end{aligned}$$

thus $\Psi_n^r\left(\left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)^n\right)=t^{-n}I_{n-1}$, for all $n\geq 2$.

Implementing a similar analysis as the one made for braids of the form $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$, let $d=p'+nk$, $k\in\mathbb{N}\cup 0'$ or $k\in\mathbb{N}$ with $\gcd(n,p')=1$, where p' indicates the position of the row equal to $(-t^{-1},-t^{-1},...,-t^{-1},...,-t^{-1},-t^{-1})$ of length $n-1$. Therefore, $\Psi_n^r(\alpha)$ is given by:

- If $d=1+nk$, $k\in\mathbb{N}$, then $\Psi_n^r(\alpha)$ is

$$\begin{pmatrix}-t^{-d} & -t^{-d} & \cdots & -t^{-d} & -t^{-d}\\ t^{-d} & 0 & \cdots & 0 & 0\\ 0 & t^{-d} & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & t^{-d} & 0\end{pmatrix},$$

in the last row, the term t^{-d} is in the $(n-1,n-2)\,th$ position.

- If $d = p' + nk$, $k \in \mathbb{N} \cup 0'$, $1 < p' < n$, then $\Psi_n^r(\alpha)$ is

$$\begin{pmatrix}
0 & \cdots & t^{-d} & \cdot & \cdot & 0 & 0 \\
\vdots & & & & & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & t^{-d} & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & t^{-d} \\
-t^{-d} & -t^{-d} & \cdots & -t^{-d} & \cdots & -t^{-d} & -t^{-d} \\
t^{-d} & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & t^{-d} & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & & & & & \vdots & \vdots \\
0 & 0 & \cdot & \cdot & t^{-d} & \cdots & 0
\end{pmatrix}_{n-1},$$

in the first row, the term t^{-d} is in the $(1, n-p+1)\, th$ and in the last row the term t^{-d} is in the $(n-1, n-p-1)\, th$ position. The proof of this can be made by induction on k.

Now, let I_{n-1} be the unit $(n-1) \times (n-1)$ matrix, and consider the matrices defined above, then

$$\det\left(\Psi_n^r(\beta) - I_{n-1}\right) = (-1)^{n+1} t^{(n-1)d} \left(1 + t^d + t^{2d} + ... + t^{d(n-2)} + t^{d(n-1)}\right). \quad (3.5)$$

Note that, since α is the form $\left(\sigma_1^{-1}\sigma_2^{-1}...\sigma_{n-1}^{-1}\right)^d$, then $\langle\alpha\rangle = (n-1)\, d$. Replacing the determinant estimated in (3.5) in the rational function defined in (2.1), one obtains:

$$f_n(\alpha) = (-1)^{n+1} \frac{s^{-(n-1)d}\left(s - s^{-1}\right)}{\left(s^n - s^{-n}\right)} g\left((-1)^{n+1} t^{-(n-1)d}\left(1 + t^d + t^{2d} + t^{3d} + ... + t^{d(n-1)}\right)\right)$$

hence,

$$f_n(\alpha) = \frac{\left(1 + s^{2d} + s^{4d} + s^{6d} + ... + s^{d(2n-4)} + s^{d(2n-2)}\right)}{s^{(3d-1)(n-1)}\left(1 + s^2 + s^4 + s^6 + ... + s^{2n-4} + s^{2n-2}\right)}. \quad (3.6)$$

This equation can be algebraically reduced as:

$$f_n(\alpha) = \frac{\left(s^{2nd}-1\right)\left(s^2-1\right)}{s^{(3d-1)(n-1)}\left(s^{2d}-1\right)\left(s^{2n}-1\right)},$$

which is the Alexander Polynomial of a knot K isotopy to the mirror of $T(n,d)$. It is important to mention that the formulas (3.4) and (3.6) provide the Alexander polynomial of the resulting knot from the closure of the braids β and α, but knots exist whose mirror has the same Alexander polynomial, then (3.4) and (3.6) are not necessarily different.

Recall, the Alexander Polynomial is a knot invariant, therefore it does not change under ambient isotopy and it allows to clasify knots. Therefore, any two knot projections with diferent Alexander Polynomial are not equivalent, but two knot projections with the same Alexander Polynomial does not mean the projections are equivalent. There are many difficult ways to compute this polynomial, however, the computations and proofs conducted in this chapter expose an alternative easy route to compute that polynomial for torus knots and provide an explicit form for the Burau matrices for the braids $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$ and $\left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)^d$ whose clousures are knots isotopy to $T(n,d)$, and the mirror of $T(n,d)$ respectively was provided.

CHAPTER 4
CONCLUSION AND FUTURE WORK

4.1 Conclusion

This work, presents a pattern in the reduced Burau representation for braids of the form $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$, and $\left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)^d$ with $gcd(n,d)=1$ and $n,d\geq 2$. Analyzing the reduced Burau representation for these braids, a characterization of the reduced Burau matrices for the braids mentioned above was described, and a general formula for the Alexander polynomial for the knot associated to the closure of the braids $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$, and $\left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)^d$ was provided, reaching all the objectives proposed in this thesis.

4.2 Future work

The following future work is proposed:

- To characterize the reduced Burau matrices for braids of the form $\left(\sigma_1^{\pm1}\sigma_2^{\pm1}\cdots\sigma_{n-1}^{\pm1}\right)^d$
- To find a general formula for the Alexander polynomial for the knot Associated to the closure of the braids $\left(\sigma_1^{\pm1}\sigma_2^{\pm1}\cdots\sigma_{n-1}^{\pm1}\right)^d$
- To extend the formulas given for the braids $(\sigma_1\sigma_2\cdots\sigma_{n-1})^d$ and $\left(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\right)^d$ to any periodic braid.

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ALEXANDER POLYNOMIAL FOR TORUS KNOTS VIA BURAU MATRICES FOR PERIODIC BRAIDS

Raúl Alfonso Beltrán Hoyos

raul.beltran1@upr.edu

Department of Mathematical Sciences

Chair: Juan A. Ortiz-Navarro

Degree: Master of Science

Graduation Date: 2017