COMPARISON OF ELASTIC PLATE THEORIES FOR MICROPOLAR MATERIALS

by

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ABSTRACT

The purpose of this thesis is to develop a comparison between two theories of micro-polar plates with moderate thickness. A comparison between the proposed and Eringen models is shown in the thesis.

We used a special technique of the method of separation variables to obtain analytical solutions for the plate models. This special technique is general and can be used for any elliptic system of linear partial differential equations.

The stress- strain polynomial approximations of the proposed model have been checked for consistency with the elastic equilibrium, boundary conditions and the constitutive relationships. The formulation of the variational principle for the proposed model is based on the generalized Hellinger-Prange-Reissner principle, which incorporates the proposed stress and strain-displacement approximations for the micro- polar plates. The proposed model produces a new theory of Cosserat plate, which includes a new form of constitutive relationships.

The proposed and Eringen models are described by elliptic systems of partial differential equations. The differences in the systems are due to the different orders of polynomial approximations of asymmetric stress, couple stress, displacement, and micro-rotation over the plate thickness.

The obtained analytical solutions for the micro- polar plate boundary value problem, have been used for numerical results and comparisons for a special case of syntactic foam plate.

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COMPARACION DE TEORIAS DE PLACAS ELASTICAS PARA MICROPOLAR MATERIALS

Por

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RESUMEN

El objetivo de esta tesis es mostrar una comparación numérica entre dos teorías de placas micro-polares con un grosor moderado. Esta comparación esta hecha entre el modelo propuesto y el modelo de Eringen.
Hemos usado una técnica especial del método de separación de variables para obtener soluciones analíticas para los modelos de placas. Esta técnica especial es general y puede ser usada para cualquier sistema elíptico de ecuaciones diferenciales parciales lineales.
Se ha comprobado la consistencia de las aproximaciones a través de polinomios para la tensión- deformación del modelo propuesto con el equilibrio elástico, las condiciones de frontera y las relaciones constitutivas. La formulación del principio variacional para el modelo propuesto esta basada en el principio generalizado de Hellinger-Prange-Reissner, el cual incorpora las aproximaciones propuestas para el stress y deformación- desplazamiento para placas micro- polares. El modelo propuesto produce una nueva teoría de placas de Cosserat, el cual incluye una nueva forma de relaciones constitutivas.
Los modelos propuesto y de Eringen son descritos por sistemas elípticos de ecuaciones diferenciales parciales. La diferencia entre estos sistemas es en el uso de diferentes aproximaciones polinomicas para la tensión asimétrica, pareja de tensiones, el desplazamiento y la micro-rotación sobre el espesor de la placa.
Las soluciones analíticas obtenidas para el problema de valor de contorno de placas micro-polares han sido usadas para resultados numéricos y comparaciones para un caso especial de placa de espuma sintáctica.

To my mother

María Melania Carranza Goicochea de Reyes,

to my sister

Ingrid Jhovanna

and

to my baby

Araceli

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To my ***Araceli,*** my baby, she spells my name and she knows a theorem about the "exchange between the limit and a continuous function".

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LIST OF SYMBOLS

t	External momentum per unit area
p	External force per unit area
Ω_i^0	Micro- rotation in the middle plane
W	Vertical deflection of the middle plane
Ψ_i	Macro- rotation of the middle plane
ν	Poisson's ratio
G	Shear Modulus
E	Young's Modulus
D	Flexural rigidity
σ_{ij}	Stress tensor
γ_{ij}	Strain tensor
μ_{ij}	Couple stress tensor
χ_{ij}	Gradient of micro- rotation vector
φ_i	Micro- rotation vector
u_i	Displacement vector
F	Free elastic energy
C	Bulk energy
λ, μ	Lame constants
$\alpha, \beta, \gamma, \varepsilon$	Complementary elasticity constants
ε_{ijk}	Levi Civita tensor
δ_{ij}	Kronecker Delta tensor
Γ	Boundary of the middle plane of a plate
N	Coupling number
Ψ	Polar ratio
l_t	Characteristic number
l_b	Characteristic bending
$w_E^{0,I}$	Maximum vertical deflection of the middle plane of the plate corresponding to the Eringen Model I for the Elastic (Classic) Materials
$w_E^{0,II}$	Maximum vertical deflection of the middle plane of the plate corresponding to the Eringen Model II for the Elastic (Classic) Materials
w_R	Maximum vertical deflection of the middle plane of the plate corresponding to the Reissner Model.

CHAPTER 1

ELASTICITY THEORIES

1.1 Brief History of Classical and Asymmetric Elasticity Theory

The well known classical bending theory of elastic plates **[6]**, **[7]**, **[17]**, was first presented by Kirchhoff in his thesis (1850) and is described by a bi-harmonic differential equation **[2]**, **[17]**. The usual assumption of this theory is that the normal to the middle plane remains normal during deformation. Thus the theory neglects transverse shear strain effects. A system of equations, which takes into account the transverse shear deformation, has been developed by E. Reissner (1945) **[13]**, **[14]**.

One of the advantages of Reissner's model is that it is able to determine the reactions along the edges of a simply supported rectangular plate, where classical theory leads to a concentrated reaction at the corners of the plate. The Reissner theory has been applied to thin walled structures with moderate thickness. The study of the relationships between these two models has proved that the solution of the clamped Reissner plate approaches the solution of the Kirchhoff plate as the thickness approaches zero **[1]** and that the maximum bending can reach up to 20% for moderate plate thickness **[2]**. The numerical calculations of bending behavior of the plate of moderate thickness, **[16]** show high level agreement between 3D and Reissner models. More remarks on the history of the modeling of classic linear elastic plates can be found in **[6]**, **[16]**.

In order to describe deformation of elastic plates with microstructure that posses grains, particles, fibers, and cellular structures **[10]**, **[11]** A. C. Eringen (1967) was the first to propose a theory of plates in the framework of Cosserat (micro-polar) Elasticity **[3]**. His Theory is based on a direct technique of integration of the Cosserat Elasticity. The Eringen plate theory does not consider a transverse variation of the micro-rotation over the thickness, which might be necessary for rather thick plates under vertical load and pure twisting momentum. In order to develop a theory of plates, which can be used for thin wall structures with moderate thickness, we propose to use the classic Reissner plate theory as a foundation for the modeling of Cosserat elastic plates. Our approach, in addition to the traditional model, takes into account the second order approximation of couple stresses and the variation of three components of micro rotation in the thickness direction.

The classical elasticity theory showed satisfactory results with experimentation in many structural materials such as aluminum, steel and iron. There were other cases of elastic materials in which theory had discrepancies with experimentation. Some of these are polymers, biological materials, cellular materials and nano materials. These differences seemed to become significative for problems where large stress gradients occur (near holes or cracks), for vibrational problems where waves have a very high frequency or small wavelength and for materials that possess granular structure. These type of observations suggested that the influence of microstructure should be taken into account.

1.1.1 Classical Theory

In the classical theory of elasticity only macroscopic effects are taken into consideration, that is, all solid bodies are assumed to be made of a continuous medium, and only the force per unit area is taken into consideration.

1.2 Micro-polar Theory

In the asymmetric theory of elasticity a force and a momentum per unit area are considered, and the stress and the couple stress tensors are in general asymmetric. When the couple stress effect is neglected then the stress tensor becomes symmetric.

1.2.1 Micro-polar Linear (Cosserat)

The Cosserat elasticity equilibrium equations without body forces represent the balance of linear and angular momentums and have the following form **[3]:**

$$div\sigma = 0, \quad \textbf{(1.1)}$$

$$\varepsilon \cdot \sigma + div\mu = 0, \quad \textbf{(1.2)}$$

where: $\sigma = \{\sigma_{ji}\}$ is the stress tensor, $\mu = \{\mu_{ji}\}$ the couple stress tensor, $\varepsilon = \{\varepsilon_{ijk}\}$ is the Levi-Civita tensor, where ε_{ijk} equals 1 or -1 according as (i, j, k) is an even or odd permutation of 1,2,3 and zero otherwise, and $\varepsilon \cdot \sigma = \{\varepsilon_{ijk}\sigma_{jk}\}$.
The constitutive equations can be written in Nowacki's form **[12]:**

$$\sigma = (\mu + \alpha)\gamma + (\mu - \alpha)\gamma^T + \lambda(tr\gamma)1, \quad \textbf{(1.3)}$$

$$\mu = (\gamma + \varepsilon)\chi + (\gamma - \varepsilon)\chi^T + \beta(tr\chi)1, \quad \textbf{(1.4)}$$

and the strain-displacement and torsion-rotation relations are:

$$\gamma = (\nabla u)^T + \varepsilon \cdot \varphi \quad \text{and} \quad \chi = \nabla\varphi \;, \quad \textbf{(1.5)}$$

where γ and χ are the micro- polar strain and torsion tensors, u and φ the displacement and rotation vectors respectively, **1** the identity tensor, and μ, λ the symmetric and $\alpha, \beta, \gamma, \varepsilon$ the asymmetric Cosserat elasticity constants.

The constitutive equations in the reversible form are given by:

$$\gamma = (\mu' + \alpha')\sigma + (\mu' - \alpha')\sigma^T + \lambda'(tr\sigma)1,$$

$$\chi = (\gamma' + \varepsilon')\mu + (\gamma' - \varepsilon')\mu^T + \beta(tr\mu)1,$$

where

$$\mu' = \frac{1}{4\mu},\ \alpha' = \frac{1}{4\alpha},\ \gamma' = \frac{1}{4\gamma},\ \varepsilon' = \frac{1}{4\varepsilon},\ \lambda' = \frac{-\lambda}{6\mu(\lambda + \frac{2\mu}{3})} \text{ and } \beta' = \frac{-\beta}{6\mu(\beta + \frac{2\gamma}{3})}.$$

We consider a Cosserat elastic body B_0. In this case the equilibrium equations **(1.1) - (1.2)** with constitutive formulas **(1.3)-(1.4)** and kinematics formulas **(1.5)** should be accompanied by the following mixed boundary conditions:

$$u = u_0\ ,\ \varphi = \varphi_0 \qquad \text{on}\, G_1 = \partial B_0 - \partial B_\sigma, \tag{1.6}$$

$$\sigma_n = \sigma \cdot n = \sigma_0,\ \ \mu_n = \mu.n = \mu_0 \qquad \text{on}\, G_2 = \partial B_\sigma, \tag{1.7}$$

where u_0, φ_0 are prescribed on G_1, σ_0 and μ_0 on G_2, and n denotes the outward unit normal vector to ∂B_0.

1.2.2 Cosserat Elastic Energy

The strain stored energy U_C of the body B_0 is defined by the integral **[11]**:

$$U_C = \int_{B_0} W\{\gamma, \chi\} dv, \tag{1.8}$$

where

$$\begin{gathered} W\{\gamma, \chi\} = \frac{\mu+\alpha}{2}\gamma_{ij}\gamma_{ij} + \frac{\mu-\alpha}{2}\gamma_{ij}\gamma_{ji} + \frac{\lambda}{2}\gamma_{kk}\gamma_{nn} + \\ \frac{\gamma+\varepsilon}{2}\chi_{ij}\chi_{ij} + \frac{\gamma-\varepsilon}{2}\chi_{ij}\chi_{ji} + \frac{\beta}{2}\chi_{kk}\chi_{nn}. \end{gathered} \tag{1.9}$$

Then the constitutive relations **(1.3)- (1.4)** can be written in the form:

$$\sigma = \nabla_\gamma W \text{ and } \mu = \nabla_\chi W. \tag{1.10}$$

The function W is positive iff **[12]:**

$$\begin{gathered} \mu > 0, \quad 3\lambda + 2\mu > 0, \\ \gamma > 0, \quad 3\beta + 2\gamma > 0, \\ \alpha > 0, \quad \mu + \alpha > 0, \\ \varepsilon > 0, \quad \gamma + \varepsilon > 0, \end{gathered} \tag{1.11}$$

The *coercivity conditions* **[9]** for Cosserat elastic energy.

$$\begin{gathered}\mu > 0, \quad 3\lambda + 2\mu > 0, \\ \gamma > 0, \quad 3\beta + 2\gamma > 0, \\ \alpha \geq 0, \ \ \varepsilon \geq 0, \ \ \gamma + \varepsilon > 0,\end{gathered} \tag{1.12}$$

are enough to provide the uniqueness of the solution of the elasticity boundary value problems.

The stress energy is given by:

$$U_K = \int_{B_0} \Phi\{\sigma, \mu\} dv,$$

where

$$\Phi\{\sigma, \mu\} = \frac{\mu' + \alpha'}{2}\sigma_{ij}\sigma_{ij} + \frac{\mu' - \alpha'}{2}\sigma_{ij}\sigma_{ji} + \frac{\lambda'}{2}\sigma_{kk}\sigma_{nn} + \\ \frac{\gamma' + \varepsilon'}{2}\mu_{ij}\mu_{ij} + \frac{\gamma' - \varepsilon'}{2}\mu_{ij}\mu_{ji} + \frac{\beta'}{2}\mu_{kk}\mu_{nn},$$

1.2.3 The Hellinger-Prange-Reissner (HPR) Principle

Steinberg **[21]** proposed the **HPR principle [5]** for the case of Cosserat elasticity in the following form:

For any set $\mathcal{A}$ of all admissible states $s = [u, \varphi, \gamma, \chi, \sigma, \mu]$ that satisfy the ***strain-displacement relation*** and torsion-rotation relations **(1.5)**, the zero variation ($\delta\ \Theta(s) = 0$) of the functional Θ defined by:

$$\Theta(s) = U_K - \int_{B_0} [\sigma \cdot \gamma + \mu \cdot \chi] dv + \int_{\varsigma_1} [\sigma_n \cdot (u - u_0) + \mu_n \cdot (\varphi - \varphi_0)] da + \int_{\varsigma_2} [\sigma_0 \cdot u + \mu_0 \cdot \varphi] da \tag{1.13}$$

at $s \in \mathcal{A}$ is equivalent to s to be a solution of the system of equilibrium equations **(1.1)-(1.2)**, constitutive relations from **(1.3)- (1.4)**, which satisfies the mixed boundary conditions **(1.6)- (1.7) .**

CHAPTER 2

THE PLATE THEORIES

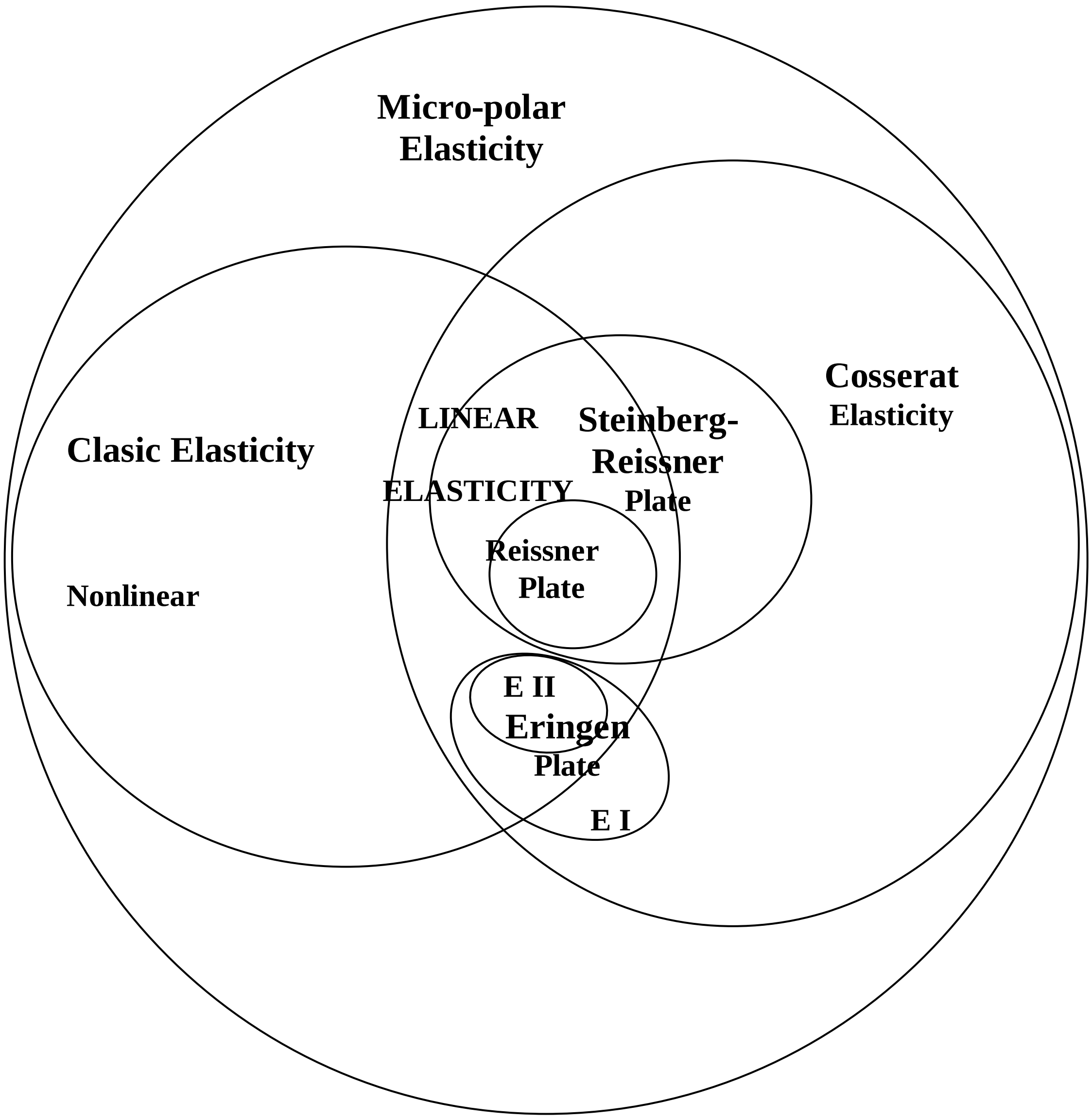

Figure 2.1 Classification of Models of Elastic Plates. This Venn Euler diagram shows that Steinberg-Reissner Model and Eringen Model II are rather close.

2.1. Steinberg-Reissner Plate Theory for Cosserat Materials

2.1.1 Plate Assumptions

We formulate the stress, couple stress and kinematic assumptions of the Cosserat plate. We consider the thin plate P, where $h = h^P$ is the thickness of the plate and $x_3 = 0$ contains its middle plane. The sets T and B are the top and bottom surfaces contained in the planes $x_3 = \frac{h}{2}$, $x_3 = \frac{-h}{2}$ respectively and the curve Γ is the boundary of the middle plane of the plate.

The set of points $\mathrm{P} = \left\{\Gamma \times [\frac{-h}{2}, \frac{h}{2}]\right\} \cup T \cup B$ forms the entire surface of the plate and $\left\{\Gamma_u \times [\frac{-h}{2}, \frac{h}{2}]\right\}$ is the lateral part of the boundary, where displacements and micro-rotations are prescribed. The notation $\Gamma_\sigma = \Gamma - \Gamma_u$ of the remainder we use to describe the lateral part of the boundary edge $\left\{\Gamma_\sigma \times [\frac{-h}{2}, \frac{h}{2}]\right\}$ where stress and couple stress are prescribed. We also use notation P_0 for the middle plane internal domain of the plate.

In our case we consider the vertical load and pure twisting momentum **boundary conditions** (**B.C.**) at the top and bottom of the plate, which can be written in the form:

$$\sigma_{33}(x_1, x_2, \frac{h}{2}) = \sigma^t(x_1, x_2), \quad \sigma_{33}(x_1, x_2, \frac{-h}{2}) = \sigma^b(x_1, x_2), \tag{2.1}$$

$$\sigma_{3\beta}(x_1, x_2, \frac{\pm h}{2}) = 0, \tag{2.2}$$

$$\mu_{33}(x_1, x_2, \frac{h}{2}) = \mu^t(x_1, x_2), \quad \mu_{33}(x_1, x_2, \frac{-h}{2}) = \mu^b(x_1, x_2), \tag{2.3}$$

$$\mu_{3\beta}(x_1, x_2, \frac{\pm h}{2}) = 0, \tag{2.4}$$

where $(x_1, x_2) \in \mathrm{P}_0$.

2.1.2 Stress, Couple Stress and Kinematics Assumptions

The Reissner's theory of plates **[13]** assumes that the variation of stress σ_{kl} and couple stress μ_{kl} components across the thickness can be represented by means of polynomials of x_3 in such a way that it will be consistent with the equilibrium equations **(1.1)** and **(1.2)**. We adapt the expressions for the stress and couple-stress components in the following form **[18]**:

$$\sigma_{\alpha\beta} = n_{\alpha\beta}(x_1, x_2) + \frac{h}{2}\varsigma_3 m_{\alpha\beta}(x_1, x_2), \tag{2.5}$$

$$\sigma_{3\beta} = q_\beta(x_1, x_2)(1-\varsigma_3^2), \tag{2.6}$$

$$\sigma_{\beta 3} = q_\beta^*(x_1, x_2)(1-\varsigma_3^2), \tag{2.7}$$

$$\sigma_{33} = \frac{-3}{4}(\frac{1}{3}\varsigma_3^3 - \varsigma_3)p + \sigma_0, \tag{2.8}$$

$$\mu_{\alpha\beta} = (1-\varsigma_3^2)r_{\alpha\beta}(x_1, x_2), \tag{2.9}$$

$$\mu_{\beta 3} = \varsigma_3 s_\beta^*(x_1, x_2) + m_\beta^*(x_1, x_2), \tag{2.10}$$

$$\mu_{3\beta} = 0, \tag{2.11}$$

$$\mu_{33} = \varsigma_3 \nu + t, \tag{2.12}$$

where:

$$\alpha, \beta \in \{1,2\},\ \varsigma_3 = \frac{2}{h}x_3,\ p = P = \sigma^t(x_1, x_2) - \sigma^b(x_1, x_2)\ ,\ \sigma_0 = \frac{1}{2}(\sigma^t(x_1, x_2) + \sigma^b(x_1, x_2)),$$

$$\nu(x_1, x_2) = \frac{1}{2}(\mu^t(x_1, x_2) - \mu^b(x_1, x_2)) \text{ and } t(x_1, x_2) = \frac{1}{2}(\mu^t(x_1, x_2) + \mu^b(x_1, x_2)).$$

We note that expression **(2.8)** is identical to the expression of σ_{33} given in **[13]** in the case of $\sigma^b = 0$.

The displacements u_α are distributed linearly over the thickness of the plate **[3]** and that u_3 does not vary over the thickness of the plate, i.e.

$$u_\alpha = U_\alpha(x_1, x_2) - \frac{h}{2}\varsigma_3 V_\alpha(x_1, x_2),$$
$$u_3 = w(x_1, x_2), \tag{2.13}$$

where the terms $V_\alpha(x_1, x_2)$ represent the rotations in middle plane.

The variation of micro-rotation with respect to x_3 is represented by means of the second and third order polynomials:

$$\varphi_\alpha = \Theta_\alpha^0(x_1, x_2)(1-\varsigma_3^2), \tag{2.14}$$

$$\varphi_3 = \Theta_3^0(x_1, x_2) + \varsigma_3(1-\frac{1}{3}\varsigma_3^2)\Theta_3(x_1, x_2). \tag{2.15}$$

The functions Θ_i^0 in **(2.14)** and **(2.15)** describe micro- rotation components in the middle plane of the plate and Θ_3 is the slope at the middle plane, i.e.

$$\Theta_3(x_1,x_2)=\frac{h}{2}\frac{\partial\varphi_3(x_1,x_2,x_3)}{\partial x_3}\Big|_{x_3=0}.$$

2.1.3 Specification of the HPR Variational Principle in the case of Cosserat Elastic Plates

The **HPR** variational principle for a Cosserat plate is most appropriately expressed in terms of corresponding integrands calculated across the whole thickness. The weighted characteristics of displacements, micro-rotations, strains and stresses of the plate are introduced, which will be used to produce the explicit forms of these integrands.

2.1.3.1 The Cosserat plate stress energy density

The plate stress energy density is given by the following formula:

$$\Phi(\mathcal{S})=\frac{h}{2}\int_{-1}^{1}\Phi\{\sigma,\mu\}d\varsigma_3, \qquad \textbf{(2.16)}$$

where the Cosserat stress set

$$\mathcal{S}=\left[M_{\alpha\beta},Q_\alpha,Q_\alpha^*,R_{\alpha\beta},S_\beta^*,N_{\alpha\beta},M_\alpha^*\right], \qquad \textbf{(2.17)}$$

and

$$M_{\alpha\beta}=\left(\frac{h}{2}\right)^2\int_{-1}^{1}\varsigma_3\sigma_{\alpha\beta}d\varsigma_3=\frac{h^3}{12}m_{\alpha\beta},$$

$$Q_\alpha=\frac{h}{2}\int_{-1}^{1}\sigma_{3\alpha}d\varsigma_3=\frac{2h}{3}q_\alpha,\qquad Q_\alpha^*=\frac{h}{2}\int_{-1}^{1}\sigma_{\alpha3}d\varsigma_3=\frac{2h}{3}q_\alpha^*,$$

$$R_{\alpha\beta}=\frac{h}{2}\int_{-1}^{1}\mu_{\alpha\beta}d\varsigma_3=\frac{2h}{3}r_{\alpha\beta}, \qquad \textbf{(2.18)}$$

$$S_\alpha^*=\left(\frac{h}{2}\right)^2\int_{-1}^{1}\varsigma_3\mu_{\alpha3}d\varsigma_3=\frac{h^2}{6}s_\alpha^*,$$

$$N_{\alpha\beta}=\frac{h}{2}\int_{-1}^{1}\sigma_{\alpha\beta}d\varsigma_3=hn_{\alpha\beta},\qquad M_\alpha^*=\frac{h}{2}\int_{-1}^{1}\mu_{\alpha3}d\varsigma_3=hm_\alpha^*,$$

Taking into account the stress and couple stress assumptions **(2.5)-(2.12)** by the integrating $\Phi\{\sigma, \mu\}$ with respect ς_3 in [-1, 1] the plate stress energy density expression is in the form [5] :

$$\begin{aligned}
\Phi(\mathcal{S}) = {} & \frac{\lambda+\mu}{2h\mu(3\lambda+2\mu)}[N_{\alpha\alpha}^2 + \frac{12}{h^2}M_{\alpha\alpha}^2] - \frac{\lambda}{2h\mu(3\lambda+2\mu)}[N_{11}N_{22} + \frac{12}{h^2}M_{11}M_{22}] \\
& + \frac{\alpha+\mu}{8h\alpha\mu}[(1-\delta_{\alpha\beta})(N_{\alpha\beta}^2 + \frac{12}{h^2}M_{\alpha\beta}^2) + \frac{6}{5}(Q_\alpha Q_\alpha + Q_\beta^* Q_\beta^*)] \\
& + \frac{3\ \alpha-\mu}{10h\alpha\mu}[Q_\alpha Q_\alpha^* + \frac{5}{6}N_{12}N_{21} + \frac{10}{h^2}M_{12}M_{21}] + \frac{3\lambda}{5h\mu(3\lambda+2\mu)}Q_{\alpha,\alpha}^* M_{\beta\beta} \\
& + \frac{3}{5h\gamma(3\beta+2\gamma)}[(\beta+\gamma)R_{\alpha\alpha}^2 - \beta R_{11}R_{22}] + \frac{3}{10h}(\frac{1}{\gamma}-\frac{1}{\varepsilon})R_{12}R_{21} \\
& + \frac{17h(\lambda+\mu)}{280\mu(3\lambda+2\mu)}(Q_{\alpha,\alpha}^*)^2 + \frac{\lambda}{2\mu(3\lambda+2\mu)}(N_{\alpha\alpha})\sigma_0 \\
& + \frac{h(\lambda+\mu)}{2\mu(3\lambda+2\mu)}\sigma_0^2 - \frac{\gamma+\varepsilon}{h\gamma\varepsilon}[\frac{1}{8}M_\alpha^* M_\alpha^* + \frac{3}{2h^2}S_\alpha^* S_\alpha^* + \frac{3}{20}(1-\delta_{\beta\gamma})R_{\beta\gamma}^2] \\
& - \frac{\beta}{2\gamma(3\beta+2\gamma)}R_{\alpha\alpha}t + \frac{h(\beta+\gamma)}{2\gamma(3\beta+2\gamma)}t^2 + \frac{h(\beta+\gamma)}{6\gamma(3\beta+2\gamma)}\nu^2.
\end{aligned} \qquad \textbf{(2.19)}$$

Here M_{11} and M_{22} are the bending moments, M_{12} and M_{21} the twisting moments, Q_α the shear forces, Q_α^* the transverse shear forces, R_{11} and R_{22} the micro-polar bending moments, R_{12} and R_{21} the micro-polar twisting moments, S_α^* the micro-polar couple moments, all defined per unit length, N_{11} and N_{22} are the bending forces, N_{12} and N_{21} the twisting forces, M_α^* the micro-polar shear couple-stress resultants.

Then the stress energy of the plate P is given by the following formula:

$$U_K^{\mathcal{S}} = \int_{P_0} \Phi(\mathcal{S})da, \qquad \textbf{(2.20)}$$

where P_0 is the internal domain of the middle plane of the plate P.

2.1.3.2 The density of the work done over the Cosserat plate boundary

The proposed stress, couple stress, and kinematic assumptions are valid for the lateral boundary of the plate P as well.
The density of the work over the boundary $\Gamma_u \times [-0.5h, 0.5h]$:

$$\mathcal{W}_1 = \frac{h}{2}\int_{-1}^{1}[\sigma_n \cdot u + \mu_n \cdot \varphi]d\varsigma_3 \qquad \textbf{(2.21)}$$

Taking into account the stress and the couple stress assumptions **(2.5)-(2.12)** and kinematic assumptions **(2.13)-(2.15)** $\mathcal{W}_1$ is represented by the following expression:

$$\mathcal{W}_1 = \mathcal{S}_n.\mathcal{U} = \breve{M}_\alpha \Psi_\alpha + \breve{Q}^* W + \breve{R}_\alpha \Omega_\alpha^0 + \breve{S}^* \Omega_3 + \breve{N}_\alpha U_\alpha + \breve{M}^* \Omega_3^0, \qquad \textbf{(2.22)}$$

where $\mathcal{S}_n$ and $\mathcal{U}$ are defined as

$$\mathcal{S}_n = [\breve{M}_\alpha, \breve{Q}^*, \breve{R}_\alpha, \breve{S}^*, \breve{N}_\alpha, \breve{M}^*],$$
$$\mathcal{U} = \left[\Psi_\alpha, W, \Omega_\alpha^0, \Omega_3, U_\alpha, \Omega_3^0\right]$$

and

$$\breve{M}_\alpha = M_{\alpha\beta} n_\beta, \breve{Q}^* = Q_\beta^* n_\beta, \breve{R}_\alpha = R_{\alpha\beta} n_\beta,$$
$$\breve{S}^* = S_\beta^* n_\beta, \breve{N}_\alpha = N_{\alpha\beta} n_\beta, \breve{M}^* = M_\beta^* n_\beta.$$

In the above n_β is the outward unit normal vector to Γ_u, and

$$\begin{aligned}
\Psi_\alpha &= \frac{3}{h}\int_{-1}^{1}\varsigma_3 u_\alpha d\varsigma_3, & W &= \frac{3}{4}\int_{-1}^{1}(1-\varsigma^2) u_3 d\varsigma_3, \\
\Omega_\alpha^0 &= \frac{3}{4}\int_{-1}^{1}(1-\varsigma^2)\varphi_\alpha d\varsigma_3, & \Omega_3 &= \frac{3}{h}\int_{-1}^{1}\varsigma_3 \varphi_3 d\varsigma_3, \\
U_\alpha &= \frac{1}{2}\int_{-1}^{1} u_\alpha d\varsigma_3, & \Omega_3^0 &= \frac{1}{2}\int_{-1}^{1}\varphi_3 d\varsigma_3,
\end{aligned} \qquad \textbf{(2.23)}$$

where:

Ψ_α : Rotation vector (axis: x_α) in the middle plane of the plate ,
U_α : Horizontal Displacement of the middle plane along axis x_α,
Ω_α^0 : Micro-rotation vector in the middle plane,
Ω_3 : Instant rate of micro-rotation change along x_3,
W : Vertical deflection of the middle plane of the plate.

The correspondence between the weighted displacement w and the micro- rotations **(2.23)** and the kinematic variables by applying **(2.13)** and **(2.15)** in integration of expressions **(2.23)** is given by the following expressions:

$$\begin{aligned} \Psi_\alpha &= V_\alpha(x_1, x_2), & W &= w(x_1, x_2), \\ \Omega_\alpha^0 &= k_1^* \Theta_\alpha^0(x_1, x_2), & \Omega_3 &= \frac{k_2^*}{h}\Theta_3(x_1, x_2), \\ U_\alpha &= U_\alpha(x_1, x_2), & \Omega_3^0 &= \Theta_3^0(x_1, x_2), \end{aligned} \tag{2. 24}$$

where coefficients k_1^* and k_2^* depend on the variation of micro-rotations.

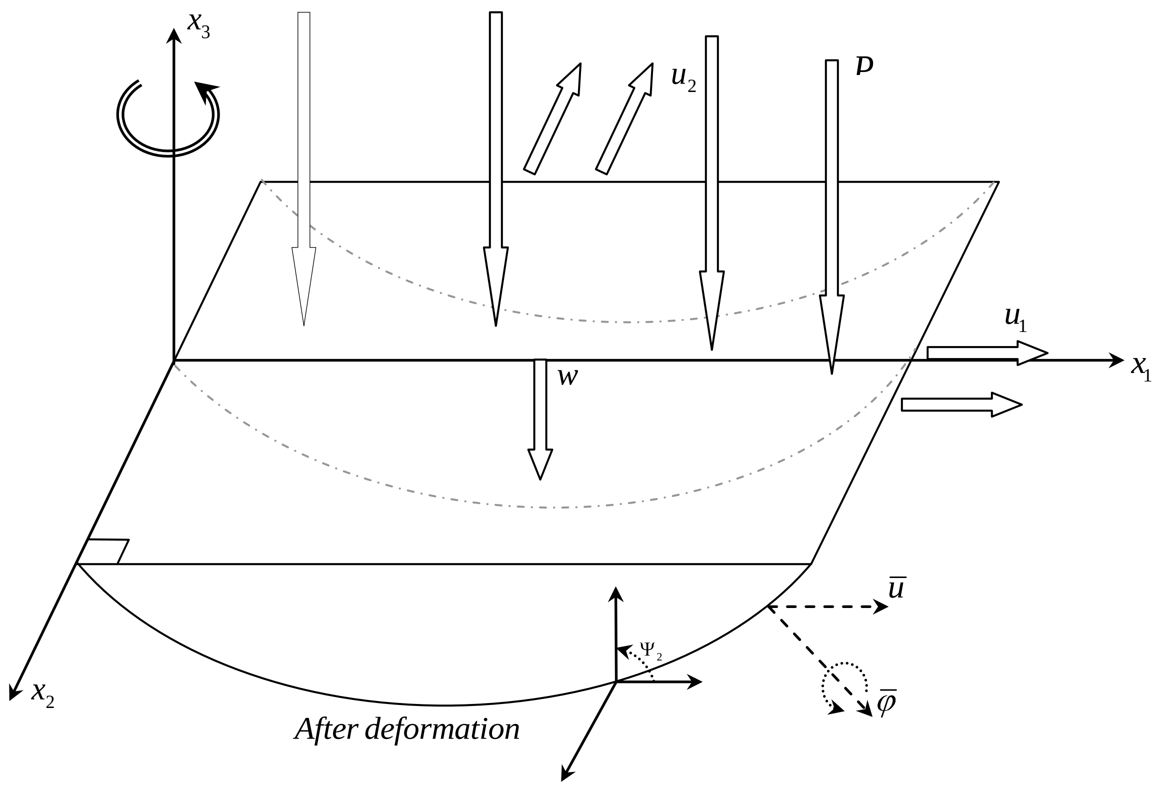

Figure 2.2 Vertical deflection of the middle plane of the plate

The density of the work over the boundary $\Gamma_\sigma \times [-0.5h, 0.5h]$ given by:

$$\mathcal{W}_2 = \frac{h}{2}\int_{-1}^{1}(\sigma_{0\alpha}\cdot u_\alpha + \mu_{0\alpha}\cdot\varphi_\alpha)n_\alpha d\varsigma_3,$$

can be presented in the form

$$\mathcal{W}_2 = \mathcal{S}_0.\mathcal{U} = \Pi_{0\alpha}\Psi_\alpha + \Pi_{03}W + M_{0\alpha}\Omega_\alpha^0 + M_{03}^*\Omega_3 + \Sigma_{0,\alpha}U_\alpha + \Upsilon_{03}\Omega_3^0,$$

where the sets $\mathcal{S}_0$ and $\mathcal{U}$ are defined as

$$\mathcal{S}_0 = [\Pi_{0\alpha}, \Pi_{03}, M_{0\alpha}, M_{03}^*, \Sigma_{0,\alpha}, \Upsilon_{03}],$$

$$\mathcal{U} = [\Psi_\alpha, W, \Omega_\alpha^0, \Omega_3, U_\alpha, \Omega_3^0],$$

and

$$\begin{aligned} M_{\alpha\beta}n_\beta &= \Pi_{0\alpha}, & R_{\alpha\beta}n_\beta &= M_{0\alpha}, \\ Q_\alpha^* n_\alpha &= \Pi_{03}, & S_\beta^* n_\beta &= M_{03}^*. \end{aligned} \quad \textbf{(2.25)}$$

$$N_{\alpha\beta}n_\beta = \Sigma_\alpha, \quad \textbf{(2.26)}$$

$$M_\alpha^* n_\alpha = \Upsilon_{03}. \quad \textbf{(2.27)}$$

, n_β is the outward unit normal vector to Γ_σ, and

$$\begin{aligned} \Pi_{0\alpha} &= \left(\frac{h}{2}\right)^2\int_{-1}^{1}\varsigma_3\sigma_{0\alpha}d\varsigma_3, & M_{0\alpha} &= \frac{h}{2}\int_{-1}^{1}\mu_{0\alpha}d\varsigma_3, \\ \Pi_{03} &= \frac{h}{2}\int_{-1}^{1}\ \sigma_{03} - \sigma_0\ d\varsigma_3, & M_{03}^* &= \frac{h}{2}\int_{-1}^{1}\ \mu_{03} - tn_3\ d\varsigma_3, \\ \Sigma_{0,\alpha} &= \frac{h}{2}\int_{-1}^{1}\sigma_{0\alpha}d\varsigma_3, & \Upsilon_{03} &= \left(\frac{h}{2}\right)^2\int_{-1}^{1}\varsigma_3\ \ \mu_{03} - \varsigma_3\nu\ d\varsigma_3. \end{aligned} \quad \textbf{(2.28)}$$

We are able to evaluate the work done at the top and bottom of the Cosserat plate by using boundary conditions **(2.1)** and **(2.3)**:

$$\int_{T\cup B}(\sigma_{03}\cdot u_3 + \mu_{03}\cdot\varphi_{03})n_3 da = \int_{P_0}(pW + \nu\Omega_3^0)da.$$

2.1.3.3 The Cosserat plate internal work density

The density of the work done by the stress and couple stress over the Cosserat strain field is defined by the following expression:

$$\mathcal{W}_3 = \frac{h}{2}\int_{-1}^{1}(\sigma\cdot\gamma+\mu\cdot\chi)d\varsigma_3. \tag{2.29}$$

Substituting the stress and couple stress assumptions **(2.5)-(2.12)** and integrating the expression **(2.29)** we obtain the following expression:

$$\mathcal{W}_3 = \mathcal{S}\cdot\varepsilon = M_{\alpha\beta}e_{\alpha\beta} + Q_\alpha\omega_\alpha + Q^*_{3\alpha}\omega^*_\alpha + R_{\alpha\beta}\tau_{\alpha\beta} + S^*_\alpha\tau_{3\alpha} + N_{\alpha\beta}\nu_{\alpha\beta} + M^*_\alpha\tau^0_{3,\alpha}, \tag{2.30}$$

where ε is the Cosserat plate strain set of the weighted averages of strain and torsion tensors

$$\varepsilon = [e_{\alpha\beta}, \omega_\beta, \omega^*_\alpha, \tau_{3\alpha}, \tau_{\alpha\beta}, \nu_{\alpha\beta}, \tau^0_{3,\alpha}].$$

Here the components of ε are:

$$\begin{gathered}
e_{\alpha\beta} = \frac{3}{h}\int_{-1}^{1}\varsigma_3\gamma_{\alpha\beta}d\varsigma_3, \qquad \omega_\alpha = \frac{3}{4}\int_{-1}^{1}\gamma_{\alpha 3}(1-\varsigma^2)d\varsigma_3,\\
\omega^*_\alpha = \frac{3}{4}\int_{-1}^{1}\gamma_{3\alpha}(1-\varsigma^2)d\varsigma_3, \qquad \tau_{3\alpha} = \frac{3}{h}\int_{-1}^{1}\varsigma_3\chi_{3\alpha}d\varsigma_3,\\
\tau^0_{\alpha\beta} = \frac{3}{4}\int_{-1}^{1}\chi_{\alpha\beta}(1-\varsigma^2)d\varsigma_3, \qquad \nu_{\alpha\beta} = \frac{1}{2}\int_{-1}^{1}\gamma_{\alpha\beta}d\varsigma_3,\\
\tau^0_{3\alpha} = \frac{1}{2}\int_{-1}^{1}\chi_{3\alpha}d\varsigma_3.
\end{gathered} \tag{2.31}$$

The components of Cosserat plate strain **(2.31)** can also be represented in terms of the components of set $\mathscr{U}$ by the following formulas:

$$\begin{gathered}
e_{\alpha\beta} = \Psi_{\beta,\alpha} + \varepsilon_{3\alpha\beta}\Omega_3, \quad \omega_\alpha = \Psi_\alpha + \varepsilon_{3\alpha\beta}\Omega^0_\beta,\\
\omega^*_\alpha = W_{,\alpha} + \varepsilon_{3\alpha\beta}\Omega^0_\beta, \quad \tau_{3\alpha} = \Omega_{3,\alpha}, \tau^0_{\alpha\beta} = \Omega^0_{\beta,\alpha},\\
\nu_{\alpha\beta} = U_{\beta,\alpha} + \varepsilon_{3\alpha\beta}\Omega^0_3,\\
\tau^0_{3,\alpha} = \Omega^0_{3,\alpha}.
\end{gathered} \tag{2.32}$$

We call the relation **(2.32)** the Cosserat plate ***strain-displacement relation***.

2.1.3.4 The HPR principle in the case of Cosserat elasticity

Steinberg proposed the **HPR principle** for the case of Cosserat Plate in the following form **[21]**. Let $\mathcal{A}$ be any set of all admissible states $s=[u,\varphi,\gamma,\chi,\sigma,\mu]$ that satisfy the ***strain-displacement relation*** **(2.32)** and torsion-rotation relations **(1.5)**. The admissible state s that minimizes (zero variation) the functional Θ defined by:

$$\Theta(s)=U_K^S-\int_{P_0}(\mathcal{S}\cdot\mathcal{E}-pW+\upsilon\Omega_3^0)da+\int_{\Gamma_0}\mathcal{S}_0\cdot(\mathcal{U}-\mathcal{U}_0)ds+\int_{\Gamma_u}\mathcal{S}_n\cdot\mathcal{U}ds\,, \tag{2.33}$$

where $s\equiv[\mathcal{U},\mathcal{E},\mathcal{S}]\in$ $\mathcal{A}$, $\mathcal{U}$ the displacement set, $\mathcal{E}$ the stress set, $\mathcal{S}$ the strain set, is equivalent to s to be a solution of the system of equilibrium equations **(1.1)- (1.2)**, constitutive relations, which satisfies the mixed boundary conditions **(1.6)- (1.7) [5]. I**n other words, the admissible state s that minimizes such functional Θ (when derivative of this functional is zero) is the solution of the plate bending (A) and twisting (B) mixed problems.

Following we show the plate bending and twisting mixed problems.

(A) PLATE BENDING PROBLEM

The Bending Equilibrium System of Equations:

$$\begin{aligned}
&M_{\alpha\beta,\alpha}-Q_\beta=0,\\
&Q^*_{\alpha,\alpha}+p=0,\\
&R_{\alpha\beta,\alpha}+\varepsilon_{3\beta\gamma}(Q^*_\gamma-Q_\gamma)=0,\\
&S^*_{\alpha,\alpha}+\varepsilon_{3\beta\gamma}M_{\beta\gamma}=0,
\end{aligned} \tag{2.34}$$

with the resultant **traction boundary conditions:**

$$\begin{aligned}
&M_{\alpha\beta}n_\beta=\Pi_{0\alpha}, \qquad R_{\alpha\beta}n_\beta=M_{0\alpha},\\
&Q^*_\alpha n_\alpha=\Pi_{03}, \qquad S^*_\alpha n_\alpha=\Upsilon_{03},
\end{aligned} \tag{2.35}$$

at the part Γ_σ and the resultant **displacement boundary conditions:**

$$\begin{aligned}
&\Psi_\alpha=\Psi_{0\alpha}, \quad W=W_0,\\
&\Omega^0_\alpha=\Omega^0_{0\alpha}, \quad \Omega_3=\Omega_{03},
\end{aligned} \tag{2.36}$$

at the part Γ_u.

The Constitutive Formulas:

$$
\begin{aligned}
e_{\alpha\alpha} &= \frac{\partial \Phi}{\partial M_{\alpha\alpha}} = \frac{12(\lambda+\mu)}{h^3\mu(3\lambda+2\mu)} M_{\alpha\alpha} - \left|\varepsilon_{\alpha\beta 3}\right| \frac{6\lambda}{h^3\mu(3\lambda+2\mu)} M_{\beta\beta} + \frac{3\lambda}{5h\mu(3\lambda+2\mu)} (Q^*_{\beta,\beta}), \\
e_{\alpha\beta} &= \frac{\partial \Phi}{\partial M_{\alpha\beta}} = \frac{3(\alpha+\mu)}{h^3\alpha\mu} M_{\alpha\beta} + \frac{3(\alpha-\mu)}{h^3\alpha\mu} M_{\beta\alpha}, \quad \alpha \neq \beta,
\end{aligned} \tag{2.37}
$$

$$
\begin{aligned}
\omega_\alpha &= \frac{\partial \Phi}{\partial Q_\alpha} = \frac{3(\alpha-\mu)}{10h\alpha\mu} Q^*_\alpha + \frac{3(\alpha+\mu)}{10h\alpha\mu} Q_\alpha, \\
\omega^*_\alpha &= \frac{\partial \Phi}{\partial Q^*_\alpha} = \frac{3(\alpha-\mu)}{10h\alpha\mu} Q_\alpha + \frac{3(\alpha+\mu)}{10h\alpha\mu} Q^*_\alpha,
\end{aligned} \tag{2.38}
$$

$$
\begin{aligned}
\tau^0_{\alpha\alpha} &= \frac{\partial \Phi}{\partial R_{\alpha\alpha}} = \frac{6(\beta+\gamma)}{5h\gamma(3\beta+2\gamma)} R_{\alpha\alpha} - \left|\varepsilon_{\alpha\beta 3}\right| \frac{3\beta}{5h\gamma(3\beta+2\gamma)} R_{\beta\beta} - \frac{\beta}{2\gamma(3\beta+2\gamma)} t, \\
\tau^0_{\alpha\beta} &= \frac{\partial \Phi}{\partial R_{\beta\alpha}} = \frac{3(\varepsilon-\gamma)}{10h\gamma\varepsilon} R_{\alpha\beta} + \frac{3(\varepsilon+\gamma)}{10h\gamma\varepsilon} R_{\beta\alpha}, \quad \alpha \neq \beta,
\end{aligned} \tag{2.39}
$$

$$
\tau_{3\alpha} = \frac{\partial \Phi}{\partial S^*_\alpha} = \frac{3(\varepsilon+\gamma)}{h^3\gamma\varepsilon} S^*_\alpha. \tag{2.40}
$$

(B) PLATE TWISTING PROBLEM

The Twisting Equilibrium System of Equations:

$$
\begin{aligned}
& N_{\alpha\beta,\alpha} = 0, \\
& M^*_{\alpha,\alpha} + \varepsilon_{3\beta\gamma} N_{\beta\gamma} + \nu = 0,
\end{aligned} \tag{2.41}
$$

with the resultant **traction boundary conditions** at Γ_σ**:**

$$
\begin{aligned}
& N_{\alpha\beta} n_\beta = \Sigma_\alpha, \\
& M^*_\alpha n_\alpha = M^*_{03},
\end{aligned} \tag{2.42}
$$

and the resultant **displacement boundary conditions** at Γ_u :

$$
U_\alpha = U_{0\alpha}, \qquad \Omega^0_3 = \Omega^0_{03}. \tag{2.43}
$$

The Constitutive Formulas:

$$\omega_{\alpha\alpha}=\frac{\partial\Phi}{\partial N_{\alpha\alpha}}=\frac{\lambda+\mu}{h\mu(3\lambda+2\mu)}N_{\alpha\alpha}-\frac{\lambda}{2h\mu(3\lambda+2\mu)}N_{\alpha'\alpha'}-\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_0,$$
$$\omega_{\alpha\beta}=\frac{\partial\Phi}{\partial N_{\alpha\beta}}=\frac{\alpha+\mu}{4h\alpha\mu}N_{\alpha\beta}+\frac{\alpha-\mu}{4h\alpha\mu}N_{\beta\alpha},\quad \alpha\neq\beta, \tag{2.44}$$

$$\tau^0_{3\alpha}=\frac{\partial\Phi}{\partial M^*_\alpha}=\frac{\varepsilon+\gamma}{4h\gamma\varepsilon}M^*_\alpha. \tag{2.45}$$

The constitutive relations can be written in the following form **[21]**:

$$M_{\alpha\alpha}=D(\Psi_{\alpha,\alpha}+\nu\Psi_{\alpha',\alpha'})+\frac{\nu h^2}{10(1-\nu)}P,$$
$$M_{\alpha'\alpha}=\frac{D}{2}\frac{(1+\nu)}{(1-N^2)}(\Psi_{\alpha',\alpha}+\Psi_{\alpha,\alpha'}+2N^2(-1)^{\alpha+1}(\Omega_3-\Psi_{\alpha',\alpha})),$$
$$R_{\alpha'\alpha}=\frac{5Gh(l_t^2-2l_b^2)}{3}\Omega^0_{\alpha',\alpha}+\frac{10Ghl_b^2}{3}\Omega^0_{\alpha,\alpha'},$$
$$R_{\alpha\alpha}=\frac{5Ghl_t^2}{3}(\Omega^0_{\alpha,\alpha}+(1-\Psi)(\Omega^0_{\alpha,\alpha}+\Omega^0_{\alpha',\alpha'}))+\frac{2Gl_t^2(1-\Psi)}{\Psi}t,$$
$$Q_\alpha=\frac{5Gh}{6(1-N^2)}(W_{,\alpha}+\Psi_\alpha-2N^2(W_{,\alpha}+(-1)^{\alpha'}\Omega^0_{\alpha'})),$$
$$Q^*_\alpha=\frac{5Gh}{6(1-N^2)}(W_{,\alpha}+\Psi_\alpha-2N^2(\Psi_\alpha+(-1)^{\alpha}\Omega^0_{\alpha'})),$$
$$S^*_\alpha=\frac{Gl_t^2(4l_b^2-2l_t^2)h^3}{12l_b^2}\Omega_{3,\alpha},$$
$$N_{\alpha\alpha}=\frac{Eh}{(1-\nu^2)}(U_{\alpha,\alpha}+\nu U_{\alpha',\alpha'})+\frac{h\nu}{1-\nu}\sigma_0,$$
$$N_{\alpha'\alpha}=\frac{Gh}{1-N^2}(U_{\alpha',\alpha}+U_{\alpha,\alpha'}-2N^2(U_{\alpha',\alpha}+(-1)^{\alpha}\Omega^0_3)), \tag{2.46}$$
$$M^*_\alpha=\frac{Gl_t^2(4l_b^2-l_t^2)h}{l_b^2}\Omega^0_{3,\alpha},$$

where α' & α are sub-index such that $\alpha'=1$ if $\alpha=2$ and $\alpha'=2$ if $\alpha=1$.

Here the following technical constants **[4]** have been used:

the Young's modulus $E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$, the Poisson's ratio $\nu=\frac{\lambda}{2(\lambda+\mu)}$, the shear modulus $G=\frac{E}{2(1+\nu)}$, the flexural rigidity of the plate $D=\frac{Eh^3}{12(1-\nu^2)}$, the characteristic length for torsion $l_t=\sqrt{\frac{\gamma}{\mu}}$, the characteristic length for bending $l_b=\frac{1}{2}\sqrt{\frac{\gamma+\varepsilon}{\mu}}$, the coupling number $N=\sqrt{\frac{\alpha}{\mu+\alpha}}$, the polar ratio $\Psi=\frac{2\gamma}{\beta+2\gamma}$.

After substitution **(2.46)** into **(2.34)** and **(2.41)** the **bending and twisting governing systems** can be obtained.[21].

We write the system corresponding for the bending case in the matrix form:

$$L(\partial_x)H=F,\quad x\in \mathrm{P}_0, \qquad \textbf{(2.47)}$$

where $L(\partial_x)=L\left(\frac{\partial}{\partial x_a}\right)$ is a matrix differential operator, H is the vector solution defined by $H^T=[\Psi_1,\Psi_2,W,\Omega_3,\Omega_1^0,\Omega_2^0]$ and F the source depending of pressure basically. L. Steinberg demonstrated the **uniqueness** of the solution for the deformation of Cosserat elastic plate which satisfies equilibrium equations, constitutive and kinematics formulas with boundary conditions (i.e. the formulas: **(2.32)** and **(2.34) - (2.46))**, if there is a solution **[21]**.

Expanding the bending equilibrium system of equations **(2.34),** we obtain the following system of ***Field Equations*** [22] and [23]:

$$I: \quad (k_1\partial_x^2 + k_2\partial_y^2 + k_3)\Psi_1 + k_{10}\partial_x\partial_y\Psi_2 + k_{11}\partial_x w + k_{12}\partial_y\Omega_3 + k_{13}\Omega_2^0 = k_{15}\partial_x P,$$

$$II: \quad k_{10}\partial_x\partial_y\Psi_1 + (k_2\partial_x^2 + k_1\partial_y^2 + k_3)\Psi_2 + k_{11}\partial_y w - k_{12}\partial_x\Omega_3 + (-k_{13})\Omega_1^0 = k_{15}\partial_y P,$$

$$III: \; -k_{11}\partial_x\Psi_1 + (-k_{11}\partial_y)\Psi_2 + (-k_3\Delta)w + (-k_{13}\partial_y)\Omega_1^0 + 0.5k_9\partial_x\Omega_2^0 = k_{17}P,$$

$$IV: \; (-k_{12}\partial_y)\Psi_1 + k_{12}\partial_x\Psi_2 + (k_5\Delta - k_6)\Omega_3 = 0,$$

$$V: \quad k_{13}\Psi_2 + k_{13}\partial_y w + (k_7\partial_x^2 + k_8\partial_y^2 - k_9)\Omega_1^0 + k_{14}\partial_x\partial_y\Omega_2^0 = \frac{-5h(1-N^2)}{6}(1-\Psi)\partial_x t,$$

$$VI: \; k_{13}\Psi_1 + (-k_{13}\partial_x w) + k_{14}\partial_x\partial_y\Omega_1^0 + (k_7\partial_y^2 + k_8\partial_x^2 - k_9)\Omega_2^0 = \frac{-5h(1-N^2)}{6}(1-\Psi)\partial_y t.$$

(2.48)

where:

$$k_1 = D(1-N^2), \;\; k_2 = \frac{D(1-\nu)}{2}, \;\; k_3 = \frac{-5Gh}{6}, \;\; k_4 = \frac{5Gh}{6},$$

$$k_5 = \frac{D(1-\nu)l_t^2(4l_b^2 - l_t^2)(1-N^2)}{2l_b^2}, \;\; k_6 = 2N^2D(1-\nu), \;\; k_7 = \frac{5h(1-N^2)Gl_t^2(2-\Psi)}{3},$$

$$k_8 = \frac{10h(1-N^2)Gl_b^2}{3}, \;\; k_9 = \frac{10hGN^2}{3}, \;\; k_{10} = \frac{D(1+\nu-2N^2)}{2},$$

$$k_{11} = \frac{5Gh(2N^2-1)}{6}, \;\; k_{12} = N^2D(1-\nu), \;\; k_{13} = \frac{5GhN^2}{3}, \; k_{14} = \frac{5hG(l_t^2(2-\Psi)-2l_b^2)(1-N^2)}{3}$$

$k_{15} = \dfrac{-h^2\nu(1-N^2)}{10(1-\nu)}$, $k_{17} = -(1-N^2)$. Note: $k_{12} = \dfrac{k_2k_9}{2k_4}$.

From **(2.35)** and **(2.36)** the correspondent boundary conditions are given in the following form:

$$T(\partial_x)H = F^*, \quad x \in \Gamma_\sigma,$$
$$H = H_0, \quad x \in \Gamma_u,$$

where $T(\partial_x)$ denote a differential operator $T(\partial_x) = T\left(\dfrac{\partial}{\partial x_\alpha}\right)$. These **B.C.** are listed below.

Boundary Conditions

For $x = 0$:

$$\Psi_2(0, y) = 0, \; w(0, y) = 0, \tag{2.49}$$

$$0 = R_{12}(0, y) = \frac{5Gh(l_t^2 - 2l_b^2)}{3}\Omega_{1,2}^0(0, y) + \frac{10Ghl_b^2}{3}\Omega_{2,1}^0(0, y)), \tag{2.50}$$

$$0 = M_{11}(0, y) = D(\Psi_{1,1}(0, y) + \nu\Psi_{2,2}(0, y)), \tag{2.51}$$

$$0 = S_1^*(0, y) = \frac{Gl_t^2(4l_b^2 - 2l_t^2)h^3}{12l_b^2}\Omega_{3,1}(0, y). \tag{2.52}$$

For $x = a$:

$$\Psi_2(a, y) = 0, \; w(a, y) = 0, \tag{2.53}$$

$$0 = R_{12}(a, y) = \frac{5Gh(l_t^2 - 2l_b^2)}{3}\Omega_{1,2}^0(a, y) + \frac{10Ghl_b^2}{3}\Omega_{2,1}^0(a, y)), \tag{2.54}$$

$$0 = M_{11}(a, y) = D(\Psi_{1,1}(a, y) + \nu\Psi_{2,2}(a, y)), \tag{2.55}$$

$$0 = S_1^*(a, y) = \frac{Gl_t^2(4l_b^2 - 2l_t^2)h^3}{12l_b^2}\Omega_{3,1}(a, y). \tag{2.56}$$

For $y = 0$:

$$\Psi_1(x, 0) = 0, \; w(x, 0) = 0, \tag{2.57}$$

$$0 = M_{22}(x, 0) = D(\Psi_{2,2}(x, 0) + \nu\Psi_{1,1}(x, 0)), \tag{2.58}$$

$$0 = R_{21}(x, 0) = \frac{5Gh(l_t^2 - 2l_b^2)}{3}\Omega_{2,1}^0(x, 0) + \frac{10Ghl_b^2}{3}\Omega_{1,2}^0(x, 0)), \tag{2.59}$$

$$0 = S_2^*(x, 0) = \frac{Gl_t^2(4l_b^2 - 2l_t^2)h^3}{12l_b^2}\Omega_{3,2}(x, 0). \tag{2.60}$$

For $y = b$:

$$\Psi_1(x, b) = 0, \; w(x, b) = 0, \tag{2.61}$$

$$0 = M_{22}(x, b) = D(\Psi_{2,2}(x, b) + \nu\Psi_{1,1}(x, b)), \tag{2.62}$$

$$0 = R_{21}(x, b) = \frac{5Gh(l_t^2 - 2l_b^2)}{3}\Omega_{2,1}^0(x, b) + \frac{10Ghl_b^2}{3}\Omega_{1,2}^0(x, b)), \tag{2.63}$$

$$0 = S_2^*(x, b) = \frac{Gl_t^2(4l_b^2 - 2l_t^2)h^3}{12l_b^2}\Omega_{3,2}(x, b). \tag{2.64}$$

We are interested in solve the ***Field Equations*** **(2.48)** with their boundary conditions (**B.C.**) **(2.49)-(2.64)** *corresponding for the bending problem*, in the section **(3.4.1)** we discuss the analytical solution of this Steinberg-Reissner model.

2.2. Reissner Plate Model

2.2.1 The HPR principle for the Plate

HPR principle for the case of elasticity is given in the following form. For any set $\mathcal{A}$ of all admissible states $s=[U,\varphi,\sigma]$ that satisfy the ***strain-displacement relation*** and torsion-rotation relations, the zero variation ($\delta\ [\Theta(s)]=0$) of the functional Θ defined by:

$$\Theta(s)=U_K-\int_{B_0}\sigma.\gamma dv+\int_{\varsigma_1}\sigma_n.(u-u_0)da+\int_{\varsigma_2}\sigma_0.u da$$

at $s\in\mathcal{A}$ is equivalent to s to be a solution of the system of equilibrium equations **(1.1)-(1.2)**, constitutive relations, which satisfies the mixed boundary conditions **(1.6)- (1.7) [5].**

2.2.2 Bending System for Reissner Model

Using the **HPR** Variational Principle for the classic plate it is easy to obtain the Reissner Plate Theory, which can be written as the following system:

$$D\Psi_{1,11}+\frac{D(1-\nu)}{2}\Psi_{1,22}+\frac{D(1+\nu)}{2}\Psi_{2,12}-\frac{5hE}{12(1+\nu)}w_{,1}-\frac{5hE}{12(1+\nu)}\Psi_1=-\frac{h^2\nu}{10(1-\nu)}P_{,1}, \quad \textbf{(2.65)}$$

$$D\Psi_{2,22}+\frac{D(1-\nu)}{2}\Psi_{2,11}+\frac{D(1+\nu)}{2}\Psi_{1,12}-\frac{5hE}{12(1+\nu)}w_{,2}-\frac{5hE}{12(1+\nu)}\Psi_2=-\frac{h^2\nu}{10(1-\nu)}P_{,2}, \quad \textbf{(2.66)}$$

$$\frac{5hE}{12(1+\nu)}\Delta w+\frac{5hE}{12(1+\nu)}(\Psi_{1,1}+\Psi_{2,2})=-P. \quad \textbf{(2.67)}$$

2.3 Eringen Plate Theory

A. Cemal Eringen **[3]** employs equations very similar to equations appearing in Reissner Theory to develop a theory of micro- polar plates.

Eringen uses the following physical quantities:

u : Displacement vector
t_α : Stress vector,
ρ : Mass density,
m_α : Couple stress vector,
φ : Micro- rotation vector,
$t_{\alpha\beta}$: Stress tensor,
$m_{\alpha\beta}$: Couple stress tensor,
$e_{\alpha\beta}$: Strain tensor,
$\varepsilon_{\alpha\beta}$: Micro polar strain tensor,

The stress tensor $t_{\alpha\beta}$ and the couple stress $m_{\alpha\beta}$ are defined through

$$t_\alpha = t_{\alpha\beta} e_\beta, \;\; m_\alpha = m_{\alpha\beta} e_\beta, \text{ where } \{e_\gamma\}_{\gamma=1}^{3} \text{ is the canonical basis of } R^3, \text{ and } \alpha, \beta \in \{1,2,3\}.$$

In the linear theory of micro- polar elasticity the equations of motion **(1.1)** and **(1.2)** are supplemented by two sets of constitutive equations, one for the stress and one for the couple stress **[3]**:

$$t_{\alpha\beta} = \lambda^E e_{\gamma\gamma} \delta_{\alpha\beta} + 2(\mu^E + \kappa^E) e_{\alpha\beta} - \kappa^E \varepsilon_{\alpha\beta}, \tag{2.68}$$

$$m_{\alpha\beta} = \alpha^E \varphi_{\gamma,\gamma} \delta_{\alpha\beta} + \beta^E \varphi_{\alpha,\beta} + \gamma^E \varphi_{\beta,\alpha}, \tag{2.69}$$

where coefficients $\lambda^E, \mu^E, \kappa^E, \alpha^E, \beta^E$ and γ^E are elastic constants appropriate to the theory and the strain and micro- polar strain tensors are respectively given by **[3]:**

$$e_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \tag{2.70}$$

$$\varepsilon_{\alpha\beta} = u_{\alpha,\beta} + \varepsilon_{\alpha\beta\gamma} \varphi_\gamma \tag{2.71}$$

2.3.1 Equations of Balance for Micro- polar Plates

A plate theory is constructed on the fundamental assumptions that **[3]**:

(a) The plate thickness $2h_E = h^P = h$ is small as compared to any characteristic length in the median plane and

(b) The stress and displacement fields do not vary rapidly across the thickness.

Consequently we may use the average and the first moments of various quantities over the thickness in integrating the field equations with respect to x_3 from $x_3 = -h^E$ to $x_3 = h^E$.

We have:

$$\bar{t}_{kl,k} = 0, \qquad \textbf{(2.72)}$$

$$\bar{t}_{k3,k} + \frac{P}{2h^E} = 0, \qquad \textbf{(2.73)}$$

$$\bar{m}_{kl,k} + \varepsilon_{kl}(\bar{t}_{3k} - \bar{t}_{k3}) = 0, \qquad \textbf{(2.74)}$$

$$\bar{m}_{k3,k} + \varepsilon_{kl}\bar{t}_{kl} = 0. \qquad \textbf{(2.75)}$$

where :

$$\bar{t}_k = \frac{1}{2h^E}\int_{-h^E}^{h^E} t_k dx_3, \quad \bar{m}_k = \frac{1}{2h^E}\int_{-h^E}^{h^E} m_k dx_3,$$

and

$$\begin{aligned}
\tau &= t_3(x_1, x_2, h) - t_3(x_1, x_2, -h) = \tau_k e_k + pe_3, \\
\mu &= m_3(x_1, x_2, h) - m_3(x_1, x_2, -h) = \mu_k e_k + me_3. \\
t_k &= t_{kl}e_l + t_{k3}e_3, \qquad t_3 = t_{3l}e_l + t_{33}e_3, \\
m_k &= m_k e_l + m_{k3}e_3, \qquad m_3 = m_{3l}e_l + m_{33}e_3.
\end{aligned}$$

Equations **(2.72)** to **(2.75)** must be supplemented with the equations of stress couples which result from the vector multiplication of balance of momentum of micro-polar elasticity by $x_3 e_3$ and integration over x_3 from $x_3 = -h^E$ to $x_3 = h^E$.

Hence

$$M_{kl,k} + 2h^E\bar{t}_{3l} = 0, \qquad \textbf{(2.76)}$$

where M_{kl} is the couples stress defined by

$$M_{kl} = \int_{-h^E}^{h^E} t_{kl}x_3 dx_3$$

Here $I = \frac{2}{3}(h_E)^3$ is the area moment of a normal cross section of the plate, having thickness $2h_E$ and unit length, with respect to the median line.

The union of equations **(2.72)** to **(2.75)** and **(2.76)** constitute the ***equations of balance for micro- polar plates.***

2.3.2 Displacements, Rotations, Strains, Constitutive Equations

Basic assumptions of the plate theory allow a power series representation in x_3 for the displacement vector u and the micro-rotation vector φ**,** the following equations includes both the stretching of the median plane and the flexure of the plate **[3]**:

$$u = [\bar{u}_k(x_1, x_2) + x_3 \Psi_k(x_1, x_2)] e_k + w(x_1, x_2) e_3, \qquad \textbf{(2.77)}$$

$$\varphi = \Omega_k^0(x_1, x_2) e_k + \varphi(x_1, x_2) e_3, \qquad \textbf{(2.78)}$$

where $\bar{u}_k$ is the two-dimensional displacement field in the median plane, Ψ_k is the angular rotation field defined by $I\Psi_l = \int_{-h^E}^{h^E} u_l x_3 dx_3$, and w is the transverse deflection of the median plane of the plate. The micro-rotation φ is decomposed by $\varphi = \Omega_k^0 e_k + \varphi e_3$ into a plane micro-rotation vector Ω_k^0 and one in the x_3-direction. The strain and micro-strain tensors are calculated by using **(2.77)** and **(2.78)** in **(2.70)** and **(2.71)** respectively **[3]**

$$\begin{gathered} e_{kl} = \bar{e}_{kl} + x_3 \tilde{e}_{kl}, \quad 2e_{k3} = 2e_{3k} = \Psi_k + w_{,k}, \quad e_{33} = 0, \\ 2\bar{e}_{kl} = \bar{u}_{k,l} + \bar{u}_{l,k}, \quad 2\tilde{e}_{kl} = \Psi_{k,l} + \Psi_{l,k}, \end{gathered} \qquad \textbf{(2.79)}$$

$$\begin{gathered} \varepsilon_{kl} = \bar{u}_{k,l} + x_3 \Psi_{k,l} + \varepsilon_{kl3}\varphi, \quad \varepsilon_{k3} = \Psi_k - \varepsilon_{kl3}\Omega_l^0, \\ \varepsilon_{3k} = w_{,k} + \varepsilon_{kl3}\Omega_l^0, \quad \varepsilon_{33} = 0. \end{gathered} \qquad \textbf{(2.80)}$$

Substituting **(2.79)** and **(2.80)** into **(2.68),** we obtain the stress constitutive equations

$$\begin{gathered} t_{kl} = \bar{t}_{kl} + \frac{x_3}{I} M_{kl}, \quad t_{k3} = \bar{t}_{k3} = (\mu^{\mathrm{E}} + \kappa^{\mathrm{E}})(\Psi_k + w_{,k}) - \kappa^{\mathrm{E}}(\Psi_k - \varepsilon_{kl}\Omega_l^0), \\ t_{3k} = \bar{t}_{3k} = (\mu^{\mathrm{E}} + \kappa^{\mathrm{E}})(\Psi_k + w_{,k}) - \kappa^{\mathrm{E}}(w_{,k} - \varepsilon_{kl}\Omega_l^0), \quad t_{33} = \lambda^{\mathrm{E}} \bar{u}_{r,r}\delta_{kl} + \lambda^{\mathrm{E}} x_3 \Psi_{r,r}\delta_{kl}, \end{gathered} \qquad \textbf{(2.81)}$$

where the plane stress and the stress couples are respectively given by

$$\bar{t}_{kl} = \lambda^E \bar{u}_{r,r}\delta_{kl} + (\mu^E + \kappa^E)(\bar{u}_{k,l} + \bar{u}_{l,k}) - \kappa^E(\bar{u}_{k,l} + \mathcal{E}_{kl}\varphi),$$
$$I\, M_{kl} = \lambda^E \Psi_{rr}\delta_{kl} + (\mu^E + \kappa^E)(\Psi_{k,l} + \Psi_{l,k}) - \kappa^E \Psi_{k,l}. \quad \textbf{(2.82)}$$

Similarly for the couple stress m_{kl}, through **(2.69)** and **(2.78)** we have

$$m_{kl} = \alpha^E \Omega^0_{r,r}\delta_{kl} + \beta^E \Omega^0_{k,l} + \gamma^E \Omega^0_{l,k}, \quad m_{k3} = \gamma^E \varphi_{,k}, \quad m_{3k} = \beta^E \varphi_{,k}, \quad \bar{m}_{33} = 0. \quad \textbf{(2.83)}$$

We consider the stress constitutive equations of the plate theory (here $t_{33} = 0$), where the plane stress $\bar{t}_{kl}$ and the stress couples M_{kl} are respectively given by **[3]**:

$$\bar{t}_{kl} = \frac{E}{1-\nu^2}[\nu \bar{u}_{r,r}\delta_{kl} + \frac{1-\nu}{2}(\bar{u}_{k,l} + \bar{u}_{l,k})] - \frac{\kappa^E}{2}(\bar{u}_{k,l} - \bar{u}_{l,k}),$$
$$M_{kl} = \frac{EI}{1-\nu^2}[\nu \Psi_{r,r}\delta_{kl} + \frac{1-\nu}{2}(\Psi_{k,l} + \Psi_{l,k})] - \frac{\kappa^E I}{2}(\Psi_{k,l} - \Psi_{l,k}). \quad \textbf{(2.84)}$$

The field equations of stretching and bending of plates may thus be constructed by using set of Equations **(2.84) [3].**

2.3.3 Field Equations and Boundary Conditions

The partial differential equations of the displacement and micro- rotation fields are obtained by substituting the constitutive equations **(2.81)** and **(2.83)** into the equations of balance **(2.72)** to **(2.75)** and **(2.76) [3]**. The system of equations is grouped into two sets: one representing the symmetrical stress distribution about the median plane x_3, the other the antisymmetrical one. These two groups respectively represent the two-dimensional elastic state (or extensional motion) of the plate and the *bending.*

2.3.4 Eringen Theory I ($e_{33} = 0$)

In this case we consider $e_{33} = 0$**,** and **(2.82)** is considered. Thus for two-dimensional problem of micro- elasticity (or the extensional motions of plates) we have **[3]**:

$$(\lambda^E + \mu^E)u_{k,lk} + (\mu^E + \kappa^E)u_{l,kk} - \kappa^E \mathcal{E}_{kl}\varphi_{,k} = 0, \quad \textbf{(2.85)}$$

$$\gamma^E \varphi_{,kk} - 2\kappa^E \varphi - \kappa^E \mathcal{E}_{kl}\bar{u}_{k,l} = 0, \quad \textbf{(2.86)}$$

where: l:1,2 and k:1,2.

For the *bending*, we have **[3]**:

$$
\begin{gathered}
I(\lambda^E+\mu^E)\Psi_{k,lk}+I(\mu^E+\kappa^E)\Psi_{l,kk} \\
-2h^E(\mu^E+\kappa^E)(\Psi_l+w_{,l})+2h^E\kappa^E(w_{,l}+\varepsilon_{lk}\Omega_k^0)=0,
\end{gathered} \qquad \textbf{(2.87)}
$$

$$
\mu^E\Psi_{k,k}+(\mu^E+\kappa^E)w_{,kk}+\kappa^E\varepsilon_{kl}\Omega_{l,k}^0=0, \qquad \textbf{(2.88)}
$$

$$
(\alpha^E+\beta^E)\Omega_{k,lk}^0+\gamma^E\Omega_{l,kk}^0+\kappa^E\varepsilon_{kl}(\Psi_k-w_{,k})-2\kappa^E\Omega_l^0=0, \qquad \textbf{(2.89)}
$$

where: l:1,2 and k:1,2.

Equations **(2.85)** to **(2.86)** constitute three partial differential equations for the plane displacement fields $\bar{u}_1,\ \bar{u}_2$ and the plane micro- rotation field φ. Equations **(2.87)** to **(2.89)** of the flexure and five equations for the unknowns $\Psi_k,\ \Omega_k^0$ and w. Since the boundary conditions are similarly uncoupled the problem of the plane micro- polar elasticity can be treated separately from that of the *bending*.

2.3.4.1 Behaviour of Eringen Model I ($e_{33}=0$) for the Elastic (Classic) Materials

From the reduction (the restriction is considering $\alpha=\beta=\gamma=\varepsilon=0$ into the equations for the *bending* **(2.87)** to **(2.89)**) of Eringen Model I ($e_{33}=0$) and using the equivalences **(3.83)** we obtain the following equations:

$$
\begin{gathered}
\frac{(h^p)^2}{12}(\lambda+\mu)\Psi_{k,lk}+\frac{(h^p)^2}{12}\mu\Psi_{l,kk}-\mu(\Psi_l+w_{,l})=0, \\
\mu\Psi_{k,k}+\mu w_{,kk}+\frac{P}{h^p}=0, \\
\text{for} \quad \text{l:1,2} \ \ \& \ \ \text{k:1,2},
\end{gathered}
$$

then expanding these three equations:

$$
\frac{(h^p)^2}{12}(\lambda+\mu)(\Psi_{1,11}+\Psi_{2,12})+\frac{(h^p)^2}{12}\mu(\Psi_{1,11}+\Psi_{1,22})-\mu(\Psi_1+w_{,1})=0, \qquad \textbf{(2.90)}
$$

$$
\frac{(h^p)^2}{12}(\lambda+\mu)(\Psi_{1,21}+\Psi_{2,22})+\frac{(h^p)^2}{12}\mu(\Psi_{2,11}+\Psi_{2,22})-\mu(\Psi_2+w_{,2})=0, \qquad \textbf{(2.91)}
$$

$$
\mu(\Psi_{1,1}+\Psi_{2,2})+\mu(w_{,11}+w_{,22})+\frac{P}{h^p}=0,. \qquad \textbf{(2.92)}
$$

2.3.5 Eringen Theory II $(t_{33} = 0)$

In this case **(2.84)** is considered and the corresponding equations of the extensional motions of plates are **[3]**:

$$\frac{1}{2}[\frac{E}{1-\nu} - \kappa^E]\bar{u}_{k,lk} + \frac{1}{2}[\frac{E}{1-\nu} - \kappa^E]\bar{u}_{l,kk} - \kappa^E \mathcal{E}_{kl}\varphi_{,k} = 0, \quad \textbf{(2.93)}$$

$$\gamma^E \varphi_{,kk} - 2\kappa^E \varphi - \kappa^E \mathcal{E}_{kl}\bar{u}_{k,l} = 0, \quad \textbf{(2.94)}$$

where: l:1,2 and k:1,2.

And those of *bending* are **[3]**:

$$\frac{I}{2}[\frac{E}{1-\nu} - \kappa^E]\Psi_{k,lk} + \frac{I}{2}[\frac{E}{1+\nu} + \kappa^E]\Psi_{l,kk} - 2h^E[G - \frac{\kappa^E}{2}]w_{,l} - 2h^E[G + \frac{\kappa^E}{2}]\Psi_l + 2\kappa^E h^E \mathcal{E}_{lk}\Omega_k^0 = 0, \quad \textbf{(2.95)}$$

$$[G - \frac{\kappa^E}{2}]\Psi_{k,k} + [G + \frac{\kappa^E}{2}]w_{,kk} + \kappa^E \mathcal{E}_{kl}\Omega_{l,k}^0 + \frac{P}{2h^E} = 0, \quad \textbf{(2.96)}$$

$$(\alpha^E + \beta^E)\Omega_{k,lk}^0 + \gamma^E \Omega_{l,kk}^0 + \kappa^E \mathcal{E}_{kl}(\Psi_k - w_{,k}) - 2\kappa^E \Omega_l^0 = 0, \quad \textbf{(2.97)}$$

where: l:1,2 and k:1,2.

We are interested in the boundary conditions for *bending*:

$$\begin{aligned} &Mt_{kl}n_k = \bar{M}_l, \quad \bar{t}_{k3}n_k = \bar{t}_3, \quad m_{kl}n_k = \bar{m}_l \quad \text{on } \mathrm{C_L}, \\ &\Psi_k = \bar{\Psi}_{0k}, \quad w = \bar{w}_0, \quad \Omega_k^0 = \bar{\Omega}_{0k}^0, \quad \text{on } \mathrm{C\text{-}C_L} \end{aligned} \quad \textbf{(2.98)}$$

The quantities $\bar{t}_l, \bar{m}_3, \bar{u}_{0k}, \varphi_0, \bar{M}_l, \bar{t}_3, \bar{m}_l, \bar{\Psi}_{0k}, \bar{w}_{0k}$, and $\bar{\Omega}_{0k}^0$ are prescribed functions along indicated portions $\mathrm{C_L}$ and $\mathrm{C\text{-}C_L}$ of the boundary C of the median plane.

2.3.5.1 Behaviour of Eringen Model II $(t_{33} = 0)$ for the Elastic (Classic) Materials

To obtain the model corresponding for the Elastic (Classic) Materials, it is considering the following reduction of Eringen Model II:

The restriction considers that $(\alpha, \beta, \gamma) = (0,0,0)$ in Eringen Model II. Substituting the restriction $\alpha = \beta = \gamma = 0$ into the equations of *bending* **(2.95)** to **(2.98)** corresponding to the Eringen Model II $(t_{33} = 0)$ we obtain the following equations:

$$\frac{(h^p)^3}{12} \frac{\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)} (\Psi_{1,11} + \Psi_{2,12}) + \frac{(h^p)^3}{12} \mu(\Psi_{1,11} + \Psi_{1,22}) - h^p \mu w_{,1} - h^p \mu \Psi_1 = 0, \quad \textbf{(2.99)}$$

$$\frac{(h^p)^3}{12} \frac{\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)} (\Psi_{1,21} + \Psi_{2,22}) + \frac{(h^p)^3}{12} \mu(\Psi_{2,11} + \Psi_{2,22}) - h^p \mu w_{,2} - h^p \mu \Psi_2 = 0, \quad \textbf{(2.100)}$$

$$\mu(\Psi_{1,1} + \Psi_{2,2}) + \mu(w_{,11} + w_{,22}) + \frac{P}{h^p} = 0, \quad \textbf{(2.101)}$$

CHAPTER 3

THE SEPARATION OF VARIABLES METHOD FOR PDE SYSTEMS APPEARING IN THE PLATE THEORIES

3.1 Analytical Solution Based on Separation of Variables

According to the method of separation of variables we assume solutions to be in the following form:

$$f^j(x_1,...,x_n) = f_1^j(x_1)...f_n^j(x_n),$$

where $f_i^j(x_i)$ is a function of one variable x_i, $i = 1,...,n$, $j = 1,...,m$. *Daniel Bernoulli* developed this technique in the 1700's **[19].**

We obtain a system of Sturm-Liouville problems for $f_i^j(x_i)$ with specific boundary conditions with respect to each variable x_i.

Finally, using the superposition principle, we construct the solution as an expansion, in the following form:

$$f^j = \sum_{k=1}^{\infty}(f_1^j)_k...(f_n^j)_k, \quad \forall j \in N,$$

where $(f_i^j)_k$ are Eigen- functions relative to the corresponding Sturm-Liouville boundary value problem for f_i^j.

3.2 Application of the Separation of Variables Method for PDE Systems of the Plates Theories

3.2.1 Analytical Solution of Steinberg-Reissner Model

We consider a rectangular domain. Due to the geometry of the domain, it is feasible to use the **method of separation of variables**. We replace each unknown function of "x" and "y" by a product of two functions, each depending of "x" or "y".

So we assume that:

$\Psi_1 = \Psi_{1x}\Psi_{1y}$, $\Psi_2 = \Psi_{2x}\Psi_{2y}$, $w = w_x w_y$, $\Omega_3 = \Omega_{3x}\Omega_{3y}$, $\Omega_1 = \Omega_{1x}\Omega_{1y}$, $\Omega_2 = \Omega_{2x}\Omega_{2y}$. Here these functions are represented by the products of functions of "x" and "y"variables.

After substitution to the partial differential equation (PDE) system and boundary conditions we obtain systems of ordinary differential equations for functions with respect of the variable "x" and "y", and the corresponding boundary conditions for them.

Here we substitute each product:

$\Psi_1 = \Psi_{1x}\Psi_{1y}$, $\Psi_2 = \Psi_{2x}\Psi_{2y}$, $w = w_x w_y$, $\Omega_3 = \Omega_{3x}\Omega_{3y}$, $\Omega_1 = \Omega_{1x}\Omega_{1y}$, $\Omega_2 = \Omega_{2x}\Omega_{2y}$

into the ***Field Equations*** **(2.48)**. Now, dividing the equation (IV) of the ***Field Equations*** **(2.48)** by the product $\Omega_{3x}\Omega_{3y}$, we obtain the following:

$$-k_{12}\Psi_{1x}\Psi_{1y}^{'} + k_{12}\Psi_{2x}^{'}\Psi_{2y} + k_5(\Omega_{3x}^{''}\Omega_{3y} + \Omega_{3x}\Omega_{3y}^{''}) - (k_6 + \lambda^{\Omega_3})\Omega_{3x}\Omega_{3y} = 0$$

$$-k_{12}\frac{\Psi_{1x}}{\Omega_{3x}}\frac{\Psi_{1y}^{'}}{\Omega_{3y}} + k_{12}\frac{\Psi_{2x}^{'}}{\Omega_{3x}}\frac{\Psi_{2y}}{\Omega_{3y}} + k_5(\frac{\Omega_{3x}^{''}}{\Omega_{3x}} + \frac{\Omega_{3y}^{''}}{\Omega_{3y}}) - (k_6 + \lambda^{\Omega_3}) = 0$$

Taking the derivative with respect to $"x"$ and $"y"$, we obtain:

$$-k_{12}\left(\frac{\Psi_{1x}}{\Omega_{3x}}\right)^{'}\left(\frac{\Psi_{1y}^{'}}{\Omega_{3y}}\right)^{'} + k_{12}\left(\frac{\Psi_{2x}^{'}}{\Omega_{3x}}\right)^{'}\left(\frac{\Psi_{2y}}{\Omega_{3y}}\right)^{'} = 0$$

$$\left(\frac{\Psi_{1x}}{\Omega_{3x}}\right)^{'}\left(\frac{\Psi_{1y}^{'}}{\Omega_{3y}}\right)^{'} = \left(\frac{\Psi_{2x}^{'}}{\Omega_{3x}}\right)^{'}\left(\frac{\Psi_{2y}}{\Omega_{3y}}\right)^{'}$$

$$\frac{\left(\frac{\Psi_{1x}}{\Omega_{3x}}\right)^{'}}{\left(\frac{\Psi_{2x}^{'}}{\Omega_{3x}}\right)^{'}} = \frac{\left(\frac{\Psi_{2y}}{\Omega_{3y}}\right)^{'}}{\left(\frac{\Psi_{1y}^{'}}{\Omega_{3y}}\right)^{'}} = \text{ constant}$$

$$\left(\frac{\Psi_{1x}}{\Omega_{3x}}\right)^{'} \cong \left(\frac{\Psi_{2x}^{'}}{\Omega_{3x}}\right)^{'} \quad \& \quad \left(\frac{\Psi_{1y}^{'}}{\Omega_{3y}}\right)^{'} \cong \left(\frac{\Psi_{2y}}{\Omega_{3y}}\right)^{'}$$

(<u>Where</u>: $f \cong g$ means that the functions f and g are such that $af = bg$, for some constant $a, b \in R$.)

By integrating both sides of each equation we obtain the following that*:*

$$\Psi_{1x} \in Span(\Psi'_{2x}, \Omega_{3x}),\ \ \Psi_{2y} \in Span(\Psi'_{1y}, \Omega_{3y})\,, \qquad \textbf{(3.3)}$$

where we used the following definition for the span**[27]**:

Span definition Let V be a vector space over a field K, then the span of a set S denoted by $Span(S)$ is defined to be the intersection W of all subspaces of V which contain S.

Now dividing (IV) by $\Psi_{1x}\Psi_{2y}$:

$$-k_{12}\frac{\Psi'_{1y}}{\Psi_{2y}} + k_{12}\frac{\Psi'_{2x}}{\Psi_{1x}} + k_5\left(\frac{\Omega''_{3x}}{\Psi_{1x}}\frac{\Omega_{3y}}{\Psi_{2y}} + \frac{\Omega_{3x}}{\Psi_{1x}}\frac{\Omega''_{3y}}{\Psi_{2y}}\right) - (k_6 + \lambda^{\Omega_3})\frac{\Omega_{3x}}{\Psi_{1x}}\frac{\Omega_{3y}}{\Psi_{2y}} = 0,$$

Taking the derivatives with respect to $"x"$ and $"y"$ we obtain:

$$k_5\left(\left(\frac{\Omega''_{3x}}{\Psi_{1x}}\right)'\left(\frac{\Omega_{3y}}{\Psi_{2y}}\right)' + \left(\frac{\Omega_{3x}}{\Psi_{1x}}\right)'\left(\frac{\Omega''_{3y}}{\Psi_{2y}}\right)'\right) - (k_6 + \lambda^{\Omega_3})\left(\frac{\Omega_{3x}}{\Psi_{1x}}\right)'\left(\frac{\Omega_{3y}}{\Psi_{2y}}\right)' = 0$$

Dividing by $\left(\frac{\Omega_{3x}}{\Psi_{1x}}\right)'\left(\frac{\Omega_{3y}}{\Psi_{2y}}\right)'$, we obtain:

$$k_5\left(\frac{\left(\frac{\Omega''_{3x}}{\Psi_{1x}}\right)'}{\left(\frac{\Omega_{3x}}{\Psi_{1x}}\right)'} + \frac{\left(\frac{\Omega''_{3y}}{\Psi_{2y}}\right)'}{\left(\frac{\Omega_{3y}}{\Psi_{2y}}\right)'}\right) - (k_6 + \lambda^{\Omega_3}) = 0$$

$$\frac{\left(\frac{\Omega''_{3x}}{\Psi_{1x}}\right)'}{\left(\frac{\Omega_{3x}}{\Psi_{1x}}\right)'} = \frac{k_6 + \lambda^{\Omega_3}}{k_5} - \frac{\left(\frac{\Omega''_{3y}}{\Psi_{2y}}\right)'}{\left(\frac{\Omega_{3y}}{\Psi_{2y}}\right)'} = \text{ constant}$$

$$\left(\frac{\Omega''_{3x}}{\Psi_{1x}}\right)' \cong \left(\frac{\Omega_{3x}}{\Psi_{1x}}\right)' \qquad \& \quad \left(\frac{\Omega''_{3y}}{\Psi_{2y}}\right)' \cong \left(\frac{\Omega_{3y}}{\Psi_{2y}}\right)'$$

, therefore $$\Psi_{1x} \in Span(\Omega_{3x}, \Omega''_{3x}),\ \ \Psi_{2y} \in Span(\Omega_{3y}, \Omega''_{3y}). \qquad \textbf{(3.4)}$$

Dividing (IV) by $\Psi'_{1y}\Psi'_{2x}$:

$$-k_{12}\frac{\Psi_{1x}}{\Psi'_{2x}}+k_{12}\frac{\Psi_{2y}}{\Psi'_{1y}}+k_5\left(\frac{\Omega''_{3x}}{\Psi'_{2x}}\frac{\Omega_{3y}}{\Psi'_{1y}}+\frac{\Omega_{3x}}{\Psi'_{2x}}\frac{\Omega''_{3y}}{\Psi'_{1y}}\right)-(k_6+\lambda^{\Omega_3})\frac{\Omega_{3x}}{\Psi'_{2x}}\frac{\Omega_{3y}}{\Psi'_{1y}}=0$$

Taking the derivatives with respect to $"x"$ and $"y"$ we obtain:

$$k_5\left(\left(\frac{\Omega''_{3x}}{\Psi'_{2x}}\right)'\left(\frac{\Omega_{3y}}{\Psi'_{1y}}\right)'+\left(\frac{\Omega_{3x}}{\Psi'_{2x}}\right)'\left(\frac{\Omega''_{3y}}{\Psi'_{1y}}\right)'\right)-(k_6+\lambda^{\Omega_3})\left(\frac{\Omega_{3x}}{\Psi'_{2x}}\right)'\left(\frac{\Omega_{3y}}{\Psi'_{1y}}\right)'=0$$

Then

$$\left(\frac{\Omega''_{3x}}{\Psi''_{2x}}\right)\cong\left(\frac{\Omega_{3x}}{\Psi'_{2x}}\right)', \qquad \left(\frac{\Omega''_{3y}}{\Psi_{1y}}\right)'\cong\left(\frac{\Omega_{3y}}{\Psi'_{1y}}\right)'$$

and $\Psi'_{2x}\in Span(\Omega_{3x},\Omega''_{3x})$, $\quad\Psi'_{1y}\in Span(\Omega_{3y},\Omega''_{3y})$ **(3.5)**

More precisely:

$$\frac{\left(\frac{\Omega''_{3x}}{\Psi'_{2x}}\right)'}{\left(\frac{\Omega_{3x}}{\Psi'_{2x}}\right)'}=\frac{k_6+\lambda^{\Omega_3}}{k_5}-\frac{\left(\frac{\Omega''_{3y}}{\Psi'_{1y}}\right)'}{\left(\frac{\Omega_{3y}}{\Psi'_{1y}}\right)'}=\text{ constant.}$$

From **(3.3)**, **(3.4)** and **(3.5),** we obtain:

$$\Psi_{1x}=a_1\Psi'_{2x}+b_1\Omega_{3x}, \tag{3.6}$$
$$\Psi_{2y}=\tilde{a}_1\Psi'_{1y}+\tilde{b}_1\Omega_{3y}, \tag{3.7}$$
$$\Psi_{1x}=a_2\Omega_{3x}+b_2\Omega''_{3x}, \tag{3.8}$$
$$\Psi_{2y}=\tilde{a}_2\Omega_{3y}+\tilde{b}_2\Omega''_{3y}, \tag{3.9}$$
$$\Psi'_{2x}=a_3\Omega_{3x}+b_3\Omega''_{3x}, \tag{3.10}$$
$$\Psi'_{1y}=\tilde{a}_3\Omega_{3y}+\tilde{b}_3\Omega''_{3y}. \tag{3.11}$$

for some real constants $a_1,a_2,a_3,b_1,b_2,b_3,\tilde{a}_1,\tilde{a}_2\tilde{a}_3,\tilde{b}_1,\tilde{b}_2,\tilde{b}_3$.

Substituting **(3.8)** and **(3.10)** into **(3.6)**:

$a_2\Omega_{3x} + b_2\Omega''_{3x} = a_3\Omega_{3x} + b_3\Omega''_{3x} + b_1\Omega_{3x}$, for some real constants a_2, a_3, b_1, b_2, b_3.

Then:

$$a_2\Omega_{3x} = b_2\Omega''_{3x}, \text{ for some real constants } a_2 \ \& \ b_2.$$

$$\frac{\Omega''_{3x}}{\Omega_{3x}} = \text{constant.} \tag{3.12}$$

Replacing **(3.9)** and **(3.11)** into **(3.7):**

$\tilde{a}_2\Omega_{3y} + \tilde{b}_2\Omega''_{3y} = \tilde{a}_3\Omega_{3y} + \tilde{b}_3\Omega''_{3y} + \tilde{b}_1\Omega_{3y}$, for some real constants $\tilde{a}_2, \tilde{a}_3, \tilde{b}_1, \tilde{b}_2 \ \& \ \tilde{b}_3$.

$$\frac{\Omega''_{3y}}{\Omega_{3y}} = \text{constant.} \tag{3.13}$$

Since **(3.8)** and **(3.12):**

$$\Psi_{1x} \cong \Omega_{3x} \tag{3.14}$$

$$\Psi''_{1x} \cong \Omega''_{3x} \cong \Omega_{3x} \cong \Psi_{1x}$$

$$\Psi''_{1x} \cong \Psi_{1x} \tag{3.15}$$

Since **(3.20), (3.10)** and **(3.14):**

$$\Psi'_{2x} \cong \Omega_{3x} \cong \Psi_{1x}$$

$$\Psi'_{2x} \cong \Psi_{1x} \tag{3.16}$$

Since **(3.9)** and **(3.13)**:

$$\Psi_{2y} \cong \Omega_{3y} \tag{3.17}$$

$$\Psi''_{2y} \cong \Omega''_{3y} \cong \Omega_{3y} \cong \Psi_{2y}$$

$$\Psi''_{2y} \cong \Psi_{2y} \tag{3.18}$$

Since **(3.11)** and **(3.13):**

$$\Psi'_{1y} \cong \Omega_{3y} \cong \Psi_{2y}$$

$$\Psi'_{1y} \cong \Psi_{2y} \tag{3.19}$$

Dividing (VI) by $\Omega_{2x}\Omega_{2y}$:

$$-k_{13}\Psi_{1x}\Psi_{1y}-k_{13}w_x^{'}w_y+k_{14}\Omega_{1x}^{'}\Omega_{1y}^{'}+k_7\Omega_{2x}\Omega_{2y}^{''}+k_8\Omega_{2x}^{''}\Omega_{2y}-(k_9+\lambda^{\Omega_2})\Omega_{2x}\Omega_{2y}=0$$

$$k_{13}\frac{\Psi_{1x}}{\Omega_{2x}}\frac{\Psi_{1y}}{\Omega_{2y}}-k_{13}\frac{w_x^{'}}{\Omega_{2x}}\frac{w_y}{\Omega_{2y}}+k_{14}\frac{\Omega_{1x}^{'}}{\Omega_{2x}}\frac{\Omega_{1y}^{'}}{\Omega_{2y}}+k_7\frac{\Omega_{2y}^{''}}{\Omega_{2y}}+k_8\frac{\Omega_{2x}^{''}}{\Omega_{2x}}-(k_9+\lambda^{\Omega_2})=0$$

Taking the derivatives with respect to "x" and "y" we obtain:

$$k_{13}\left(\frac{\Psi_{1x}}{\Omega_{2x}}\right)^{'}\left(\frac{\Psi_{1y}}{\Omega_{2y}}\right)^{'}-k_{13}\left(\frac{w_x^{'}}{\Omega_{2x}}\right)^{'}\left(\frac{w_y}{\Omega_{2y}}\right)^{'}+k_{14}\left(\frac{\Omega_{1x}^{'}}{\Omega_{2x}}\right)^{'}\left(\frac{\Omega_{1y}^{'}}{\Omega_{2y}}\right)^{'}=0$$

, and then

$$\left(\frac{\Psi_{1x}}{\Omega_{2x}}\right)^{'}\in Span\left(\left(\frac{w_x^{'}}{\Omega_{2x}}\right)^{'},\left(\frac{\Omega_{1x}^{'}}{\Omega_{2x}}\right)^{'}\right),\qquad \left(\frac{\Psi_{1y}}{\Omega_{2y}}\right)^{'}\in Span\left(\left(\frac{w_y}{\Omega_{2y}}\right)^{'},\left(\frac{\Omega_{1y}^{'}}{\Omega_{2y}}\right)^{'}\right)$$

$$\Psi_{1x}\in Span(w_x^{'},\Omega_{1x}^{'},\Omega_{2x}),\qquad \Psi_{1y}\in Span(w_y,\Omega_{1y}^{'},\Omega_{2y}) \qquad \textbf{(3.20)}$$

Dividing (I) by $\Psi_{1x}\Psi_{1y}$:

$$k_1\Psi_{1x}^{''}\Psi_{1y}+k_2\Psi_{1x}\Psi_{1y}^{''}-(k_3+\lambda^{\Psi_1})\Psi_{1x}\Psi_{1y}+k_{10}\Psi_{2x}^{'}\Psi_{2y}^{'}+k_{11}w_x^{'}w_y+k_{12}\Omega_{3x}\Omega_{3y}^{'}+k_{13}\Omega_{2z}\Omega_{2y}=0$$

$$k_1\frac{\Psi_{1x}^{''}}{\Psi_{1x}}+k_2\frac{\Psi_{1y}^{''}}{\Psi_{1y}}-(k_3+\lambda^{\Psi_1})+k_{10}\frac{\Psi_{2x}^{'}}{\Psi_{1x}}\frac{\Psi_{2y}^{'}}{\Psi_{1y}}+k_{11}\frac{w_x^{'}}{\Psi_{1x}}\frac{w_y}{\Psi_{1y}}+k_{12}\frac{\Omega_{3x}}{\Psi_{1x}}\frac{\Omega_{3y}^{'}}{\Psi_{1y}}+k_{13}\frac{\Omega_{2z}}{\Psi_{1x}}\frac{\Omega_{2y}}{\Psi_{1y}}=0$$

Taking the derivatives with respect to "x" and "y" we obtain:

$$k_{10}\left(\frac{\Psi_{2x}^{'}}{\Psi_{1x}}\right)^{'}\left(\frac{\Psi_{2y}^{'}}{\Psi_{1y}}\right)^{'}+k_{11}\left(\frac{w_x^{'}}{\Psi_{1x}}\right)^{'}\left(\frac{w_y}{\Psi_{1y}}\right)^{'}+k_{12}\left(\frac{\Omega_{3x}}{\Psi_{1x}}\right)^{'}\left(\frac{\Omega_{3y}^{'}}{\Psi_{1y}}\right)^{'}+k_{13}\left(\frac{\Omega_{2z}}{\Psi_{1x}}\right)^{'}\left(\frac{\Omega_{2y}}{\Psi_{1y}}\right)^{'}=0$$

By **(3.14)**, we obtain that $\left(\frac{\Omega_{3x}}{\Psi_{1x}}\right)^{'}=0$. And then:

$$k_{10}\left(\frac{\Psi^{'}_{2x}}{\Psi_{1x}}\right)^{'}\left(\frac{\Psi^{'}_{2y}}{\Psi_{1y}}\right)^{'}+k_{11}\left(\frac{w^{'}_{x}}{\Psi_{1x}}\right)^{'}\left(\frac{w_{y}}{\Psi_{1y}}\right)^{'}+k_{13}\left(\frac{\Omega_{2z}}{\Psi_{1x}}\right)^{'}\left(\frac{\Omega_{2y}}{\Psi_{1y}}\right)^{'}=0$$

$$\left(\frac{\Omega_{2z}}{\Psi_{1x}}\right)^{'}\in Span\left(\left(\frac{\Psi^{'}_{2x}}{\Psi_{1x}}\right)^{'},\left(\frac{w^{'}_{x}}{\Psi_{1x}}\right)^{'}\right),\qquad \left(\frac{\Omega_{2y}}{\Psi_{1y}}\right)^{'}\in Span\left(\left(\frac{\Psi^{'}_{2y}}{\Psi_{1y}}\right)^{'},\left(\frac{w_{y}}{\Psi_{1y}}\right)^{'}\right)$$

$$\Omega_{2z}\in Span(w^{'}_{x},\Psi^{'}_{2x},\Psi_{1x}),\qquad \Omega_{2y}\in Span(w_{y},\Psi^{'}_{2y},\Psi_{1y}) \qquad \textbf{(3.21)}$$

Dividing (II) by $\Psi^{'}_{2x}\Psi^{'}_{1y}$:

$$k_{10}\frac{\Psi^{'}_{1x}}{\Psi^{'}_{2x}}+k_{2}\frac{\Psi^{''}_{2x}}{\Psi^{'}_{2x}}\frac{\Psi_{2y}}{\Psi^{'}_{1y}}+k_{1}\frac{\Psi_{2x}}{\Psi^{'}_{2x}}\frac{\Psi^{''}_{2y}}{\Psi^{'}_{1y}}-(k_{3}+\lambda^{\Psi_{2}})\frac{\Psi_{2x}}{\Psi^{'}_{2x}}\frac{\Psi_{2y}}{\Psi^{'}_{1y}}+k_{11}\frac{w_{x}}{\Psi^{'}_{2x}}\frac{w^{'}_{y}}{\Psi^{'}_{1y}}+k_{12}\frac{\Omega^{'}_{3x}}{\Psi^{'}_{2x}}\frac{\Omega_{3y}}{\Psi^{'}_{1y}}-k_{13}\frac{\Omega_{1z}}{\Psi^{'}_{2x}}\frac{\Omega_{1y}}{\Psi^{'}_{1y}}=0$$

(By **(3.17), (3.18)** and **(3.19),** $\frac{\Psi_{2y}}{\Psi^{'}_{1y}},\frac{\Psi^{''}_{2y}}{\Psi^{'}_{1y}}$ and $\frac{\Omega_{3y}}{\Psi^{'}_{1y}}$ are constants.)

Taking the derivatives with respect to $"x"$ and $"y"$ we obtain:

$$k_{11}\frac{w_{x}}{\Psi^{'}_{2x}}\left(\frac{w^{'}_{y}}{\Psi^{'}_{1y}}\right)^{'}=k_{3}\frac{\Omega_{1z}}{\Psi^{'}_{2x}}\left(\frac{\Omega_{1y}}{\Psi^{'}_{1y}}\right)^{'}$$

$$\frac{w_{x}}{\Omega_{1x}}=\frac{k_{3}\left(\frac{\Omega_{1y}}{\Psi^{'}_{1y}}\right)^{'}}{k_{11}\left(\frac{w^{'}_{y}}{\Psi^{'}_{1y}}\right)^{'}}=\text{ constant}$$

$$w_{x}\cong\Omega_{1x},\qquad \Omega_{1y}\in Span(w^{'}_{y},\Psi^{'}_{1y}) \qquad \textbf{(3.22)}$$

Dividing (V) by $\Omega_{1x}\Omega_{1y}$:

$$k_{13}\Psi_{2x}\Psi_{2y}+k_{13}w_x w_y^{'}+k_7\Omega_{1x}^{"}\Omega_{1y}+k_8\Omega_{1x}\Omega_{1y}^{"}-(k_9+\lambda^{\Omega_1})\Omega_{1x}\Omega_{1y}+k_{14}\Omega_{2x}^{'}\Omega_{2y}^{'}=0$$

$$k_{13}\left(\frac{\Psi_{2x}}{\Omega_{1x}}\right)^{'}\left(\frac{\Psi_{2y}}{\Omega_{1y}}\right)^{'}+k_{13}\left(\frac{w_x}{\Omega_{1x}}\right)^{'}\left(\frac{w_y^{'}}{\Omega_{1y}}\right)^{'}+k_{14}\left(\frac{\Omega_{2x}^{'}}{\Omega_{1x}}\right)^{'}\left(\frac{\Omega_{2y}^{'}}{\Omega_{1y}}\right)^{'}=0$$

$$\left(\frac{\Omega_{2x}^{'}}{\Omega_{1x}}\right)^{'}\in Span\left(\left(\frac{\Psi_{2x}}{\Omega_{1x}}\right)^{'},\left(\frac{w_x}{\Omega_{1x}}\right)^{'}\right),\qquad \left(\frac{\Omega_{2y}^{'}}{\Omega_{1y}}\right)^{'}\in Span\left(\left(\frac{\Psi_{2y}}{\Omega_{1y}}\right)^{'},\left(\frac{w_y^{'}}{\Omega_{1y}}\right)^{'}\right)$$

$$\Omega_{2x}^{'}\in Span(\Psi_{2x},w_x,\Omega_{1x}),\qquad \Omega_{2y}^{'}\in Span(\Psi_{2y},w_y^{'},\Omega_{1y}) \qquad \textbf{(3.23)}$$

Dividing (II) by $\Psi_{2x}\Psi_{2y}$ and taking the derivatives with respect to "x" and "y":

$$k_{10}\left(\frac{\Psi_{1x}^{'}}{\Psi_{2x}}\right)^{'}\left(\frac{\Psi_{1y}^{'}}{\Psi_{2y}}\right)^{'}+k_{11}\left(\frac{w_x}{\Psi_{2x}}\right)^{'}\left(\frac{w_y^{'}}{\Psi_{2y}}\right)^{'}+k_{12}\left(\frac{\Omega_{3x}^{'}}{\Psi_{2x}}\right)^{'}\left(\frac{\Omega_{3y}}{\Psi_{2y}}\right)^{'}-k_{13}\left(\frac{\Omega_{1z}}{\Psi_{2x}}\right)^{'}\left(\frac{\Omega_{1y}}{\Psi_{2y}}\right)^{'}=0,$$

because $\dfrac{\Psi_{1y}^{'}}{\Psi_{2y}}$ and $\dfrac{\Omega_{3y}}{\Psi_{2y}}$ are constants.

$$\frac{k_{11}}{k_{13}}\frac{\left(\dfrac{w_x}{\Psi_{2x}}\right)^{'}}{\left(\dfrac{\Omega_{1z}}{\Psi_{2x}}\right)^{'}}=\frac{\left(\dfrac{\Omega_{1y}}{\Psi_{2y}}\right)^{'}}{\left(\dfrac{w_y^{'}}{\Psi_{2y}}\right)^{'}}=\text{constant}$$

$$w_x\in Span(\Omega_{1x},\Psi_{2x}),\qquad w_y^{'}\in Span(\Omega_{1y},\Psi_{2y}) \qquad \textbf{(3.24)}$$

Dividing (III) by $w_x w_y$ and taking the derivatives with respect to $"x"$ and $"y"$:

$$-k_{11}(\frac{\Psi'_{1x}}{w_x})'(\frac{\Psi_{1y}}{w_y})' - k_{11}(\frac{\Psi_{2x}}{w_x})'(\frac{\Psi'_{2y}}{w_y})' - k_{13}(\frac{\Omega_{1x}}{w_x})'(\frac{\Omega'_{1y}}{w_y})' + k_{13}(\frac{\Omega'_{2x}}{w_x})'(\frac{\Omega_{2y}}{w_y})' = 0,$$

and replacing **(3.22)** and **(3.23),** we obtain the following linear combination**:**

$$\Omega'_{2x} = a\Psi_{2x} + bw_x \ , \quad \text{for some real constants } a, b.$$

$$(\frac{\Omega'_{2x}}{w_x})' = (a\frac{\Psi_{2x}}{w_x} + b)'$$

$$= a(\frac{\Psi_{2x}}{w_x})',$$

and then:

$$-k_{11}(\frac{\Psi'_{1x}}{w_x})'(\frac{\Psi_{1y}}{w_y})' + (-k_{11}(\frac{\Psi'_{2y}}{w_y})' + k_{13}a(\frac{\Omega_{2y}}{w_y})')(\frac{\Psi_{2x}}{w_x})' = 0$$

$$(\frac{\Psi'_{1x}}{w_x})' \cong (\frac{\Psi_{2x}}{w_x})', \qquad (\frac{\Psi_{1y}}{w_y})' \in Span((\frac{\Psi'_{2y}}{w_y})', (\frac{\Omega_{2y}}{w_y})')$$

$$w_x \in Span(\Psi'_{1x}, \Psi_{2x}), \qquad \Psi_{1y} \in Span(\Psi'_{2y}, \Omega_{2y}, w_y) \qquad \textbf{(3.25)}$$

Dividing (II) by $\Psi''_{2x}\Psi_{2y}$ and taking the derivatives with respect to $"x"$ and $"y"$ we obtain:

$$k_{10}\frac{\Psi'_{1x}}{\Psi''_{2x}}\frac{\Psi'_{1y}}{\Psi_{2y}} + k_2 + k_1\frac{\Psi_{2x}}{\Psi''_{2x}}\frac{\Psi''_{2y}}{\Psi_{2y}} - k_3\frac{\Psi_{2x}}{\Psi''_{2x}} + k_{11}\frac{w_x}{\Psi''_{2x}}\frac{w'_y}{\Psi_{2y}} + k_{12}\frac{\Omega'_{3x}}{\Psi''_{2x}}\frac{\Omega_{3y}}{\Psi_{2y}} - k_{13}\frac{\Omega_{1z}}{\Psi''_{2x}}\frac{\Omega_{1y}}{\Psi_{2y}} = 0\,,$$

and using **(3.14), (3.16), (3.18)** and **(3.19)**:

$$\frac{w_x}{\Psi''_{2x}} \cong 1$$

Taking the derivatives with respect to $"x"$ and $"y"$ we obtain:

$$(\frac{\Psi_{2x}}{\Psi''_{2x}})'(k_1\frac{\Psi''_{2y}}{\Psi_{2y}} - k_3) - k_{13}(\frac{\Omega_{1z}}{\Psi''_{2x}})'\frac{\Omega_{1y}}{\Psi_{2y}} = 0$$

$$(\frac{\Psi_{2x}}{\Psi''_{2x}})' \cong (\frac{\Omega_{1z}}{\Psi''_{2x}})', \qquad k_1 \frac{\Psi''_{2y}}{\Psi_{2y}} - k_3 \cong \frac{\Omega_{1y}}{\Psi_{2y}}$$

$$\Psi''_{2x} \in Span(\Omega_{1z}, \Psi_{2x}), \qquad k_1 \Psi''_{2y} \cong \Omega_1 + k_3 \Psi_{2y}$$

$$\Psi''_{2x} \in Span(\Omega_{1z}, \Psi_{2x}), \qquad \Psi''_{2y} \in Span(\Omega_{1y}, \Psi_{2y}) \quad \textbf{(3.26)}$$

Thus, replacing between them **(3.20)** to **(3.26),** we obtain:

$$\Psi_{1x} = a_4 w'_x + b_4 \Omega'_{1x} + c_4 \Omega_{2z} \quad \textbf{(3.27)}$$

$$\Psi_{1y} = d_4 w_y + e_4 \Omega'_{1y} + f_4 \Omega_{2y} \quad \textbf{(3.28)}$$

$$\Omega_{2x} = a_5 w'_x + b_5 \Psi'_{2x} + c_5 \Psi_{1x} \quad \textbf{(3.29)}$$

$$\Omega_{2y} = d_5 w_y + e_5 \Psi'_{2y} + f_5 \Psi_{1y} \quad \textbf{(3.30)}$$

$$w_x = a_6 \Omega_{1x} \quad \textbf{(3.31)}$$

$$\Omega_{1y} = d_6 w'_y + e_6 \Psi'_{1y} \quad \textbf{(3.32)}$$

$$\Omega'_{2x} = a_7 \Psi_{2x} + b_7 w_x + c_7 \Omega_{1x} \quad \textbf{(3.33)}$$

$$\Omega'_{2y} = d_7 \Psi_{2x} + e_7 w'_y + f_7 \Omega_{1y} \quad \textbf{(3.34)}$$

$$w_x = a_8 \Omega_{1x} + b_8 \Psi_{2x} \quad \textbf{(3.35)}$$

$$w'_y = d_8 \Omega_{1y} + e_8 \Psi_{2y} \quad \textbf{(3.36)}$$

$$w_x = a_9 \Psi'_{1x} + b_9 \Psi_{2x} \quad \textbf{(3.37)}$$

$$\Psi_{1y} = d_9 \Psi'_{2y} + e_9 \Omega_{2y} + f_9 w_y \quad \textbf{(3.38)}$$

$$\Psi''_{2x} = a_{10} \Omega_{1x} + b_{10} \Psi_{2x} \quad \textbf{(3.39)}$$

$$\Psi''_{2y} = d_{10} \Omega_{1y} + e_{10} \Psi_{2y} \quad \textbf{(3.40)}$$

, where a_i, b_i, c_i, d_i, e_i and f_i are constants, $i = 1,10$.

Since **(3.31)** and **(3.35)**:

$$w_x \cong \Psi_{2x} \quad \textbf{(3.41)}$$

Since **(3.31), (3.33)** and **(3.41)**:

$$\Omega'_{2x} \cong w_x \quad \textbf{(3.42)}$$

Since **(3.37)** and **(3.41)**:

$$\Psi'_{1x} \cong w_x \quad \textbf{(3.43)}$$

Since **(3.31), (3.39)** and **(3.41)**:

$$w''_x \cong w_x \quad \textbf{(3.44)}$$

Since **(3.31), (3.41)** and **(3.44)**:

$$\Omega''_{1x} \cong \Omega_{1x} \quad \textbf{(3.45)}$$

$$\Psi''_{2x} \cong \Psi_{2x} \quad \textbf{(3.46)}$$

From **(3.41)** and **(3.44),** we obtain:

$$\Psi''_{2x} \cong \Psi_{2x} \tag{3.47}$$

Substituting **(3.15)** into **(3.43)** and then substituting these results together with (**3.16)** into **(3.29) ,** we obtain the following:

$$w'_x \cong \Psi_{1x} \tag{3.48}$$

$$\Omega_{2x} \cong \Psi_{1x} \tag{3.49}$$

From **(3.15)** and **(3.49),** we obtain:

$$\Omega''_{2x} \cong \Omega_{2x} \tag{3.50}$$

Substituting **(3.18)** into **(3.40)**:

$$\Omega_{1y} \cong \Psi_{2y} \tag{3.51}$$

Substituting **(3.18)** into **(3.51)**:

$$\Omega''_{1y} \cong \Omega_{1y} \tag{3.52}$$

Substituting **(3.19)** and **(3.51)** into **(3.32)**:

$$w'_y \cong \Psi_{2y} \tag{3.53}$$

Substituting **(3.51)** and **(3.53)** into **(3.34)**:

$$\Omega'_{2y} \cong \Psi_{2y} \tag{3.54}$$

Substituting **(3.28)** into **(3.30)** and using **(3.51)**:

$$\Omega_{2y} = \tilde{d}_s w_y + \tilde{e}_s \Psi'_{2y} \tag{3.55}$$

, were $\tilde{d}_s, \tilde{e}_s$ are some real constants.

Substituting **(3.54)** into **(3.55)**

$$\Omega_{2y} = \bar{d}_s w_y + \bar{e}_s \Omega''_{2y} \ , \ \bar{d}_s, \bar{e}_s \in R. \tag{3.56}$$

In another form: $w_y = \hat{d}_s \Omega_{2y} + \hat{e}_s \Omega''_{2y} \ , \ \hat{d}_s, \hat{e}_s \in R.$

Substituting **(3.28)** into **(3.38)** and furthermore, replacing **(3.51), (3.54)** into **(3.56)** we obtain:

$$\Omega''_{2y} \cong \Omega_{2y}, \tag{3.57}$$

and, from **(3.57)** and **(3.58)**:

$$w_y \cong \Omega_{2y} \tag{3.58}$$

and

$$w''_y \cong w_y \tag{3.59}$$

Substituting **(3.54), (3.58)** and **(3.59)** into **(3.38):**

$$\Psi_{1y} \cong \Omega_{2y}, \tag{3.60}$$

and

$$\Psi_{1y} \cong \Psi''_{1y} \ . \tag{3.61}$$

In a nutshell:

$$\Psi_{1x} \cong \Omega_{3x} \cong \Omega_{2x} \ , \ \Psi'_{2x} \cong \Psi_{1x} \ , \ \Psi_{2x} \cong w_x \cong \Omega_{1x} \ , \ \Psi''_{1x} \cong \Psi_{1x} \ , \ \Psi''_{2x} \cong \Psi_{2x} \ , \ \Psi_{1y} \cong w_y \cong \Omega_{2y} \ ,$$

$$\Psi'_{1y} \cong \Psi_{2y} \ , \ \Psi_{2y} \cong \Omega_{3y} \cong \Omega_{1y} \ , \ \Psi''_{1y} \cong \Psi_{1y} \quad \& \quad \Psi''_{2y} \cong \Psi_{2y}. \tag{3.62}$$

It is interesting "how well these functions are related", only it is a function (of one variable, for example "x") and its derivative, the derivatives of any order of them belong to one of the two equivalence classes of which this function and its derivative are representatives, i.e. "the quotient set (the functions f, g are related $f \cong g$ if $af = bg$ for some non-zero real constants a, b) contains only two functions".

We obtain the following **Sturm-Liouville Eigenvalue Problems** from the boundary conditions **(2.49)-(2.64)** and the relations in **(3. 62):**

CASE I

From the system of equations **(3.62),** we consider the following assumption:

$$\frac{w''_x}{w_x} = \lambda^2 \qquad , \text{ where } \lambda \neq 0.$$

$$(D_x^2 - \lambda^2) w_x = 0$$

$$D_x = \pm\lambda \, .$$

Then $$w_x(x) = \alpha e^{\lambda x} + \beta e^{-\lambda x}, \text{ where } \alpha, \beta \in R.$$

Substituting $w = w_x w_y$ into **(2.49),** we obtain the following B.C. for the ODE $\frac{w''_x}{w_x} = \lambda^2$:

$$w_x(0) = 0$$

$$\alpha + \beta = 0 \, , \ w_x(x) = \alpha(e^{\lambda x} - e^{-\lambda x})$$

$$w_x(x) = \alpha \ sinh(\lambda x)$$

And by B.C. in $x = a$:

$$w_x(a) = 0 \text{ and then } \alpha\, sinh(\lambda a) = 0$$

Then $\alpha = 0$ and $\beta = 0$, because $\lambda a \neq 0$. Then

$$w_x = 0$$
$$\Psi_{2x} = \Omega_{1x} = w_x = 0$$

$$\Psi_2 = \Omega_1 = w = 0. \qquad \textbf{(3.63)}$$

By $\Psi_{1x}(x) = \Psi'_{2x}(x)$, we obtain:

$$\Psi_{1x} = 0$$

$$\Omega_{3x} = \Omega^0_{2x} = \Psi_{1x} = 0$$

$$\Omega_3 = \Omega^0_2 = \Psi_1 = 0 \qquad \textbf{(3.64)}$$

We obtain the trivial solution **(3.63)** and **(3.64).**

Similarly for $\dfrac{w_y''}{w_y} = \tilde{\lambda}^2$, $\tilde{\lambda} \neq 0$, with $w_y(0) = 0$, $w_y(b) = 0$.

CASE II

From the system of equations **(3.62),** we consider the following case:

$$\frac{w_x''}{w_x} = 0, \text{ where } \lambda = 0.$$
$$w_x'' = 0,\ w_x(x) = \bar{a}x + b.$$

Applying the B.C.: $w_x(0) = 0$, $b = 0$. Then $w_x(x) = \bar{a}x$; $w_x(a) = 0$

$$\bar{a}a = 0, \text{ then } \bar{a} = 0,\ w_x = 0.$$

Also as in Case I, we obtain the trivial solution $\Psi_1 = \Psi_2 = w = \Omega_3 = \Omega^0_1 = \Omega^0_2 = 0$. Similarly for w_y.

CASE III

From the system of equations **(3.62),** we consider the following case:

$$\frac{w_x''}{w_x} = -\lambda^2 \text{, where } \lambda \neq 0.$$

$$(D_x^2 + \lambda^2)w_x = 0, \quad D_x = \pm i\lambda$$

$$w_x(x) = \alpha\ cos(\lambda x) + \beta\ sin(\lambda x).$$

Applying the B.C.: $w_x(0) = 0.$

$$\alpha = 0, \quad w_x(x) = \beta\ sin(\lambda x)$$

Applying the B.C.: $w_x(a) = 0.$

$$\beta\ Sin(\lambda a) = 0 \text{, and then } \lambda a = n\pi, \quad n \in Z.$$

$$\lambda = \lambda_n = \frac{n\pi}{a}\text{. Then } w_{x_n}(x) = sin(\lambda_n x)$$

$$\Psi_{2x_n} \cong \Omega_{1x_n} \cong w_{x_n} = sin(\frac{n\pi x}{a}) \tag{3.65}$$

$$\Psi_{1x} \cong \Psi_{2x}'\text{, then } \Psi_{1x_n} = cos(\frac{n\pi x}{a}) \tag{3.66}$$

$$\Omega_{3x_n} \cong \Omega_{2x_n} \cong \Psi_{1x_n} = cos(\frac{n\pi x}{a}) \tag{3.67}$$

Similarly for w_y, $\frac{w_y''}{w_y} = -\tilde{\lambda}^2$, $\tilde{\lambda} > 0$, with $w_y(0) = w_y(b) = 0.$

$$\Psi_{1y_m} \cong \Omega_{2y_m} \cong w_{y_m} = sin(\frac{m\pi y}{b}) \tag{3.68}$$

And $$\Psi_{2y_m} \cong \Omega_{3y_m} \cong \Omega_{1y_m} = cos(\frac{m\pi y}{b})\text{, where } m \in Z. \tag{3.69}$$

From **(3.65)** to **(3.69)** we showed that the Eigen-functions for the ***Field Equations*** **(2.48),** with the boundary conditions **(2.49)** to **(2.64)** are the following:

$$\Psi_{1_{m,n}} \cong \cos(\frac{m\pi x}{a})sin(\frac{n\pi y}{b}), \quad \Psi_{2_{m,n}} \cong sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b}), \quad w_{m,n} \cong sin(\frac{m\pi x}{a})sin(\frac{n\pi y}{b}),$$

$$\Omega_{3_{m,n}} \cong \cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b}), \quad \Omega_{1_{m,n}} \cong sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b}) \text{ and } \Omega_{2_{m,n}} \cong \cos(\frac{m\pi x}{a})sin(\frac{n\pi y}{b}).$$

Now applying the ***Superposition Principle*** we obtain the following series:

$$\Psi_1(x,y)=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_1}\cos(\frac{m\pi x}{a})sin(\frac{n\pi y}{b}), \quad \textbf{(3.70)}$$

$$\Psi_2(x,y)=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_2}sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b}), \quad \textbf{(3.71)}$$

$$w(x,y)=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{w}sin(\frac{m\pi x}{a})sin(\frac{n\pi y}{b}), \quad \textbf{(3.72)}$$

$$\Omega_3(x,y)=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_3}\cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b}), \quad \textbf{(3.73)}$$

$$\Omega_1^0(x,y)=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_1^0}sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b}), \quad \textbf{(3.74)}$$

$$\Omega_2^0(x,y)=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_2^0}\cos(\frac{m\pi x}{a})sin(\frac{n\pi y}{b}). \quad \textbf{(3.75)}$$

We assume that the ***pressure*** is given by

$$P(x,y)=\sum_{m,n=1}^{\infty}\alpha_{n,m}sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})=sin(\frac{\pi x}{a})sin(\frac{\pi y}{b}),$$

where $\alpha_{n,m}=0,\ \forall(m,n)\neq(1,1)$ and $\alpha_{1,1}=1,$ **(3.76)**

Substituting **(3.70)** to **(3.76)** into the ***Field Equations*** (I) **-** (VI) **(2.48)**, we obtain the system of algebraic equations $(I)^{**}$ - $(VI)^{**}$ and (VII).

Calculus of the Equation $(I)^*$:

Using the Theorem 1 **[19]** (Term-by-term Differentiation of Fourier Cosine Series) and the Theorem 2 **[19]** (Differentiation of Fourier Sine Series) of the appendix into the equation (I) we proceed to obtain the equation $(I)^*$. Also, for the other equations $(II)^*$ to $(VI)^*$.

We obtain $k_1\partial_x^{(2)}(\Psi_1)$, $\partial_y^2\Psi_1$, $\partial_y\partial_x\Psi_2$, $\partial_x w$ and $\partial_y\Omega_3$, which appear in the equation (I):

$$\Psi_1=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_1}\cos(\frac{n\pi x}{a})sin(\frac{m\pi y}{b}),$$

$$\Psi_1=\sum_{n}\sum_{m}\alpha_{n,m}^{\Psi_1}sin(\frac{m\pi y}{b})\cos(\frac{n\pi x}{a})$$

For $k_1\partial_x(\Psi_1)$ we obtain the following series:

$$k_1\partial_x(\Psi_1)=\sum_{m,n=1}^{\infty}\{\alpha_{n,m}^{\Psi_1}(-k_1(\frac{n\pi}{a}))\sin(\frac{n\pi x}{a})\}\sin(\frac{m\pi y}{b})$$

$$\sum_n[(-k_1(\frac{n\pi}{a}))\sum_m\alpha_{n,m}^{\Psi_1}\sin(\frac{m\pi y}{b})]\sin(\frac{n\pi x}{a})$$

again, applying the operator ∂_x, we obtain the following series:

$$k_1\partial_x^{(2)}(\Psi_1)=\frac{1}{a}[\partial_x\Psi_1(a,y)-\partial_x\Psi_1(0,y)]+\sum_n[\frac{n\pi}{L}\{(-k_1(\frac{n\pi}{a}))\sum_m\alpha_{n,m}^{\Psi_1}\sin(\frac{m\pi y}{b})\}+$$
$$+\frac{2}{a}((-1)^n\partial_x\Psi_1(a,y)-\partial_x\Psi_1(0,y))]\cos(\frac{n\pi x}{a})$$

From $\Psi_1(x,y)=\sum_m[\sum_n\alpha_{n,m}^{\Psi_1}cos\frac{n\pi x}{a}]sin\frac{m\pi y}{b}$ we obtain the following series:

$$\partial_y\Psi_1(x,y)=\frac{1}{b}[\Psi_1(x,b)-\Psi_1(x,0)]+\sum_m[\frac{m\pi}{b}[\sum_n\alpha_{n,m}^{\Psi_1}cos\frac{n\pi x}{a}]+$$
$$\frac{2}{b}((-1)^m\Psi_1(x,b)-\Psi_1(x,0))]cos\frac{m\pi y}{b},$$

$$\partial_y^2\Psi_1(x,y)=-\sum_m\frac{m\pi}{b}[\frac{m\pi}{b}[\sum_n\alpha_{n,m}^{\Psi_1}cos\frac{n\pi x}{a}]+\frac{2}{b}((-1)^m\Psi_1(x,b)-\Psi_1(x,0))]sin\frac{m\pi y}{b},$$

From $\Psi_2(x,y)=\sum_n[\sum_m\alpha_{n,m}^{\Psi_2}cos\frac{m\pi y}{b}]sin\frac{n\pi x}{a}$ we obtain the following series:

$$\partial_x\Psi_2(x,y)=\frac{1}{a}[\Psi_2(a,y)-\Psi_2(0,y)]+\sum_n[\frac{n\pi}{a}[\sum_m\alpha_{n,m}^{\Psi_2}cos\frac{m\pi y}{b}]+$$
$$\frac{2}{a}((-1)^n\Psi_2(a,y)-\Psi_2(0,y))]cos\frac{n\pi x}{a},$$

$$\partial_x\Psi_2(x,y)=\frac{1}{a}[\Psi_2(a,y)-\Psi_2(0,y)]+\sum_m[[\sum_n\frac{n\pi}{a}\alpha_{n,m}^{\Psi_2}cos\frac{n\pi x}{a}]cos\frac{m\pi y}{b}+$$
$$\frac{2}{a}\sum_n((-1)^n\Psi_2(a,y)-\Psi_2(0,y))cos\frac{n\pi x}{a},$$

$$\partial_y\partial_x\Psi_2(x,y)=\frac{1}{a}[\partial_y\Psi_2(a,y)-\partial_y\Psi_2(0,y)]-\sum_m\frac{m\pi}{b}[\sum_n\frac{n\pi}{a}\alpha_{n,m}^{\Psi_2}cos\frac{n\pi x}{a}]sin\frac{m\pi y}{b}+$$
$$\frac{2}{a}\sum_n((-1)^n\partial_y\Psi_2(a,y)-\partial_y\Psi_2(0,y))cos\frac{n\pi x}{a},$$

From $w(x,y)=\sum_n[\sum_m\alpha_{n,m}^{w}sin\frac{m\pi y}{b}]sin\frac{n\pi x}{a}$ we obtain the following:

$$\partial_x w(x,y)=\frac{1}{a}[w(a,y)-w(0,y)]+$$
$$\sum_n[\frac{n\pi}{a}[\sum_m\alpha_{n,m}^{w}sin\frac{m\pi y}{b}]+\frac{2}{a}((-1)^n w(a,y)-w(0,y))]cos\frac{n\pi x}{a},$$

From $\Omega_3(x,y)=\sum_m[\sum_n\alpha_{n,m}^{\Omega_3}cos\frac{n\pi x}{a}]cos\frac{m\pi y}{b}$ we obtain the following:

$$\partial_y\Omega_3(x,y)=-\sum_m\frac{m\pi}{b}[\sum_n\alpha_{n,m}^{\Omega_3}cos\frac{n\pi x}{a}]sin\frac{m\pi y}{b},$$

P is expressed in the form $P(x,y)=\sum_n[\sum_m\alpha_{n,m}sin\frac{m\pi y}{b}]sin\frac{n\pi x}{a}$, substituting expressions in series form for $k_1\partial_x^{(2)}(\Psi_1)$, $\partial_y^2\Psi_1$, $\partial_y\partial_x\Psi_2$, $\partial_x w$ and $\partial_y\Omega_3$ into the equation (I), we obtain the following equation:

$$\frac{k_{15}}{a}[P(a,y)-P(0,y)]+k_{15}\sum_n[\frac{n\pi}{a}[\sum_m\alpha_{n,m}sin\frac{m\pi y}{b}]+\frac{2}{a}((-1)^n P(a,y)-P(0,y))]cos\frac{n\pi x}{a}=$$
$$\frac{k_1}{a}[\partial_x\Psi_1(a,y)-\partial_x\Psi_1(0,y)+k_1\sum_n[-(\frac{n\pi}{a})^2[\sum_m\alpha_{n,m}^{\Psi_1}sin\frac{m\pi y}{b}]+\frac{2}{a}((-1)^n\partial_x\Psi_1(a,y)-\partial_x\Psi_1(0,y))]$$
$$cos\frac{n\pi x}{a}-k_2\sum_m(\frac{m\pi}{b})[\frac{m\pi}{b}[\sum_n\alpha_{n,m}^{\Psi_1}cos\frac{n\pi x}{a}]+\frac{2}{a}((-1)^m\Psi_1(x,b)-\Psi_1(x,0))]sin\frac{m\pi y}{b}-$$
$$k_3\sum_n\sum_m\alpha_{n,m}^{\Psi_1}cos\frac{n\pi x}{a}sin\frac{m\pi y}{b}+\frac{k_{10}}{a}[\partial_y\Psi_2(a,y)-\partial_y\Psi_2(0,y)]-k_{10}\sum_m(\frac{m\pi}{b})[\sum_n\frac{n\pi}{a}\alpha_{n,m}^{\Psi_2}cos\frac{n\pi x}{a}]sin\frac{m\pi y}{b}$$
$$+\frac{2k_{10}}{a}\sum_n((-1)^n\partial_y\Psi_2(a,y)-\partial_y\Psi_2(0,y))cos\frac{n\pi x}{a}+\frac{k_{11}}{a}[w(a,y)-w(0,y)]+$$
$$k_{11}\sum_n[\frac{n\pi}{a}[\sum_n\alpha_{n,m}^{w}sin\frac{m\pi y}{b}]+\frac{2}{a}((-1)^n w(a,y)-w(0,y))]cos\frac{n\pi x}{a}-$$
$$k_{12}\sum_m\frac{m\pi}{b}[\sum_n\alpha_{n,m}^{\Omega_3}cos\frac{n\pi x}{a}]sin\frac{m\pi y}{b}+k_{13}\sum_n\sum_m\alpha_{n,m}^{\Omega_2^0}cos\frac{n\pi x}{a}sin\frac{m\pi y}{b},$$

$(I)^*$.

The equation $(I)^*$ contains the following zero expressions:

$$P(a,y) - P(0,y) = 0,$$
$$(-1)^n P(a,y) - P(0,y) = 0,$$
$$\partial_x \Psi_1(a,y) - \partial_x \Psi_1(0,y) = 0,$$
$$(-1)^n \partial_x \Psi_1(a,y) - \partial_x \Psi_1(0,y) = 0,$$
$$(-1)^m \Psi_1(x,b) - \Psi_1(x,0) = 0,$$
$$\partial_y \Psi_2(a,y) - \partial_y \Psi_2(0,y) = 0,$$
$$(-1)^n \partial_y \Psi_2(a,y) - \partial_y \Psi_2(0,y) = 0,$$
$$w(a,y) - w(0,y) = 0,$$
$$(-1)^n w(a,y) - w(0,y) = 0.$$

Calculus of the Equation $(II)^*$:

$$\Psi_1(x,y) = \sum_n [\sum_m \alpha_{n,m}^{\Psi_1} sin \frac{m\pi y}{b}] cos \frac{n\pi x}{a}$$
$$\partial_x \Psi_1(x,y) = -\sum_n (\frac{n\pi}{a})[\sum_m \alpha_{n,m}^{\Psi_1} sin \frac{m\pi y}{b}] sin \frac{n\pi x}{a}$$
$$\partial_x \Psi_1(x,y) = \sum_m [\sum_n -(\frac{n\pi}{a}) \alpha_{n,m}^{\Psi_1} sin \frac{n\pi x}{a}] sin \frac{m\pi y}{b},$$

$$\partial_y \partial_x \Psi_1(x,y) = \frac{1}{b}[\partial_x \Psi_1(x,b) - \partial_x \Psi_1(x,0)] + \sum_m [(\frac{m\pi}{b})[\sum_n -(\frac{n\pi}{a}) \alpha_{n,m}^{\Psi_1} sin \frac{n\pi x}{a}] + \frac{2}{b}((-1)^m \partial_x \Psi_1(x,b) - \partial_x \Psi_1(x,0))] cos \frac{m\pi y}{b},$$

From the following equation

$$\partial_x \Psi_2(x,y) = \frac{1}{a}[\Psi_2(a,y) - \Psi_2(0,y)] + \sum_m [[\sum_n \frac{n\pi}{a} \alpha_{n,m}^{\Psi_2} cos \frac{n\pi x}{a}] cos \frac{m\pi y}{b} +$$
$$+ \frac{2}{a} \sum_n ((-1)^n \Psi_2(a,y) - \Psi_2(0,y)) cos \frac{n\pi x}{a},$$

we obtain:

$$\partial_x^{(2)} \Psi_2(x,y) = -\sum_n (\frac{n\pi}{a})[\frac{n\pi}{a} \sum_m \alpha_{n,m}^{\Psi_2} cos \frac{m\pi y}{b}] sin \frac{n\pi x}{a},$$

From $\Psi_2(x,y) = \sum_n [\sum_m \alpha_{n,m}^{\Psi_2} sin \frac{n\pi x}{a}] cos \frac{m\pi y}{b}$, we obtain the following:

$$\partial_y \Psi_2(x,y) = -\sum_m \frac{m\pi}{b} [\sum_n \alpha_{n,m}^{\Psi_2} sin \frac{n\pi x}{a}] sin \frac{m\pi y}{b},$$

$$\partial_y^{(2)}\Psi_2(x,y)=\frac{1}{b}(\partial_y\Psi_2(x,b)-\partial_y\Psi_2(x,0))+$$
$$\sum_m[\frac{m\pi}{b}[\sum_m-\frac{m\pi}{b}[\sum_n\alpha_{n,m}^{\Psi_2}sin\frac{n\pi x}{a}]]+\frac{2}{b}((-1)^m\partial_y\Psi_2(x,b)-\partial_y\Psi_2(x,0))]cos\frac{m\pi y}{b},$$

From the expression **(3.72)** for w we obtain the following:

$$\partial_y w(x,y)=\frac{1}{b}(w(x,b)-w(x,0))+$$
$$\sum_m[\frac{m\pi}{b}[\sum_n\alpha_{n,m}^{w}sin\frac{n\pi x}{a}]+\frac{2}{b}(w(x,b)-w(x,0))]cos\frac{m\pi y}{b},$$

From the expression **(3.73)** for Ω_3 we obtain the following:

$$\partial_x\Omega_3(x,y)=-\sum_n\frac{n\pi}{a}[\sum_m\alpha_{n,m}^{\Omega_3}cos\frac{m\pi y}{b}]sin\frac{n\pi x}{a},$$

Substituting expressions obtained for the functions $\partial_y\partial_x\Psi_1$, $\partial_x^{(2)}\Psi_2$, $\partial_y^{(2)}\Psi_2$, $\partial_y w$ and $\partial_x\Omega_3$ in the equation (II) of the ***Field Equations*** **(2.48)**, we obtain the second equation:

$$k_{15}\partial_y P(x,y)=\frac{k_{15}}{b}(P(x,b)-P(x,0))+k_{15}\sum_m[\frac{m\pi}{b}[\sum_n\alpha_{n,m}sin\frac{n\pi x}{a}]+\frac{2}{b}[P(x,b)-P(x,0)]]cos\frac{m\pi y}{b}=$$
$$\frac{k_{10}}{b}(\partial_x\Psi_1(x,b)-\partial_x\Psi_1(x,0))+k_{10}\sum_m[\frac{m\pi}{b}[\sum_n-\frac{n\pi}{a}\alpha_{n,m}^{\Psi_1}sin\frac{n\pi x}{a}]+\frac{2}{b}((-1)^m\partial_x\Psi_1(x,b)-\partial_x\Psi_1(x,0))]$$
$$cos\frac{m\pi y}{b}-k_2\sum_n\frac{n\pi}{a}[\frac{n\pi}{a}[\sum_m\alpha_{n,m}^{\Psi_2}cos\frac{m\pi y}{b}]sin\frac{n\pi x}{a}]+\frac{k_1}{b}(\partial_y\Psi_2(x,b)-\partial_y\Psi_2(x,0))+$$
$$k_1\sum_m[\frac{m\pi}{b}[\sum_m-\frac{m\pi}{b}[\sum_n\alpha_{n,m}^{\Psi_2}sin\frac{n\pi x}{a}]]+\frac{2}{b}((-1)^m\partial_y\Psi_2(x,b)-\partial_y\Psi_2(x,0))]cos\frac{m\pi y}{b}-$$
$$k_3\sum_{n,m}\alpha_{n,m}^{\Psi_2}cos\frac{m\pi y}{b}sin\frac{n\pi x}{a}+\frac{k_{11}}{b}(w(x,b)-w(x,0))+k_{11}\sum_m[\frac{m\pi}{b}[\sum_n\alpha_{n,m}^{w}sin\frac{n\pi x}{a}]+\frac{2}{b}((-1)^m w(x,b)-w(x,0))]$$
$$cos\frac{m\pi y}{b}+k_{12}\sum_n\frac{n\pi}{a}[\sum_m\alpha_{n,m}^{\Omega_3}cos\frac{m\pi y}{b}]sin\frac{n\pi x}{a}-k_3\sum_{n,m}\alpha_{n,m}^{\Omega_1^0}sin\frac{n\pi x}{a}cos\frac{m\pi y}{b}.$$
$$(II)^*.$$

Calculus of the other Equations $(III)^*$ to $(VI)^*$:

The equations $(III)^*$ to $(VI)^*$ are calculated from the equations (III) to (VI) respectively.

From B.C.: $\Psi_2(0,y)=0,\ \ \Psi_2(a,y)=0,\ \ w(0,y)=0,\ \ w(a,y)=0,$

we obtain that the following terms are zero:

$$\Psi_{2,2}(0,y)=0,\ \ \Psi_{2,2}(a,y)=0,\ \ \partial_y w(0,y)=0,\ \ \partial_y w(a,y)=0\,.$$

From $M_{11}(0,y)=0$, we obtain that the following term is zero:

$$\Psi_{1,1}(0,y)=0.$$

From $M_{11}(a,y)=0$, we obtain that the following terms are zero:

$$\Psi_{1,1}(a,y)=0,\quad \Omega_{3,1}(0,y)=0,\quad \Omega_{3,1}(a,y)=0.$$

$$\Psi_1(x,0)=0,\quad w(x,0)=0,\quad \Psi_1(x,b)=0,\quad w(x,b)=0\,,$$

Consequently: $\Psi_{1,1}(x,0)=0,\quad \Psi_{1,1}(x,b)=0,\quad \partial_x w(x,0)=0,\quad \partial_x w(x,b)=0.$

From $M_{2,2}(x,0)=0$, we obtain that the following term is zero:

$$\Psi_{2,2}(x,0)=0$$

From $M_{2,2}(x,b)=0$, we obtain that the following terms are zero:

$$\Psi_{2,2}(x,b)=0,\quad \Omega_{3,2}(x,0)=0,\quad \Omega_{3,2}(x,b)=0.$$

Consequently $\Psi_{1,1}\big|_{\partial_B}=0,\quad \Psi_{2,2}\big|_{\partial_B}=0.$ Given the results: $\Psi_{1,1}\big|_{\partial_B}=0,\quad \Psi_{2,2}\big|_{\partial_B}=0,$ etc and from the first and second equations $(I)^*$ and $(II)^*$ respectively, we already to obtain Ω_1^0 and Ω_2^0, since $k_{13}\neq 0.$ Replacing these results into the two last equations (V) and (VI), we obtain:

$$\begin{aligned}
\Omega_1^0(a,y) &\approx \frac{-1}{k_{13}}\sum_{n,m}\gamma_{n,m}^{\Omega_1^0} sin\frac{n\pi(a)}{a}cos\frac{m\pi y}{b}=0,\\
\Omega_1^0(0,y) &\approx \frac{-1}{k_{13}}\sum_{n,m}\gamma_{n,m}^{\Omega_1^0} sin\frac{n\pi(0)}{a}cos\frac{m\pi y}{b}=0,\\
\Omega_2^0(x,b) &\approx \frac{1}{k_{13}}\sum_{n,m}\gamma_{n,m}^{\Omega_2^0} cos\frac{n\pi x}{a}sin\frac{m\pi(0)}{b}=0,\\
\Omega_2^0(x,0) &\approx \frac{1}{k_{13}}\sum_{n,m}\gamma_{n,m}^{\Omega_2^0} cos\frac{n\pi x}{a}sin\frac{m\pi(b)}{b}=0.
\end{aligned} \qquad \textbf{(3.77)}$$

Now, we will identify terms equal to zero in equations (III) to (VI).
Equation (III) contains the following zero terms**:**

$$w(a,y)-w(0,y)=0,$$
$$(-1)^n w(a,y)-w(0,y)=0,$$
$$(-1)^m w(x,b)-w(x,0)=0,$$

Equation (IV) contains the following zero terms**:**

$$\Psi_1(x,b)-\Psi_1(x,0)=0,$$
$$(-1)^m \Psi_1(x,b)-\Psi_1(x,0)=0,$$
$$\Psi_2(a,y)-\Psi_2(0,y)=0,$$
$$(-1)^n \Psi_2(a,y)-\Psi_2(0,y)=0,$$
$$\partial_x\Omega_3(a,y)-\partial_x\Omega_3(0,y)=0,$$
$$(-1)^n \partial_x\Omega_3(a,y)-\partial_x\Omega_3(0,y)=0,$$
$$\partial_y\Omega_3(x,b)-\partial_y\Omega_3(x,0)=0,$$
$$(-1)^m \partial_y\Omega_3(x,b)-\partial_y\Omega_3(x,0)=0,$$

Equation (V) contains the following zero terms**:**

$$w(x,b)-_1 w(x,0)=0,$$
$$(-1)^m w(x,b)-_1 w(x,0)=0,$$

From B.C., (3.74) and (3.75):

$$\Omega_1^0(a,y)-\Omega_1^0(0,y)=0$$
$$(-1)^n \Omega_1^0(a,y)-\Omega_1^0(0,y)=0$$

$$\partial_y\Omega_1^0(x,b)-\partial_y\Omega_1^0(x,0)=A\partial_x\Omega_2^0(x,b)-B\partial_x\Omega_2^0(x,0)=0$$
$$(-1)^m \partial_y\Omega_1^0(x,b)-\partial_y\Omega_1^0(x,0)=A\partial_x\Omega_2^0(x,b)-B\partial_x\Omega_2^0(x,0)=0$$
$$\Omega_2^0(x,b)-_1\Omega_2^0(x,0)=0$$
$$(-1)^m \Omega_2^0(x,b)-_1\Omega_2^0(x,0)=0.$$

, where $A\,\&\,B$ are real constants.

Equation (VI) contains the following zero terms**:**

$$w(a,y)-w(0,y)=0,$$
$$(-1)^n w(a,y)-w(0,y)=0,$$

From B.C., (3.74) and (3.75):

$$\Omega_1^0(a,y)-\Omega_1^0(0,y)=0$$
$$(-1)^n\Omega_1^0(a,y)-\Omega_1^0(0,y)=0$$
$$\partial_x\Omega_2^0(a,y)-\partial_x\Omega_2^0(0,y)=A\partial_y\Omega_1^0(a,y)-B\partial_y\Omega_1^0(0,y)=0$$
$$(-1)^n\partial_x\Omega_2^0(a,y)-\partial_x\Omega_2^0(0,y)=A\partial_y\Omega_1^0(a,y)-B\partial_y\Omega_1^0(0,y)=0$$
$$\Omega_2^0(x,b)-\Omega_2^0(x,0)=0$$
$$(-1)^m\Omega_2^0(x,b)-\Omega_2^0(x,0)=0,$$

where $A\,\&\,B$ are real constants.

3.2.1.1 Linear System of Algebraic Equations

In this section we obtain the linear system of algebraic equations given by $(I)^{**}$-$(VI)^{**}$ from the ***Field Equations* (2.48)**.

$(I)^{*}$:

$$-k_{15}\sum_{m,n=1}^{\infty}\alpha_{n,m}(\frac{n\pi}{a})\cos(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_1}(-k_1(\frac{n\pi}{a})^2-k_2(\frac{m\pi}{b})^2-k_3)$$
$$\cos(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_2}(-k_{10}\frac{n\pi}{a}\frac{m\pi}{b})\cos(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})+$$
$$\sum_{m,n=1}^{\infty}\alpha_{n,m}^{w}(k_{11}\frac{n\pi}{a})\cos(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_3}(-k_{12}\frac{m\pi}{b})\cos(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})+$$
$$+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_2^0}k_{13}\cos(\frac{n\pi x}{a})sin(\frac{m\pi y}{b}),$$

$(II)^{*}$:

$$-k_{15}\sum_{m,n=1}^{\infty}\alpha_{n,m}(\frac{m\pi}{b})sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_1}(-k_{10}\frac{n\pi}{a}\frac{m\pi}{b})sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+$$
$$+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_2}(-k_1(\frac{m\pi}{b})^2-k_2(\frac{n\pi}{a})^2-k_3)sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+$$
$$\sum_{m,n=1}^{\infty}\alpha_{n,m}^{w}k_{11}\frac{m\pi}{b}sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})-\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_3}(-k_{12})\frac{n\pi}{a}sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})$$
$$+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_1^0}(-k_{13})sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}),$$

$(III)*:$

$$-k_{17}\sum_{m,n=1}^{\infty}\alpha_{n,m}sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_1}(k_{11}\frac{n\pi}{a})sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})+$$

$$\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_2}(k_{11}\frac{m\pi}{b})sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{w}(-k_4)((\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2)sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})+$$

$$\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_1^0}k_{13}\frac{m\pi}{b}sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_2^0}(-k_{13}\frac{n\pi}{a})sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b}),$$

$(IV)*:$

$$0=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_1}(-k_{12}\frac{m\pi}{b})\cos(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_2}k_{12}\frac{n\pi}{a}\cos(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+$$

$$+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_3}(-k_5((\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2)-k_6)\cos(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}),$$

$(V)*:$

$$0=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_2}k_{13}sen(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{w}k_{13}\frac{m\pi}{b}sen(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+$$

$$+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_1^0}(-k_7(\frac{n\pi}{a})^2-k_8(\frac{m\pi}{b})^2-k_9)sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+$$

$$\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_2^0}(-k_{14}\frac{n\pi}{a}\frac{m\pi}{b})sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}),$$

$(VI)*:$

$$0=\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_1}k_{13}\cos(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{w}(-k_{13}\frac{n\pi}{a})\cos(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})+$$

$$+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_1^0}(-k_{14}\frac{n\pi}{a}\frac{m\pi}{b})\cos(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})+$$

$$\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_2^0}(-k_8(\frac{n\pi}{a})^2-k_7(\frac{m\pi}{b})^2-k_9)\cos(\frac{n\pi x}{a})sin(\frac{m\pi y}{b}).$$

Now we calculate the coefficients $\alpha_{n,m}^{\Psi_1}, \alpha_{n,m}^{\Psi_2}, \alpha_{n,m}^{w}, \alpha_{n,m}^{\Omega_3}, \alpha_{n,m}^{\Omega_1^0}, \alpha_{n,m}^{\Omega_2^0}$, where $m, n \in Z$, using the orthogonality of the trigonometric functions. We consider the trigonometric functions $\cos(\frac{\nu\pi x}{a})$ and $sin(\frac{\varpi\pi y}{b})$, for all $\nu,\ \varpi \in Z$, over the domain $0 \le x \le a,\ 0 \le y \le b.$

For each $\nu,\ \varpi \in N$, we do the following:

Taking the integral $\iint\limits_{\substack{0\le x\le a,\\ 0\le y\le b.}} (I)*\cos(\frac{\nu\pi x}{a})sin(\frac{\varpi\pi y}{b})dxdy$, we obtain:

$$[-k_{15}\alpha_{m,n}(\frac{n\pi}{a})]_{\substack{m=\nu\\ n=\varpi.}}(\frac{a}{2})(\frac{b}{2}) = [\alpha_{n,m}^{\Psi_1}(-k_1(\frac{n\pi}{a})^2 - k_2(\frac{m\pi}{b})^2 - k_3) + \alpha_{n,m}^{\Psi_2}(-k_{10}\frac{n\pi}{a}\frac{m\pi}{b}) + a_{n,m}^{w}(k_{11}\frac{n\pi}{a}) +$$

$$a_{n,m}^{\Omega_3}(-k_{12}\frac{m\pi}{b}) + a_{n,m}^{\Omega_2^0}k_{13}]_{\substack{m=\nu\\ n=\varpi.}}(\frac{a}{2})(\frac{b}{2}),$$

Taking the integral $\iint\limits_{\substack{0\le x\le a,\\ 0\le y\le b.}} (VI)*\cos(\frac{\nu\pi x}{a})sin(\frac{\varpi\pi y}{b})dxdy$, we obtain:

$$0 = [\alpha_{n,m}^{\Psi_1}k_{13} + \alpha_{n,m}^{w}(-k_{13}\frac{n\pi}{a}) + \alpha_{n,m}^{\Omega_1^0}(-k_{14}\frac{n\pi}{a}\frac{m\pi}{b}) + \alpha_{n,m}^{\Omega_2^0}(-k_8(\frac{n\pi}{a})^2 - k_7(\frac{m\pi}{b})^2 - k_9)]_{\substack{m=\nu,\\ n=\varpi.}}(\frac{a}{2})(\frac{b}{2}),$$

Taking the integral $\iint\limits_{\substack{0\le x\le a,\\ 0\le y\le b.}} (II)*sin(\frac{\nu\pi x}{a})\cos(\frac{\varpi\pi y}{b})dxdy$, we obtain:

$$[-k_{15}\alpha_{n,m}(\frac{m\pi}{b})]_{\substack{m=\nu,\\ n=\varpi.}}(\frac{a}{2})(\frac{b}{2}) = [\alpha_{n,m}^{\Psi_1}(-k_{10}\frac{n\pi}{a}\frac{m\pi}{b}) + \alpha_{n,m}^{\Psi_2}(-k_1(\frac{m\pi}{b})^2 - k_2(\frac{n\pi}{a})^2 - k_3) +$$

$$\alpha_{n,m}^{w}k_{11}\frac{m\pi}{b} - \alpha_{n,m}^{\Omega_3}(-k_{12})\frac{n\pi}{a} + \alpha_{n,m}^{\Omega_1^0}(-k_{13})]_{\substack{m=\nu,\\ n=\varpi.}}(\frac{a}{2})(\frac{b}{2}),$$

Taking the integral $\iint\limits_{\substack{0\le x\le a,\\ 0\le y\le b.}} (VI)*sin(\frac{\nu\pi x}{a})\cos(\frac{\varpi\pi y}{b})dxdy$, we obtain:

$$0 = [\alpha_{n,m}^{\Psi_2}k_{13} + \alpha_{n,m}^{w}k_{13}\frac{m\pi}{b} + \alpha_{n,m}^{\Omega_1^0}(-k_7(\frac{n\pi}{a})^2 - k_8(\frac{m\pi}{b})^2 - k_9) + \alpha_{n,m}^{\Omega_2^0}(-k_{14}\frac{n\pi}{a}\frac{m\pi}{b})]_{\substack{m=\nu,\\ n=\varpi.}}(\frac{a}{2})(\frac{b}{2}),$$

Taking the integral $\iint\limits_{\substack{0\le x\le a,\\ 0\le y\le b.}} (III)*sin(\frac{\nu\pi x}{a})sin(\frac{\varpi\pi y}{b})dxdy$, we obtain:

$$[-k_{17}\alpha_{m,n}]_{\substack{m=\nu,\\ n=\varpi.}}(\frac{a}{2})(\frac{b}{2}) = [\alpha_{n,m}^{\Psi_1}(k_{11}\frac{n\pi}{a}) + \alpha_{n,m}^{\Psi_2}(k_{11}\frac{m\pi}{b}) + \alpha_{n,m}^{w}(-k_4)((\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2) + \alpha_{n,m}^{\Omega_1^0}k_{13}\frac{m\pi}{b} + \alpha_{n,m}^{\Omega_2^0}(-k_{13}\frac{n\pi}{a})]_{\substack{m=\nu,\\ n=\varpi.}}(\frac{a}{2})(\frac{b}{2}),$$

Taking the integral $\iint\limits_{\substack{0\le x\le a,\\ 0\le y\le b.}} (IV)*\cos(\frac{\nu\pi x}{a})\cos(\frac{\varpi\pi y}{b})dxdy$, we obtain:

$$0 = [\alpha_{n,m}^{\Psi_1}(-k_{12}\frac{m\pi}{b}) + \alpha_{n,m}^{\Psi_2}k_{12}\frac{n\pi}{a} + \alpha_{n,m}^{\Omega_3}(-k_5((\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2) - k_6)]_{\substack{m=\nu,\\ n=\varpi.}}(\frac{a}{2})(\frac{b}{2}),$$

Note that $P|_{\partial B} = 0$.

Thus for each $n,m \in N$ we obtain the following **Linear System of Algebraic Equations:**

$(I)**$:

$$k_{15}\alpha_{n,m}\frac{n\pi}{a} = [\alpha_{n,m}^{\Psi_1}(-k_1(\frac{n\pi}{a})^2 - k_2(\frac{m\pi}{b})^2 - k_3) + \alpha_{n,m}^{\Psi_2}(-k_{10}\frac{n\pi}{a}\frac{m\pi}{b}) + \alpha_{n,m}^{w}(k_{11}\frac{n\pi}{a}) + \alpha_{n,m}^{\Omega_3}(-k_{12}\frac{m\pi}{b}) + \alpha_{n,m}^{\Omega_2^0}k_{13}],$$

$(II)**$:

$$k_{15}\alpha_{n,m}\frac{m\pi}{b} = [\alpha_{n,m}^{\Psi_1}(-k_{10})\frac{n\pi}{a}\frac{m\pi}{b} + \alpha_{n,m}^{\Psi_2}(-k_1(\frac{m\pi}{b})^2 - k_2(\frac{n\pi}{a})^2 - k_3) + \alpha_{n,m}^{w}k_{11}\frac{m\pi}{b} - \alpha_{n,m}^{\Omega_3}(-k_{12}\frac{n\pi}{a}) + \alpha_{n,m}^{\Omega_1^0}(-k_{13})],$$

$(III)**$:

$$k_{17}\alpha_{n,m} = [\alpha_{n,m}^{\Psi_1}k_{11}\frac{n\pi}{a} + \alpha_{n,m}^{\Psi_2}k_{11}\frac{m\pi}{b} + \alpha_{n,m}^{w}(-k_4)((\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2) + \alpha_{n,m}^{\Omega_1^0}k_{13}\frac{m\pi}{b} + \alpha_{n,m}^{\Omega_2^0}(-k_{13}\frac{n\pi}{a})],$$

$(IV)^{**}$:

$$0=[\alpha_{n,m}^{\Psi_1}(-k_{12}\frac{m\pi}{b})+\alpha_{n,m}^{\Psi_2}k_{12}\frac{n\pi}{a}+\alpha_{n,m}^{\Omega_3}(-k_5((\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2)-k_6)],$$

$(V)^{**}$:

$$0=[\alpha_{n,m}^{\Psi_2}k_{13}+\alpha_{n,m}^{w}k_{13}\frac{m\pi}{b}+\alpha_{n,m}^{\Omega_1^0}(-k_7(\frac{n\pi}{a})^2-k_8(\frac{m\pi}{b})^2-k_9)+\alpha_{n,m}^{\Omega_2^0}(-k_{14}\frac{n\pi}{a}\frac{m\pi}{b})],$$

$(VI)^{**}$:

$$0=[\alpha_{n,m}^{\Psi_1}k_{13}+\alpha_{n,m}^{w}(-k_{13}\frac{n\pi}{a})+\alpha_{n,m}^{\Omega_1^0}(-k_{14}\frac{n\pi}{a}\frac{m\pi}{b})+\alpha_{n,m}^{\Omega_2^0}(-k_8(\frac{n\pi}{a})^2-k_7(\frac{m\pi}{b})^2-k_9)],\ \forall n,m\in N.$$

, where $(\alpha_{n,m}^{\Psi_1},\alpha_{n,m}^{\Psi_2},\alpha_{n,m}^{w},\alpha_{n,m}^{\Omega_3},\alpha_{n,m}^{\Omega_1^0},\alpha_{n,m}^{\Omega_2^0})$ is the unknown vector. Where for each $(m,n)\in N\times N-\{(1,1)\}$, we have a ***homogeneous system*** of six equations of six variables. The coefficient matrix of the system is nonsingular, so the vector solution is the trivial vector (zero solution). These linear systems have a coefficient matrix and a vector of independent terms which can vary the nature of the solution, such as in the case for which $w_{|\partial B}=0$, $p=0$, & $p_{,1}=0$, $p_{,2}=0$, $t_{,1}=t_{,2}=0$ over the domain, the vector of independent terms is zero and so the system is homogeneous for each $(m,n)\in N\times N$, so if the determinant of coefficient matrix is nonzero, the vector solution is trivial. If we would like to have a solution for the system $(I)^{**}$- $(VI)^{**}$ in which the functions consisting of more than one term. For example, this is possible if the matrix of the homogeneous system is singular, and of course, depends of the linear independence of column vectors of the matrix of coefficients, and essentially of input data of the plate (the material elastic constants $\alpha,\beta,\varepsilon,\gamma,\lambda,\mu$ and the size of the plate h, a & b). For $m=n=1$, $(I)^{**}$- $(VI)^{**}$ is not ***homogeneous system***.

Linear System $(Case:(m,n)=(1,1))$

$$k_{15}\frac{\pi}{a}=\alpha_{1,1}^{\Psi_1}(-k_1(\frac{\pi}{a})^2-k_2(\frac{\pi}{b})^2-k_3)+\alpha_{1,1}^{\Psi_2}(-k_{10}\frac{\pi}{a}\frac{\pi}{b})+\alpha_{1,1}^{w}(k_{11}\frac{\pi}{a})+\alpha_{1,1}^{\Omega_3}(-k_{12}\frac{\pi}{b})+\alpha_{1,1}^{\Omega_2^0}k_{13},$$

$$k_{15}\frac{\pi}{b}=\alpha_{1,1}^{\Psi_1}(-k_{10}\frac{\pi}{a}\frac{\pi}{b})+\alpha_{1,1}^{\Psi_2}(-k_1(\frac{\pi}{b})^2-k_2(\frac{\pi}{a})^2-k_3)+\alpha_{1,1}^{w}k_{11}\frac{\pi}{b}+\alpha_{1,1}^{\Omega_3}k_{12}\frac{\pi}{a}+\alpha_{1,1}^{\Omega_1^0}(-k_{13}),$$

$$k_{17}=\alpha_{1,1}^{\Psi_1}(k_{11}\frac{\pi}{a})+\alpha_{1,1}^{\Psi_2}(k_{11}\frac{\pi}{b})+\alpha_{1,1}^{w}(-k_4)((\frac{\pi}{a})^2+(\frac{\pi}{b})^2)+\alpha_{1,1}^{\Omega_1^0}k_{13}\frac{\pi}{b}+\alpha_{1,1}^{\Omega_2^0}(-k_{13}\frac{\pi}{a}),$$

$$0=\alpha_{1,1}^{\Psi_1}(-k_{12}\frac{\pi}{b})+\alpha_{1,1}^{\Psi_2}k_{12}\frac{\pi}{a}+\alpha_{1,1}^{\Omega_3}(-k_5((\frac{\pi}{a})^2+(\frac{\pi}{b})^2)-k_6),$$

$$0=\alpha_{1,1}^{\Psi_2}k_{13}+\alpha_{1,1}^{w}k_{13}\frac{\pi}{b}+\alpha_{1,1}^{\Omega_1^0}(-k_7(\frac{\pi}{a})^2-k_8(\frac{\pi}{b})^2-k_9)+\alpha_{1,1}^{\Omega_2^0}(-k_{14}\frac{\pi}{a}\frac{\pi}{b}),$$

$$0=\alpha_{1,1}^{\Psi_1}k_{13}+\alpha_{1,1}^{w}(-k_{13}\frac{\pi}{a})+\alpha_{1,1}^{\Omega_1^0}(-k_{14}\frac{\pi}{a}\frac{\pi}{b})+\alpha_{1,1}^{\Omega_2^0}(-k_8(\frac{\pi}{a})^2-k_7(\frac{\pi}{b})^2-k_9). \qquad (VII)$$

3.2.2 Analytical Solution of Reissner Model

We consider the pressure defined by

$$P(x,y)=sin(\frac{\pi x}{a})sin(\frac{\pi y}{b}),\quad (x,y)\in[0,a]\times[0,b]$$

into of the bending system for Reissner **(2.65)- (2.67). S**ubstituting the derivatives ∂_x **(2.65)** and ∂_y **(2.66)** into **(2.67)** we obtain the following system:

$$\frac{5hE}{12(1+\nu)}\Delta w+D[\Psi_{1,111}+\Psi_{2,222}]+\frac{D(1-\nu)}{2}(\Psi_{1,122}+\Psi_{2,112})+\frac{D(1+\nu)}{2}(\Psi_{2,112}+\Psi_{1,122})-\frac{5hE}{12(1+\nu)}(w_{,11}+w_{,22})+$$
$$\frac{h^2\nu}{10(1-\nu)}(P_{,11}+P_{,22})=-P$$

$$\frac{5hE}{12(1+\nu)}\Delta w+D[\Psi_{1,111}+\Psi_{2,222}+\Psi_{1,122}+\Psi_{2,112}]-\frac{5hE}{12(1+\nu)}\Delta w+\frac{h^2\nu}{10(1-\nu)}\Delta P=-P$$

$$D[\Delta(\Psi_{1,1}+\Psi_{2,2})]+\frac{h^2\nu}{10(1-\nu)}\Delta P=-P \qquad \textbf{(3.78)}$$

Substituting ∂_x **(2.65)** and putting $(\Psi_{1,1}+\Psi_{2,2})$ from **(2.67)** into **(3.78):**

$$D\Delta(\frac{-12(1+\nu)}{5hE}P-\Delta w)+\frac{h^2\nu}{10(1-\nu)}\Delta P=-P$$

$$D\Delta^2 w+D\frac{12(1+\nu)}{5hE}\Delta P-\frac{h^2\nu}{10(1-\nu)}\Delta P=P$$

By definition of $\mathrm{D}=\frac{\mathrm{Eh}^3}{12(1-\nu^2)}$:

$$\Delta^2 w=\frac{1}{D}P+[\frac{12(1-\nu^2)}{Eh^3}\frac{h^2\nu}{10(1-\nu)}-\frac{12(1+\nu)}{5Eh}]\Delta P$$

$$\Delta^2 w=\frac{1}{D}P+\frac{12(1+\nu)}{10Eh}[\nu-2]\Delta P$$

From definition of the elastic constant E, given by E = $\dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$, we obtain one formula for the maximum vertical deflection of the middle plane of the plate corresponding to the Reissner Model:

$$\Delta^2 w_R = \frac{P}{D} + \frac{h^2(\nu-2)}{10D(1-\nu)}\Delta P \tag{3.79}$$

From **(3.72)** we obtain the following equation:

$$w(x,y) = \sum_{m,n=1}^{\infty} \alpha^w_{n,m} sin(\frac{n\pi x}{a}) sin(\frac{m\pi y}{b})$$

$$\Delta^2 w(x,y) = \sum_{m,n=1}^{\infty} \alpha^w_{m,n}((\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2)^2 sin(\frac{n\pi x}{a}) sin(\frac{m\pi y}{b})$$

$$= \frac{1}{D} sin(\frac{\pi x}{a}) sin(\frac{\pi y}{b}) - \frac{h^2(\nu-2)}{10D(1-\nu)}((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2) sin(\frac{\pi x}{a}) sin(\frac{\pi y}{b})$$

$$= \frac{1}{D}[1 - \frac{h^2(\nu-2)}{10D(1-\nu)}((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2)] sin(\frac{\pi x}{a}) sin(\frac{\pi y}{b})$$

Then integrating over the domain $[0,a]\times[0,b]$ we obtain the following equations:

$$\alpha^w_{n,m} = 0, \forall(n,m) \neq (1,1).$$

For $n = m = 1$ we obtain the following:

$$\alpha^w_{1,1} = \frac{1}{D}[-\frac{h^2(\nu-2)}{10(1-\nu)} + \frac{1}{(\frac{\pi}{a})^2 + (\frac{\pi}{b})^2}]\frac{1}{((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2)}, \tag{3.80}$$

and

$$w(x,y) = \frac{1}{D((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2)}[\frac{1}{(\frac{\pi}{a})^2 + (\frac{\pi}{b})^2} + \frac{h^2(2-\nu)}{10(1-\nu)}] sin(\frac{\pi x}{a}) sin(\frac{\pi y}{b}). \tag{3.81}$$

Table of Correspondence between A. C. ERINGEN and NOWACKI W. moduli elasticity

Notations for ERINGEN version for the elastic microstructure constants:

$$\alpha^E, \mu^E, \lambda^E, \kappa^E, \beta^E, \gamma^E.$$

Notations for NOWACKI version for the elastic microstructure constants:

$$\alpha, \mu, \lambda, \beta, \gamma, \varepsilon. \quad \textbf{(3.82)}$$

Sets of elastic constants:

NOWACKI:	α	μ	λ	β	γ	ε
ERINGEN:	$\frac{\kappa^E}{2}$	$\mu^E + \frac{\kappa^E}{2}$	λ^E	α^E	$\frac{\beta^E + \gamma^E}{2}$	$\frac{\gamma^E - \beta^E}{2}$

We obtain the Eringen version in terms of the Nowacki version (used in the Steinberg-Reissner Model) for the elastic microstructure constants:

$$\kappa^E = 2\alpha, \quad \mu^E = \mu - \alpha, \quad \lambda^E = \lambda, \quad \alpha^E = \beta, \quad \gamma^E = \gamma + \varepsilon, \quad \beta^E = \gamma - \varepsilon. \quad \textbf{(3.83)}$$

Other important correspondence is given by $h_E = \frac{h}{2}$, between the thickness of the plate of Eringen Theory h_E and the thickness of the plate of Steinberg- Reissner Theory $h^P (= h)$.

3.2.3 Analytical Solution of Eringen Model I ($e_{33} = 0$)

For the *bending* in the case $e_{33} = 0$, the two-dimensional problem of micro- elasticity (or the extensional motions of plates) consider the equations from **(2.87)** to **(2.89)** and using the equivalences **(3.83)** we obtain the following system of equations**:**

$$\frac{2}{3} h_E^3 (\lambda + \mu - \alpha) \Psi_{k,lk} + \frac{2}{3} h_E^3 (\mu - \alpha + 2\alpha) \Psi_{l,kk} - 2h_E (\mu - \alpha + 2\alpha)(\Psi_l + w_{,l}) +$$
$$2h_E 2\alpha (w_{,l} + \mathcal{E}_{lk} \Omega_k^0) = 0,$$
$$(\mu - \alpha)\Psi_{k,k} + (\mu - \alpha + 2\alpha) w_{,kk} + 2\alpha \mathcal{E}_{kl} \Omega_{l,k}^0 + \frac{P}{2h_E} = 0,$$
$$(\beta + \gamma - \varepsilon)\Omega_{k,lk}^0 + (\gamma + \varepsilon)\Omega_{l,kk}^0 + 2\alpha \mathcal{E}_{kl} (\Psi_k - w_{,k}) - 4\alpha \Omega_l^0 = 0.$$

for $l:1,2$ & $k:1,2$.

Expanding, we obtain five equations:

$$\frac{2}{3}h_E^3(\lambda+\mu-\alpha)(\Psi_{1,11}+\Psi_{2,12})+\frac{2}{3}h_E^3(\mu+\alpha)(\Psi_{1,11}+\Psi_{1,22})-2h_E(\mu+\alpha)(\Psi_1+w_{,1})+4\alpha h_E(w_{,1}+\Omega_2^0)=0,$$
$$\frac{2}{3}h_E^3(\lambda+\mu-\alpha)(\Psi_{1,21}+\Psi_{2,22})+\frac{2}{3}h_E^3(\mu+\alpha)(\Psi_{2,11}+\Psi_{2,22})-2h_E(\mu+\alpha)(\Psi_2+w_{,2})+4\alpha h_E(w_{,2}-\Omega_1^0)=0,$$
$$(\mu-\alpha)(\Psi_{1,1}+\Psi_{2,2})+(\mu+\alpha)(w_{,11}+w_{,22})+2\alpha(-\Omega_{l,2}^0+\Omega_{2,1}^0)+\frac{P}{2h_E}=0,$$
$$(\beta+\gamma-\varepsilon)(\Omega_{1,11}^0+\Omega_{2,12}^0)+(\gamma+\varepsilon)(\Omega_{1,11}^0+\Omega_{1,22}^0)-2\alpha(\Psi_2-w_{,2})-4\alpha\Omega_1^0=0,$$
$$(\beta+\gamma-\varepsilon)(\Omega_{1,21}^0+\Omega_{2,22}^0)+(\gamma+\varepsilon)(\Omega_{2,11}^0+\Omega_{2,22}^0)+2\alpha(\Psi_1-w_{,1})-4\alpha\Omega_2^0=0,$$

replacing the series **(3.70)** to **(3.76)** we obtain the following series:

$$\frac{2}{3}h_E^3(\lambda+\mu-\alpha)(\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})^2\alpha_{n,m}^{\Psi_1}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})(\frac{m\pi}{b})\alpha_{n,m}^{\Psi_2}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}))+$$

$$\frac{2}{3}h_E^3(\mu+\alpha)(\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})^2\alpha_{n,m}^{\Psi_1}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}-(\frac{m\pi}{b})^2\alpha_{n,m}^{\Psi_1}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}))-$$
$$2h_E(\mu+\alpha)(\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_1}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}\frac{n\pi}{a}\alpha_{n,m}^{w}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}))+$$
$$4\alpha h_E(\sum_{m,n=1}^{\infty}\frac{n\pi}{a}\alpha_{n,m}^{w}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_2^0}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}))=0,$$

$$\frac{2}{3}h_E^3(\lambda+\mu-\alpha)(\sum_{m,n=1}^{\infty}-\frac{n\pi}{a}\frac{m\pi}{b}\alpha_{n,m}^{\Psi_1}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}-(\frac{m\pi}{b})^2\alpha_{n,m}^{\Psi_2}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}))+$$

$$\frac{2}{3}h_E^3(\mu+\alpha)(\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})^2\alpha_{n,m}^{\Psi_2}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}-(\frac{m\pi}{b})^2\alpha_{n,m}^{\Psi_2}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}))-$$
$$2h_E(\mu+\alpha)(\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_2}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}\frac{m\pi}{b}\alpha_{n,m}^{w}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}))+$$
$$4\alpha h_E(\sum_{m,n=1}^{\infty}\frac{m\pi}{b}\alpha_{n,m}^{w}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})-\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_1^0}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}))=0,$$

$$(\mu-\alpha)(\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})\alpha_{n,m}^{\Psi_1}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}-(\frac{m\pi}{b})\alpha_{n,m}^{\Psi_2}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}))+$$
$$(\mu+\alpha)(\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})^2\alpha_{n,m}^{w}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}-(\frac{m\pi}{b})^2\alpha_{n,m}^{w}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}))+$$
$$2\alpha(\sum_{m,n=1}^{\infty}(\frac{m\pi}{b})\alpha_{n,m}^{\Omega_1^0}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})\alpha_{n,m}^{\Omega_2^0}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}))+\frac{sin(\frac{\pi x}{a})sin(\frac{\pi y}{b})}{2h_E}=0,$$

$$(\beta+\gamma-\varepsilon)(\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})^2\alpha_{n,m}^{\Omega_1^0}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})(\frac{m\pi}{b})\alpha_{n,m}^{\Omega_2^0}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}))+$$
$$(\gamma+\varepsilon)(\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})^2\alpha_{n,m}^{\Omega_1^0}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}-(\frac{m\pi}{b})^2\alpha_{n,m}^{\Omega_1^0}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}))-$$
$$2\alpha(\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_2}SC-\sum_{m,n=1}^{\infty}\frac{m\pi}{b}\alpha_{n,m}^{w}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}))-4\alpha\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_1^0}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})=0,$$

$$(\beta+\gamma-\varepsilon)(\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})(\frac{m\pi}{b})\alpha_{n,m}^{\Omega_1^0}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}-(\frac{m\pi}{b})^2\alpha_{n,m}^{\Omega_2^0}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}))$$
$$+(\gamma+\varepsilon)(\sum_{m,n=1}^{\infty}-(\frac{n\pi}{a})^2\alpha_{n,m}^{\Omega_2^0}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})+\sum_{m,n=1}^{\infty}-(\frac{m\pi}{b})^2\alpha_{n,m}^{\Omega_2^0}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}))$$
$$+2\alpha(\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Psi_1}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})-\sum_{m,n=1}^{\infty}\frac{n\pi}{a}\alpha_{n,m}^{w}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}))-4\alpha\sum_{m,n=1}^{\infty}\alpha_{n,m}^{\Omega_2^0}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})=0.$$

By the orthogonality of the trigonometric functions we obtain the following linear system:

$$\begin{aligned}&[-\frac{2}{3}h_E^3(\lambda+\mu-\alpha)(\frac{n\pi}{a})^2-\frac{2}{3}h_E^3(\mu+\alpha)((\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2)-2h_E(\mu+\alpha)]\alpha_{n,m}^{\Psi_1}+\\&[-\frac{2}{3}\frac{n\pi}{a}\frac{m\pi}{b}h_E^3(\lambda+\mu-\alpha)]\alpha_{n,m}^{\Psi_2}+[-2h_E(\mu+\alpha)\frac{n\pi}{a}+4\alpha h_E\frac{n\pi}{a}]\alpha_{n,m}^{w}+[4\alpha h_E]\alpha_{n,m}^{\Omega_2^0}=0,\end{aligned}\tag{3.84}$$

$$\begin{aligned}&[-\frac{2}{3}h_E^3(\lambda+\mu-\alpha)\frac{n\pi}{a}\frac{m\pi}{b}]\alpha_{n,m}^{\Psi_1}+[-(\frac{m\pi}{b})^2\frac{2}{3}h_E^3(\lambda+\mu-\alpha)-\frac{2}{3}h_E^3(\mu+\alpha)((\frac{n\pi}{a})^2+\\&(\frac{m\pi}{b})^2)-2h_E(\mu+\alpha)]\alpha_{n,m}^{\Psi_2}+[-2h_E(\mu+\alpha)\frac{m\pi}{b}+4\alpha h_E\frac{m\pi}{b}]\alpha_{n,m}^{w}+[-4\alpha h_E]\alpha_{n,m}^{\Omega_1^0}=0,\end{aligned}\tag{3.85}$$

$$[-(\mu-\alpha)\frac{n\pi}{a}]\alpha_{n,m}^{\Psi_1}+[-\frac{m\pi}{b}(\mu-\alpha)]\alpha_{n,m}^{\Psi_2}+[-(\mu+\alpha)((\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2)]\alpha_{n,m}^{w}+$$

$$[2\alpha\frac{m\pi}{b}]\alpha_{n,m}^{\Omega_1^0}+[-2\alpha\frac{n\pi}{a}]\alpha_{n,m}^{\Omega_2^0}+\frac{sin(\frac{\pi x}{a})sin(\frac{\pi y}{b})}{2h_E}\delta_{n,1}\delta_{m,1}=0, \qquad \textbf{(3.86)}$$

$$[-2\alpha]\alpha_{n,m}^{\Psi_2}+[2\alpha\frac{m\pi}{b}]\alpha_{n,m}^{w}+[-(\beta+\gamma-\varepsilon)(\frac{n\pi}{a})^2-(\gamma+\varepsilon)((\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2)$$
$$-4\alpha]\alpha_{n,m}^{\Omega_1^0}+[-\frac{n\pi}{a}\frac{m\pi}{b}(\beta+\gamma-\varepsilon)]\alpha_{n,m}^{\Omega_2^0}=0, \qquad \textbf{(3.87)}$$

$$,[2\alpha]\alpha_{n,m}^{\Psi_1}+[-2\alpha\frac{n\pi}{a}]\alpha_{n,m}^{w}+[-(\beta+\gamma-\varepsilon)\frac{n\pi}{a}\frac{m\pi}{b}]\alpha_{n,m}^{\Omega_1^0}+$$
$$[-(\beta+\gamma-\varepsilon)(\frac{m\pi}{b})^2-(\gamma+\varepsilon)((\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2)-4\alpha]\alpha_{n,m}^{\Omega_2^0}=0 \qquad \textbf{(3.88)}$$

3.2.3.1 Analytical Solution for the Elastic (Classic) Materials

Taking ∂_x **(2.90)** we obtain the following**:**

$$\frac{(h^p)^2}{12}(\lambda+\mu)(\Psi_{1,111}+\Psi_{2,112})+\frac{(h^p)^2}{12}\mu(\Psi_{1,111}+\Psi_{1,122})-\mu(\Psi_{1,1}+w_{,11})=0,$$

∂_y **(2.91)** :

$$\frac{(h^p)^2}{12}(\lambda+\mu)(\Psi_{1,122}+\Psi_{2,222})+\frac{(h^p)^2}{12}\mu(\Psi_{2,112}+\Psi_{2,222})-\mu(\Psi_{2,2}+w_{,22})=0,$$

Putting **(2.92)** into ∂_x **(2.90)** $+\partial_y$ **(2.91):**

$$\frac{(h^p)^2}{12}(\lambda+\mu)\Delta(\Psi_{1,1}+\Psi_{2,2})+\frac{(h^p)^2}{12}\mu\Delta(\Psi_{1,1}+\Psi_{2,2})+\frac{P}{h^p}=0$$

$$\Delta(\Psi_{1,1}+\Psi_{2,2})=\frac{-12P}{(\lambda+2\mu)(h^p)^3} \qquad \textbf{(3.89)}$$

Δ **(2.92):**

$$\Delta(\Psi_{1,1}+\Psi_{2,2})+\Delta(\Delta w)=\frac{-\Delta P}{\mu h^p} \qquad \textbf{(3.90)}$$

Putting **(3.89)** into **(3.90)**, we obtain a formula for the maximum vertical deflection of the middle plane of the plate corresponding to the Eringen Model I for the Elastic (Classic) Materials:

$$\Delta^2 w_E^{0,I} = \frac{-\Delta P}{\mu h^p} + \frac{12P}{(\lambda+2\mu)(h^p)^3} \quad \textbf{(3.91)}$$

$$\Delta^2 w_E^{o,I} = \frac{-\Delta P}{h^p \mu} + \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)^2}\frac{P}{D} \quad \textbf{(3.92)}$$

$$w_E(x,y) = \sum_{m,n=1}^{\infty} \alpha_{m,n}^{w_E} sin(\frac{n\pi x}{a}) sin(\frac{m\pi y}{b})$$

$$w_E^{0,I}(x,y) = \frac{1}{((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2)}[\frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)^2}\frac{1}{D((\frac{\pi}{a})^2+(\frac{\pi}{b})^2)} + \frac{1}{h^p\mu}] sin(\frac{\pi x}{a}) sin(\frac{\pi y}{b}). \quad \textbf{(3.93)}$$

3.2.4 Analytical Solution of Eringen Model II $(t_{33} = 0)$

Substituting the equivalences **(3.83)** into the corresponding equations of *bending* **(2.95)** to **(2.97),** we obtain the following system:

$$\frac{1}{2}\frac{(h^p)^3}{12}(\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}\frac{1}{1-\frac{\lambda}{2(\lambda+\mu)}} - 2\alpha)\Psi_{k,lk} + \frac{1}{2}\frac{(h^p)^3}{12}(\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}\frac{1}{1+\frac{\lambda}{2(\lambda+\mu)}} + 2\alpha)\Psi_{l,kk}$$

$$-h^p(\mu-\alpha)w_l - h^p(\mu+\alpha)\Psi_l + 2\alpha h^p \mathcal{E}_{lk}\Omega_k^0 = 0,$$

$$(\mu-\alpha)\Psi_{k,k} + (\mu+\alpha)w_{,kk} + 2\alpha\mathcal{E}_{kl}\Omega_{l,k}^0 + \frac{P}{h^p} = 0,$$

$$(\alpha+\beta)\Omega_{k,lk}^0 + \gamma\Omega_{l,kk}^0 + 2\alpha\mathcal{E}_{kl}(\Psi_k - w_{,k}) - 4\alpha\Omega_l^0 = 0.$$

for $l:1,2$ & $k:1,2.$

Expanding, we obtain six equations:

$$\frac{(h^p)^3}{12}(\frac{\mu(3\lambda+2\mu)}{(\lambda+2\mu)}-\alpha)(\Psi_{1,11}+\Psi_{2,12})+\frac{(h^p)^3}{12}(\mu+\alpha)(\Psi_{1,11}+\Psi_{1,22})-h^p(\mu-\alpha)w_{,1} \\ -h^p(\mu+\alpha)\Psi_1+2\alpha h^p\Omega_2^0=0, \quad \textbf{(3.94)}$$

$$\frac{(h^p)^3}{12}(\frac{\mu(3\lambda+2\mu)}{(\lambda+2\mu)}-\alpha)(\Psi_{1,21}+\Psi_{2,22})+\frac{(h^p)^3}{12}(\mu+\alpha)(\Psi_{2,11}+\Psi_{2,22})-h^p(\mu-\alpha)w_{,2} \\ -h^p(\mu+\alpha)\Psi_2-2\alpha h^p\Omega_1^0=0, \quad \textbf{(3.95)}$$

$$(\mu-\alpha)(\Psi_{1,1}+\Psi_{2,2})+(\mu+\alpha)(w_{,11}+w_{,22})-2\alpha\Omega_{1,2}^0+\frac{P}{h^p}=0, \quad \textbf{(3.96)}$$

$$(\mu-\alpha)(\Psi_{1,1}+\Psi_{2,2})+(\mu+\alpha)(w_{,11}+w_{,22})+2\alpha\Omega_{2,1}^0+\frac{P}{h^p}=0, \quad \textbf{(3.97)}$$

$$(\alpha+\beta)(\Omega_{1,11}^0+\Omega_{2,12}^0)+\gamma(\Omega_{1,11}^0+\Omega_{1,22}^0)+2\alpha(w_{,2}-\Psi_2)-4\alpha\Omega_1^0=0, \quad \textbf{(3.98)}$$

$$(\alpha+\beta)(\Omega_{1,21}^0+\Omega_{2,22}^0)+\gamma(\Omega_{2,11}^0+\Omega_{2,22}^0)+2\alpha(\Psi_{,1}-w_{,1})-4\alpha\Omega_2^0=0. \quad \textbf{(3.99)}$$

So we obtain one important property: $\Omega_{2,1}^0=-\Omega_{1,2}^0$.

Replacing the series **(3.70)** to **(3.76)** into **(3.94)** to **(3.99)** we obtain the following series:

$$\sum_{m,n=1}^{\infty}\{\frac{(h^p)^3}{12}(\frac{\mu(3\lambda+2\mu)}{(\lambda+2\mu)}-\alpha)[-(\frac{n\pi}{a})^2\alpha_{n,m}^{\Psi_1}-(\frac{n\pi}{a})(\frac{m\pi}{b})\alpha_{n,m}^{\Psi_2}]+\frac{(h^p)^3}{12}(\mu+\alpha) \\ [-(\frac{n\pi}{a})^2-(\frac{m\pi}{b})^2]\alpha_{n,m}^{\Psi_1}-h^p(\mu-\alpha)[\frac{n\pi}{a}]\alpha_{n,m}^{w}-h^p(\mu+\alpha)\alpha^{\Psi_1}+2\alpha h^p\alpha_{n,m}^{\Omega_2^0}\}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})=0, \quad \textbf{(3.100)}$$

$$\sum_{m,n=1}^{\infty}\{\frac{(h^p)^3}{12}(\frac{\mu(3\lambda+2\mu)}{(\lambda+2\mu)}-\alpha)[-(\frac{n\pi}{a})(\frac{m\pi}{b})\alpha_{n,m}^{\Psi_1}-(\frac{m\pi}{b})^2\alpha_{n,m}^{\Psi_2}]+ \\ \frac{(h^p)^3}{12}(\mu+\alpha)[-(\frac{n\pi}{a})^2-(\frac{m\pi}{b})^2]\alpha_{n,m}^{\Psi_2}-h^p(\mu-\alpha)[\frac{n\pi}{a}]\alpha_{n,m}^{w}-h^p(\mu+\alpha)\alpha_{n,m}^{\Psi_2}-2\alpha h^p\alpha_{n,m}^{\Omega_1^0}\}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})=0, \quad \textbf{(3.101)}$$

$$\sum_{m,n=1}^{\infty}\{(\mu-\alpha)[-\frac{n\pi}{a}\alpha_{n,m}^{\Psi_1}-\frac{m\pi}{b}\alpha_{n,m}^{\Psi_2}]+(\mu+\alpha)[-(\frac{n\pi}{a})^2-(\frac{m\pi}{b})^2]\alpha_{n,m}^{w} \\ -2\alpha[-\frac{m\pi}{b}]\alpha_{n,m}^{\Omega_1^0}\}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})+\frac{\delta_{n,1}\delta_{m,1}P}{h^p}=0, \quad \textbf{(3.102)}$$

$$\sum_{m,n=1}^{\infty}\{(\mu-\alpha)[-\frac{n\pi}{a}\alpha_{n,m}^{\Psi_1}-\frac{m\pi}{b}\alpha_{n,m}^{\Psi_2}]+(\mu+\alpha)[-(\frac{n\pi}{a})^2-(\frac{m\pi}{b})^2]\alpha_{n,m}^{w}+ \\ 2\alpha[-\frac{n\pi}{a}]\alpha_{n,m}^{\Omega_2^0}\}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})+\frac{\delta_{n,1}\delta_{m,1}P}{h^p}=0, \tag{3.103}$$

$$\sum_{m,n=1}^{\infty}\{(\alpha+\beta)[-(\frac{n\pi}{a})^2\alpha_{n,m}^{\Omega_1^0}-(\frac{n\pi}{a})(\frac{m\pi}{b})\alpha_{n,m}^{\Omega_2^0}]+\gamma[-(\frac{n\pi}{a})^2-(\frac{m\pi}{b})^2]\alpha_{n,m}^{\Omega_1^0} \\ +2\alpha[\frac{m\pi}{b}\alpha_{n,m}^{w}-\alpha_{n,m}^{\Psi_2}]-4\alpha\alpha_{n,m}^{\Omega_1^0}\}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})=0, \tag{3.104}$$

$$\sum_{m,n=1}^{\infty}\{(\alpha+\beta)[-(\frac{n\pi}{a})(\frac{m\pi}{b})\alpha_{n,m}^{\Omega_1^0}-(\frac{m\pi}{b})^2\alpha_{n,m}^{\Omega_2^0}]+\gamma[-(\frac{n\pi}{a})^2-(\frac{m\pi}{b})^2]\alpha_{n,m}^{\Omega_2^0}+ \\ 2\alpha[\alpha_{n,m}^{\Psi_1}-\frac{n\pi}{a}\alpha_{n,m}^{w}]-4\alpha\alpha_{n,m}^{\Omega_2^0}\}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})=0. \tag{3.105}$$

3.2.4.1 Analytical Solution for the Elastic (Classic) Materials

Taking ∂_x **(2.99)** $+\partial_x$ **(2.100)** we obtain the following equation:

$$\frac{(h^p)^3}{12}\frac{\mu(3\lambda+2\mu)}{(\lambda+2\mu)}(\Psi_{1,111}+\Psi_{2,121}+\Psi_{1,212}+\Psi_{2,222})+\frac{(h^p)^3}{12}\mu(\Psi_{1,111}+\Psi_{1,221}+\Psi_{2,112}+\Psi_{2,222}) \\ -h^p\mu(w_{,11}+w_{,22})-h^p\mu(\Psi_{1,1}+\Psi_{2,2})=0$$

$$\frac{(h^p)^3}{12}\frac{\mu(3\lambda+2\mu)}{(\lambda+2\mu)}\Delta(\Psi_{1,1}+\Psi_{2,2})+\frac{(h^p)^3}{12}\mu\Delta(\Psi_{1,1}+\Psi_{2,2})- \\ h^p\mu(w_{,11}+w_{,22})-h^p\mu(\Psi_{1,1}+\Psi_{2,2})=0, \tag{3.106}$$

From **(2.101)** and **(3.106)** we obtain the following**:**

$$\frac{(h^p)^3}{12}\mu(\frac{3\lambda+2\mu}{\lambda+2\mu}+1)\Delta(\Psi_{1,1}+\Psi_{2,2})-h^p\mu(\frac{-P}{\mu h^p})=0 \tag{3.107}$$

$$\frac{4(h^p)^3}{12}\mu(\frac{\lambda+\mu}{\lambda+2\mu})\Delta(\frac{-P}{\mu h^p}-\Delta w)+P=0$$

$$\Delta^2 w=\frac{3}{(h^p)^3\mu}\frac{\lambda+2\mu}{\lambda+\mu}P-\frac{1}{\mu h^p}\Delta P$$

$$\Delta^2 w=\frac{1}{\mu h^p}[\frac{3}{(h^p)^2}\frac{\lambda+2\mu}{\lambda+\mu}-\Delta]P,$$

we obtain one formula for the maximum vertical deflection of the middle plane of the plate corresponding to the Eringen Model II for the Elastic (Classic) Materials:

$$\Delta^2 w_E^{0,II} = -\frac{1}{\mu h^p}\Delta P + \frac{P}{D}, \quad \text{where} \quad D = \frac{\mu(\lambda+\mu)(h^p)^3}{3(\lambda+2\mu)} \qquad \textbf{(3.108)}$$

Note: $\mu = G$.

From $w_E^{0,II}(x,y) = \sum_{m,n=1}^{\infty} \alpha_{m,n}^w sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})$, we obtain: **(3.109)**

$$\Delta^2 w(x,y) = \sum_{m,n=1}^{\infty} \alpha_{m,n}^w ((\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2)^2 sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})$$

$$\frac{P}{D} - \frac{1}{\mu h^p}\Delta P = \frac{1}{D} sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b}) + \frac{1}{h^p \mu}((\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2) sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})$$

$$\alpha_{1,1}^w ((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2)^2 \frac{a}{2}\frac{b}{2} = [\frac{1}{D} + \frac{1}{h^p\mu}((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2)]\frac{a}{2}\frac{b}{2},$$

where $\alpha_{m,n}^w = 0, \forall (n,m) \neq (1,1)$.

$$\alpha_{1,1}^w = \frac{1}{((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2)}[\frac{1}{D((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2)} + \frac{1}{h^p\mu}] \qquad \textbf{(3.110)}$$

Then:

$$w_E^{0,II}(x,y) = \frac{1}{((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2)}[\frac{1}{D\left((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2\right)} + \frac{1}{h^p\mu}] sin(\frac{\pi x}{a})sin(\frac{\pi y}{b}). \qquad \textbf{(3.111)}$$

CHAPTER 4

COMPARISON OF THE THEORIES

4.1 Comparison Based on the Analytical Solutions

Consider the expression for $\Delta^2 w$ of the *vertical deflection of the middle plane of the plate* "w"obtained after the reduction (restriction: $\alpha = \beta = \gamma = 0$) of the Eringen Model II $(t_{33} = 0)$ and of the Reissner Model, respectively **:**

$$\nabla^4 w_E^{0,II} = -\frac{1}{\mu h^p}\nabla^2 P + \frac{P}{D}$$

and

$$\Delta^2 w_R = \frac{(h^p)^2(\nu - 2)}{10D(1-\nu)}\Delta P + \frac{P}{D}.$$

And then we proceed to compare the coefficients of ΔP:

From the definition of D, $D = \dfrac{E(h^p)^3}{12(1-\nu^2)}$ we obtain $E = \dfrac{12(1-\nu^2)}{(h^p)^3}D$ and substituting

into μ: $\mu = G = \dfrac{E}{2(1+\nu)} = \dfrac{12(1-\nu^2)D}{2(1+\nu)(h^p)^3} = \dfrac{6(1-\nu)D}{(h^p)^3}$, and then:

$\dfrac{-1}{\mu h^p} = \dfrac{-(h^p)^2}{6(1-\nu)D} = \dfrac{(h^p)^2}{(1-\nu)D}\left(\dfrac{-1}{6}\right)$. Finally we compare the coefficients of ΔP for

the reduction of Eringen II and Reissner:

$$\frac{\text{Coefficient of } \Delta P \text{ in the formula for } \Delta^2 w_R}{\text{Coefficient of } \Delta P \text{ in the formula for } \Delta^2 w_E^{0,II}} = \frac{\dfrac{(h^p)^2(\nu-2)}{10D(1-\nu)}}{\dfrac{-1}{\mu h^p}} = \frac{3(2-\nu)}{5} = 1.2 - 0.6\nu$$

$$= 0.6(2-\nu).$$

The slope of the straight line is -0.6 with intercept 1.2. In our experiments we consider $\nu = 0.339$ with $\lambda = 2186$ and $\mu = 1029$, and then we have:

$$\frac{\text{Coefficient of } \Delta P \text{ in the formula for } \Delta^2 w_R}{\text{Coefficient of } \Delta P \text{ in the formula for } \Delta^2 w_E^{0,II}} = 0.6(2 - 0.339) = 0.99 \cong 1.$$

i.e.:

$$-\frac{1}{\mu h^p}\Delta P \cong \frac{(h^p)^2(\nu-2)}{10D(1-\nu)}\Delta P$$

and then

$$-\frac{1}{\mu h^p}\Delta P + \frac{P}{D} \cong \frac{(h^p)^2(\nu-2)}{10D(1-\nu)}\Delta P + \frac{P}{D}$$

$$\Delta^2 w_E^{0,II} \cong \Delta^2 w_R \ .$$

Simple comparison of these two equations leads to the conclusion that difference between two solutions should be small.

4.2 Comparison Based on Numerical Computations

We compare the maximum vertical deflection of the middle plane of the plate- w corresponding to the six models of elastic plates. The microstructure is determined by additional elastic material constants, related with asymmetric elasticity property. We study effect of microstructure at different levels: 1%, 10%, 100% , which means that the elasticity constants related with the microstructure **(3.82)** α, β, γ & ε are divided by 1, 10, 100, or others numbers respectively. Note that elastic contacts λ, μ, related with macrostructure, we keep the same values. Also, we compare the rotation vector (with axis: x_α) in the middle plane of the plate- Ψ_α, the displacement of the middle plane along x_α-axis: U_α, the micro-rotation vector in the middle plane- Ω_α^0, and the instant rate of micro-rotation change long x_3-axis: Ω_3 corresponding to the models of elastic plates.

We consider the quadratic plate, which is made of *syntactic foam* (lightweight engineered foam consisting of glass hollow spheres embedded in a resin matrix), has the following elastic constants for our material in consideration **[24]:** $h = 0.1$ **m,** $E = 2758$ **Mpa**, $G = 1029.1$ **Mpa**, $\nu = 0.34$, $l_t = 0.065$ **mm**, $l_b = 0.033$, $N^2 = 0.1$, $\Psi = 1.5$ and then we find the other constants such as $D = 0.2598749$ and also the values of the k_i as a functions of the elasticity constants related with the microstructure.

Exist important relations between the k_i:

$$k_{14} = k_7 - k_8, \quad k_4 = -k_3, \quad k_{13} = 0.5k_9, \quad k_1 = k_2 + k_{10}, \quad k_6 = 2k_{12}.$$

The following graphs shows different values of $a(h)^{-1}$ on the X-axis. The value of "h" remains constant (equal to 0.1 in our case) while the value of "a", the side of the square begins to increase from 0.5 to 3.

For our case we consider the following values of the elastic material constants corresponding for the technical constants $h, E, G, \nu, l_t, l_b, N$ and Ψ:

$$\alpha = 114, \quad \lambda = 2186, \quad \beta = -2.898, \quad \gamma = 4.3, \quad \varepsilon = 0.135, \quad \mu = 1029.1.$$

Remarks:

The effect of microstructure is considered in 100% if the elasticity constants related with the microstructure α, β, γ & ε are multiplied by 1,

The effect of microstructure is considered in 10% if the elasticity constants related with the microstructure α, β, γ & ε are multiplied by 0.1,

The effect of microstructure is considered in 1% if the elasticity constants related with the microstructure α, β, γ & ε are multiplied by 0.01, etc.

"The continued loss" of the asymmetric part (as illustrated in the Steinberg-Reissner Model), as the elasticity constants related with the microstructure converge to zero makes such a maximum vertical deflection ratio between two models go for 1, as is the case of Steinberg-Reissner over Reissner, moreover this happens in a neighborhood of $\frac{a}{h} = 5$.

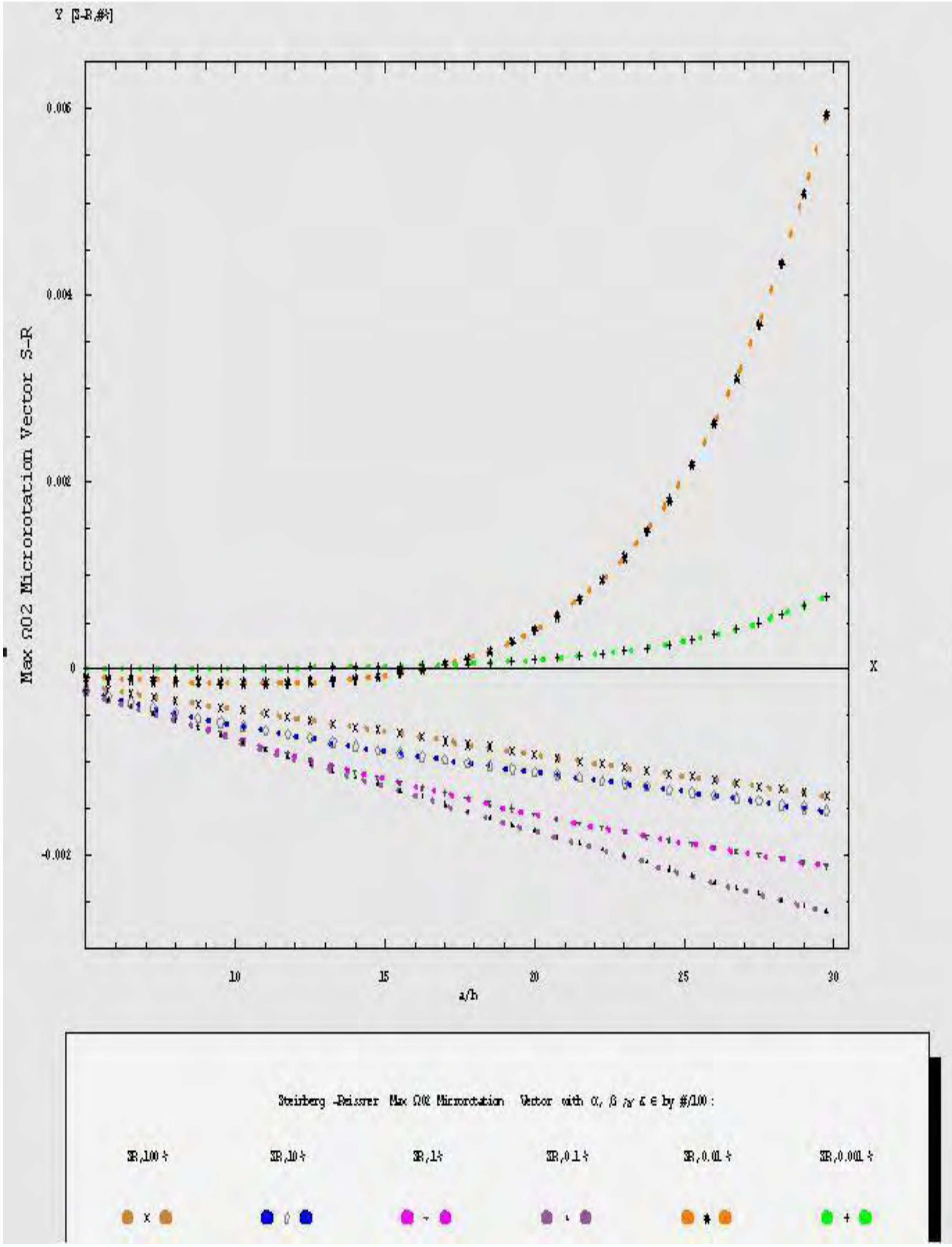

Figure 4.1 Maximum of Ω_2^0 Corresponding to Steinberg-Reissner Model

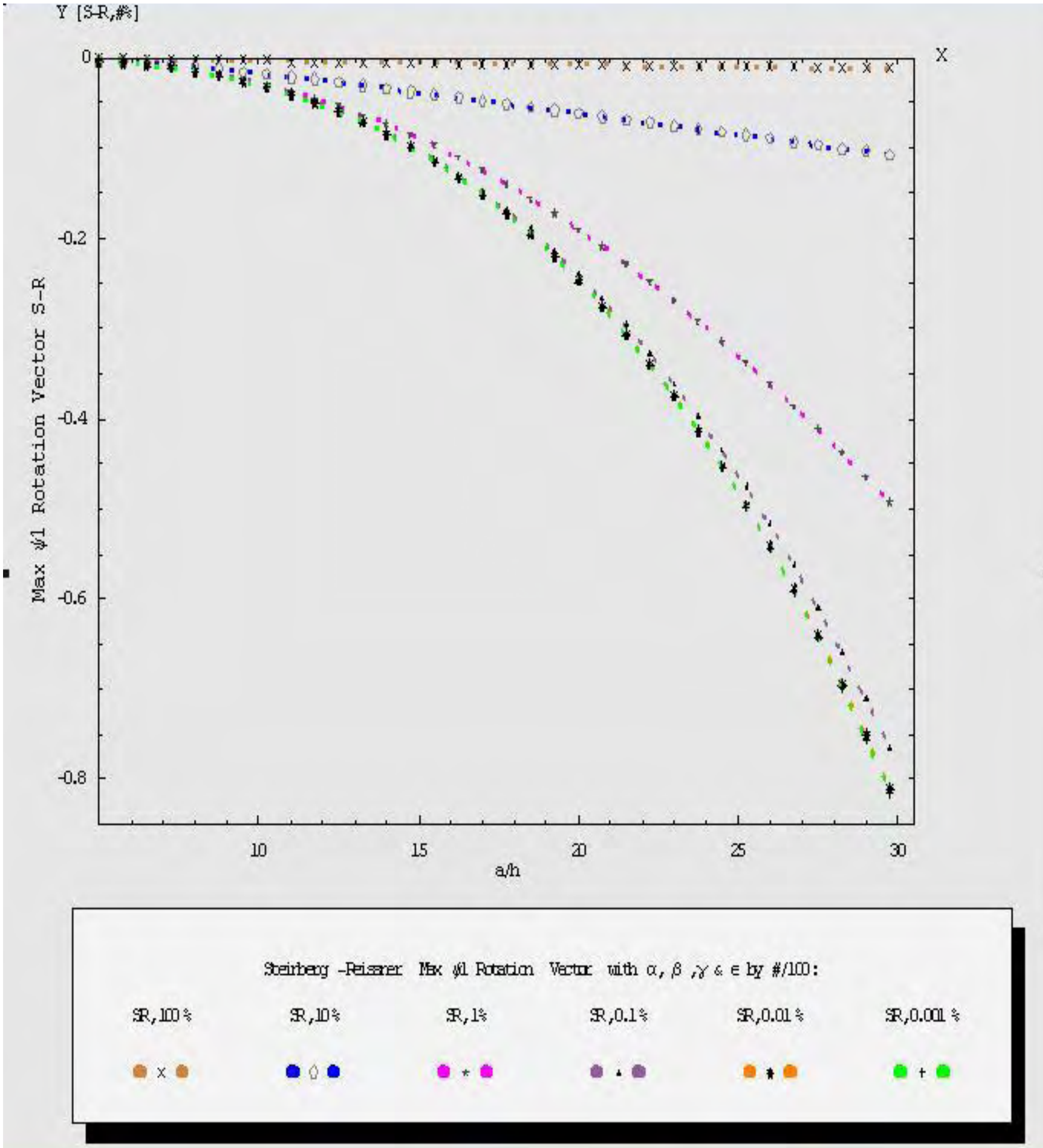

Figure 4.2 Maximum of Ψ_1 Corresponding to Steinberg-Reissner Model

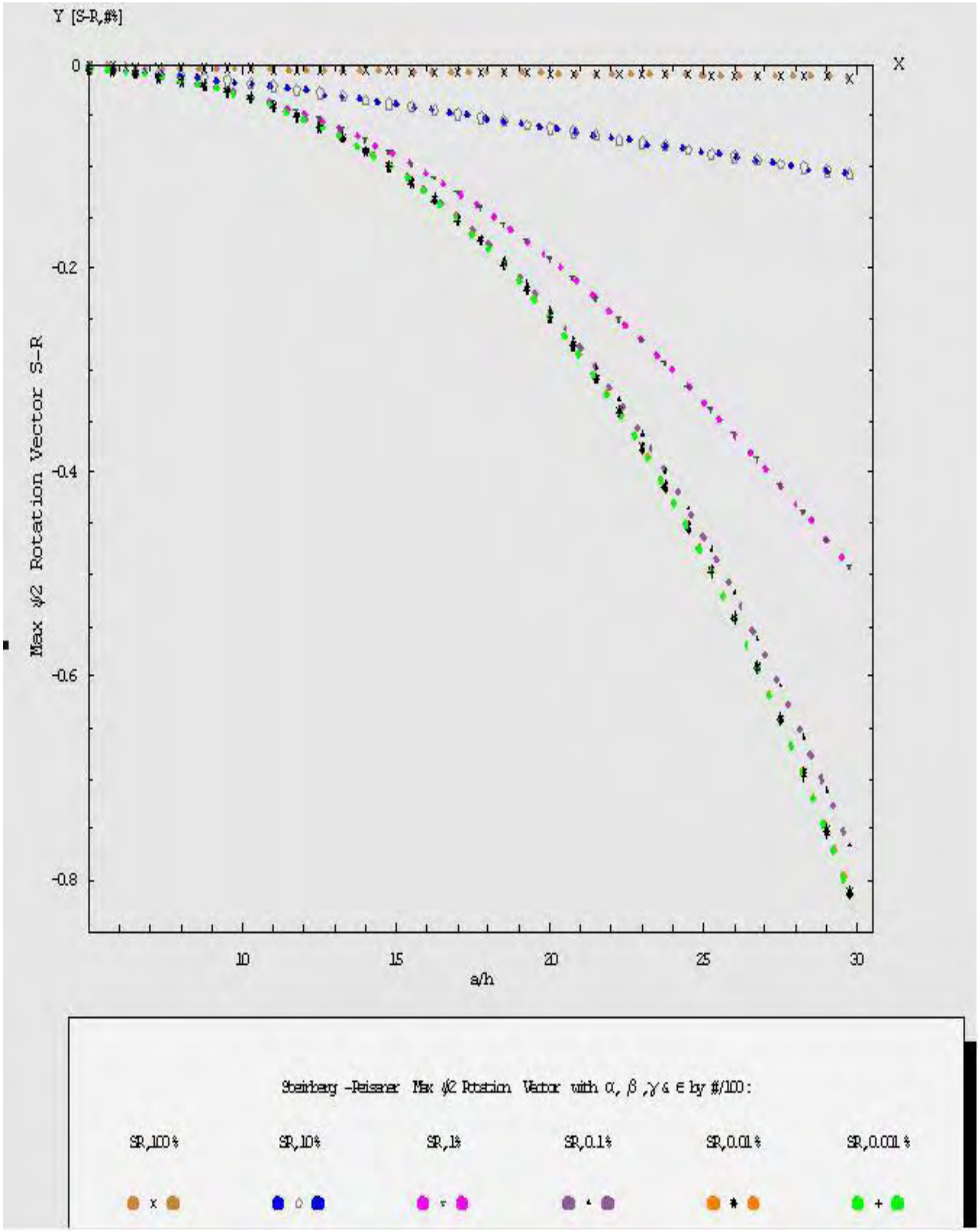

Figure 4.3 Maximum of Ψ_2 Corresponding to Steinberg-Reissner Model

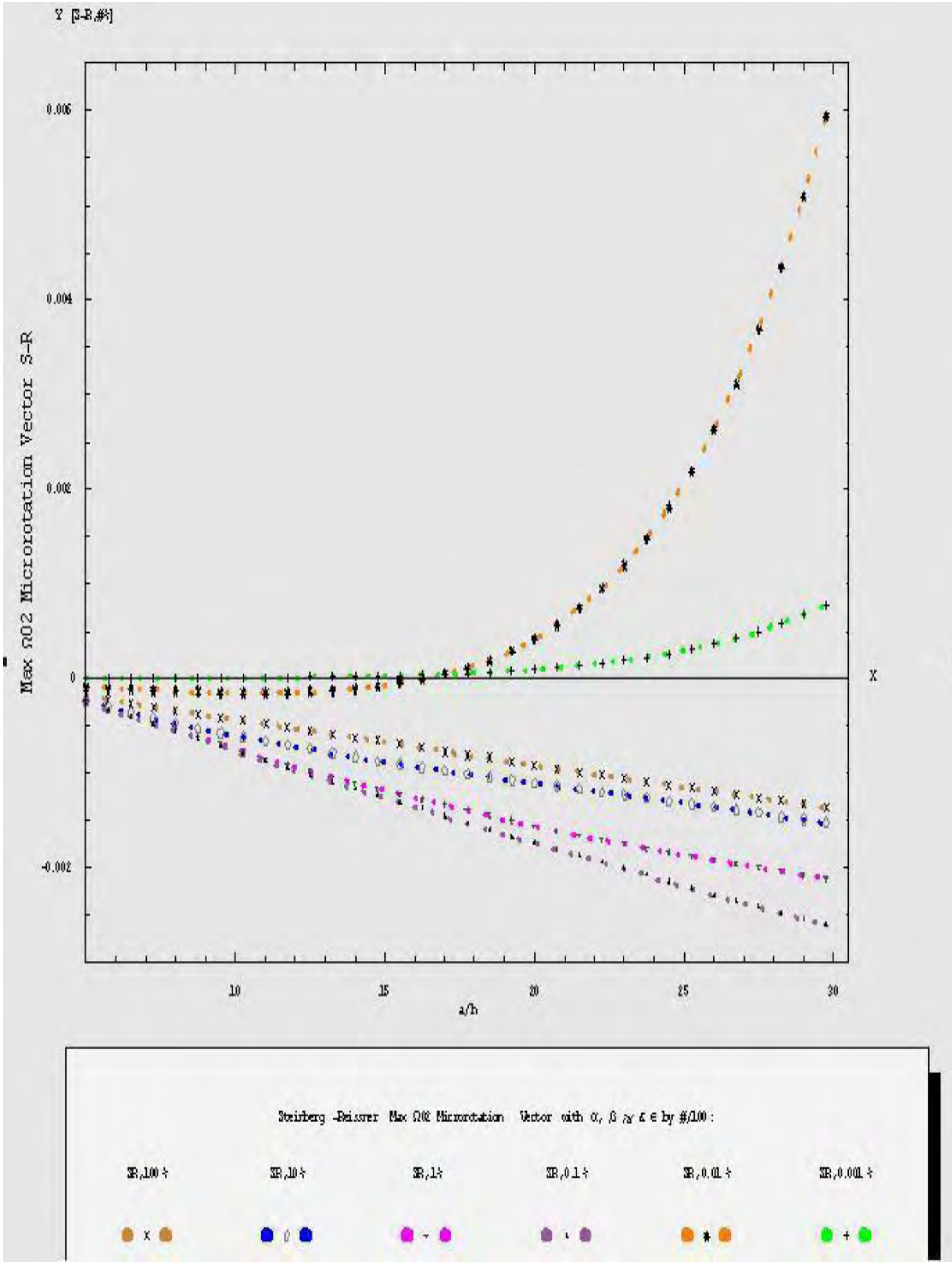

Figure 4.4 Maximum of Ψ_1 Corresponding to Eringen Model II

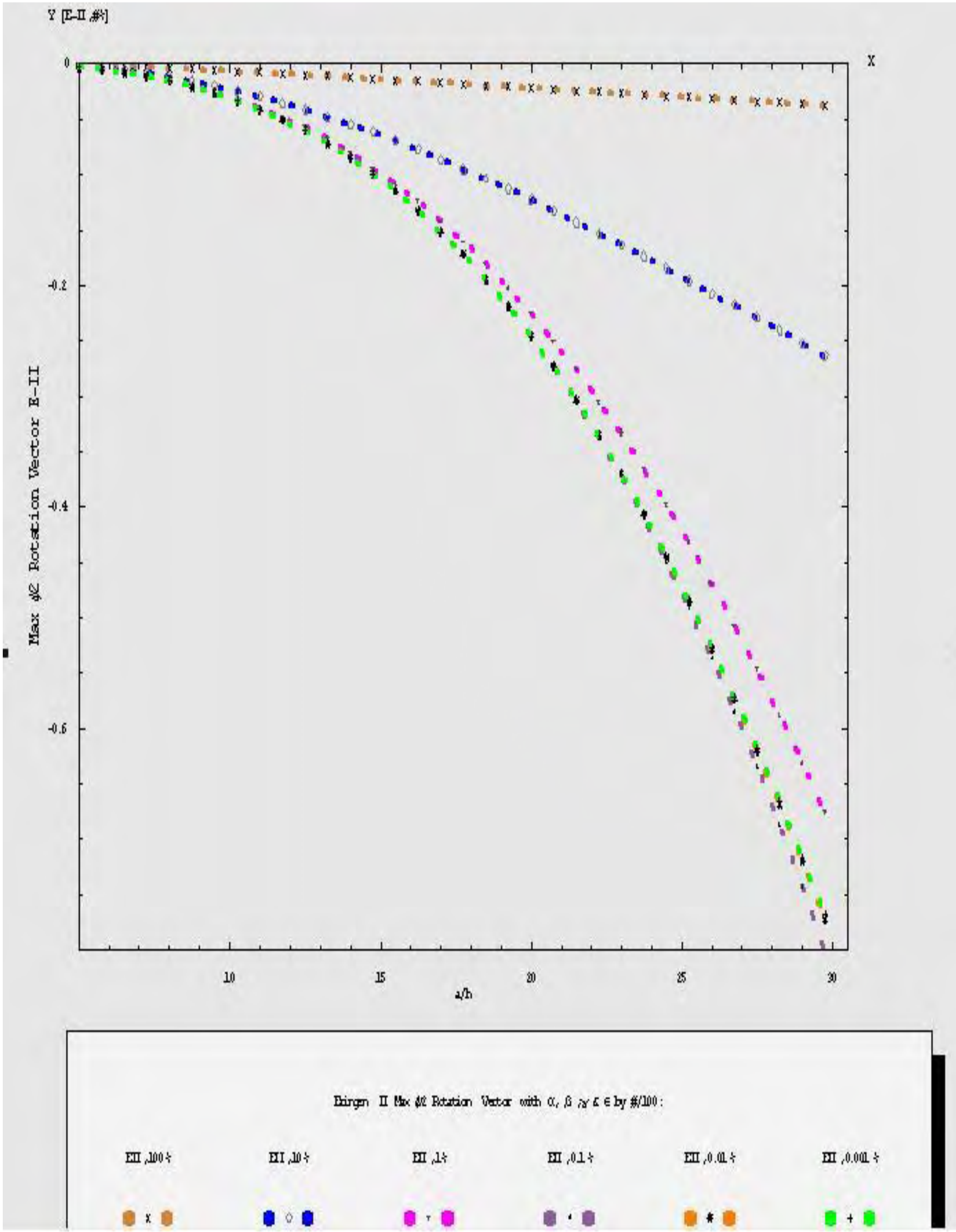

Figure 4.5 Maximum of Ψ_2 Corresponding to Eringen Model II

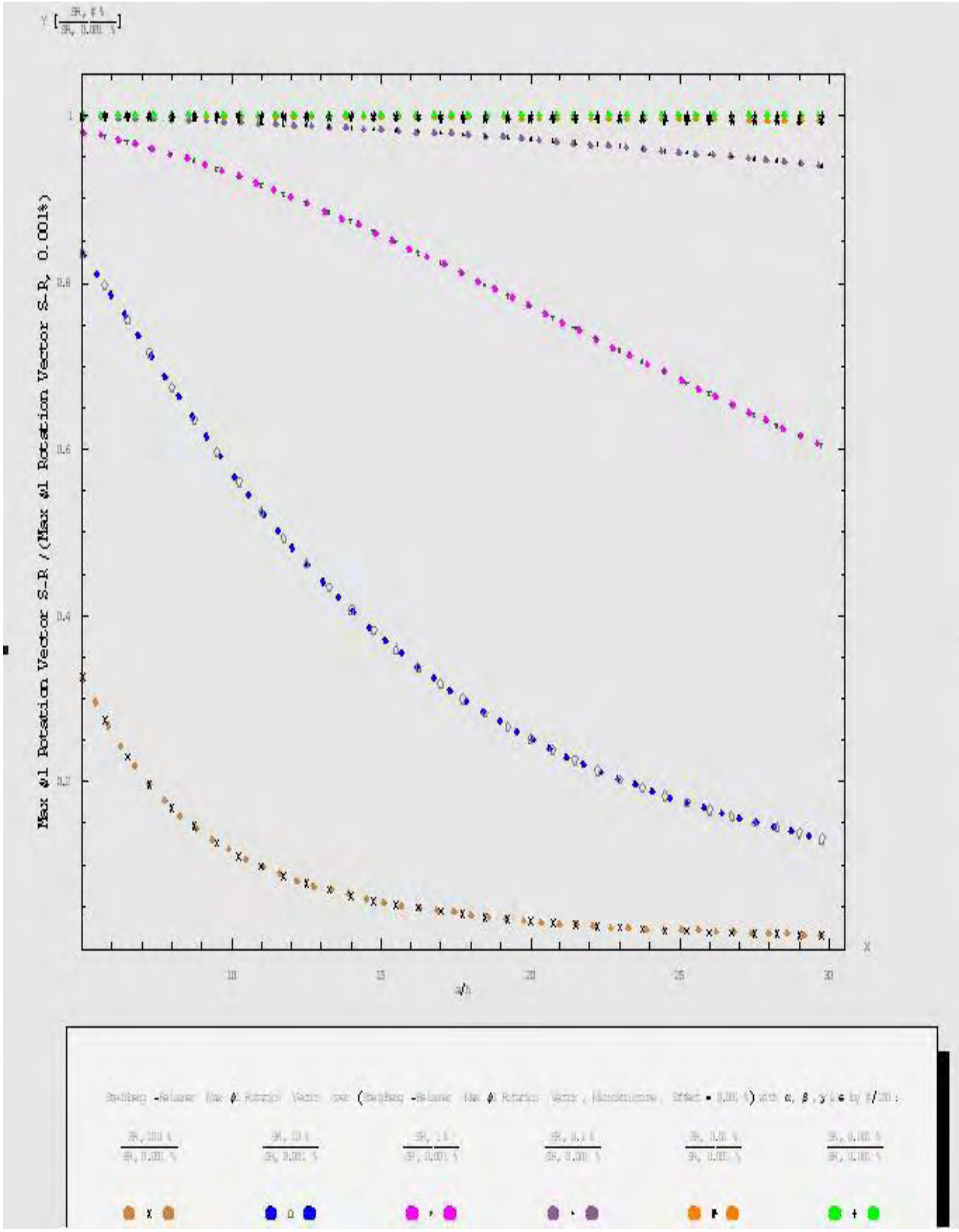

Figure 4.6 Maximum of Ψ_1 Corresponding to Steinberg-Reissner Model over Maximum of Ψ_1 Corresponding to Steinberg-Reissner Model with 0.001% (level of Microstructure)

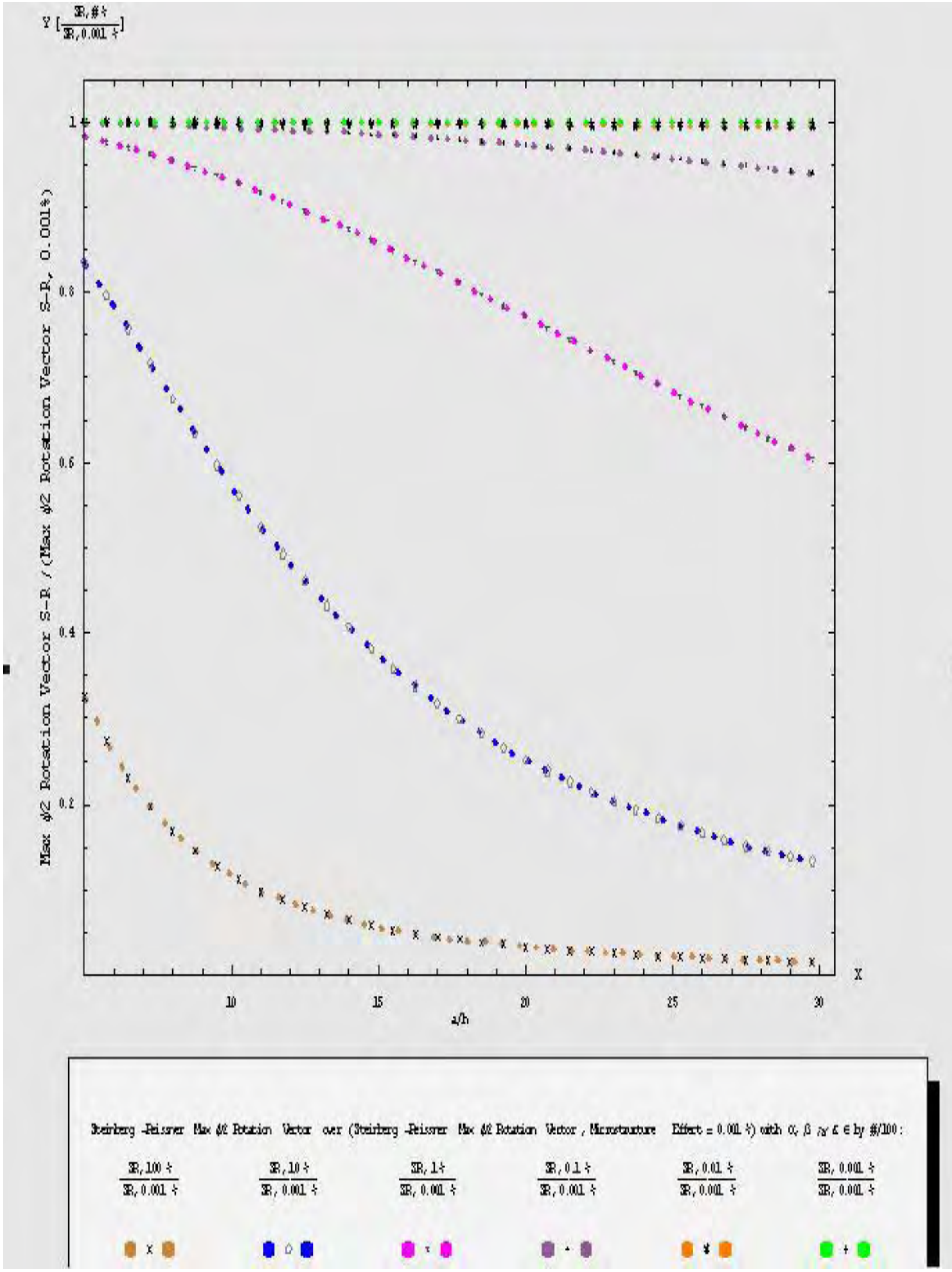

Figure 4.7 Maximum of Ψ_2 Corresponding to Steinberg-Reissner Model over Maximum of Ψ_2 Corresponding to Steinberg-Reissner Model with 0.001% (level of Microstructure)

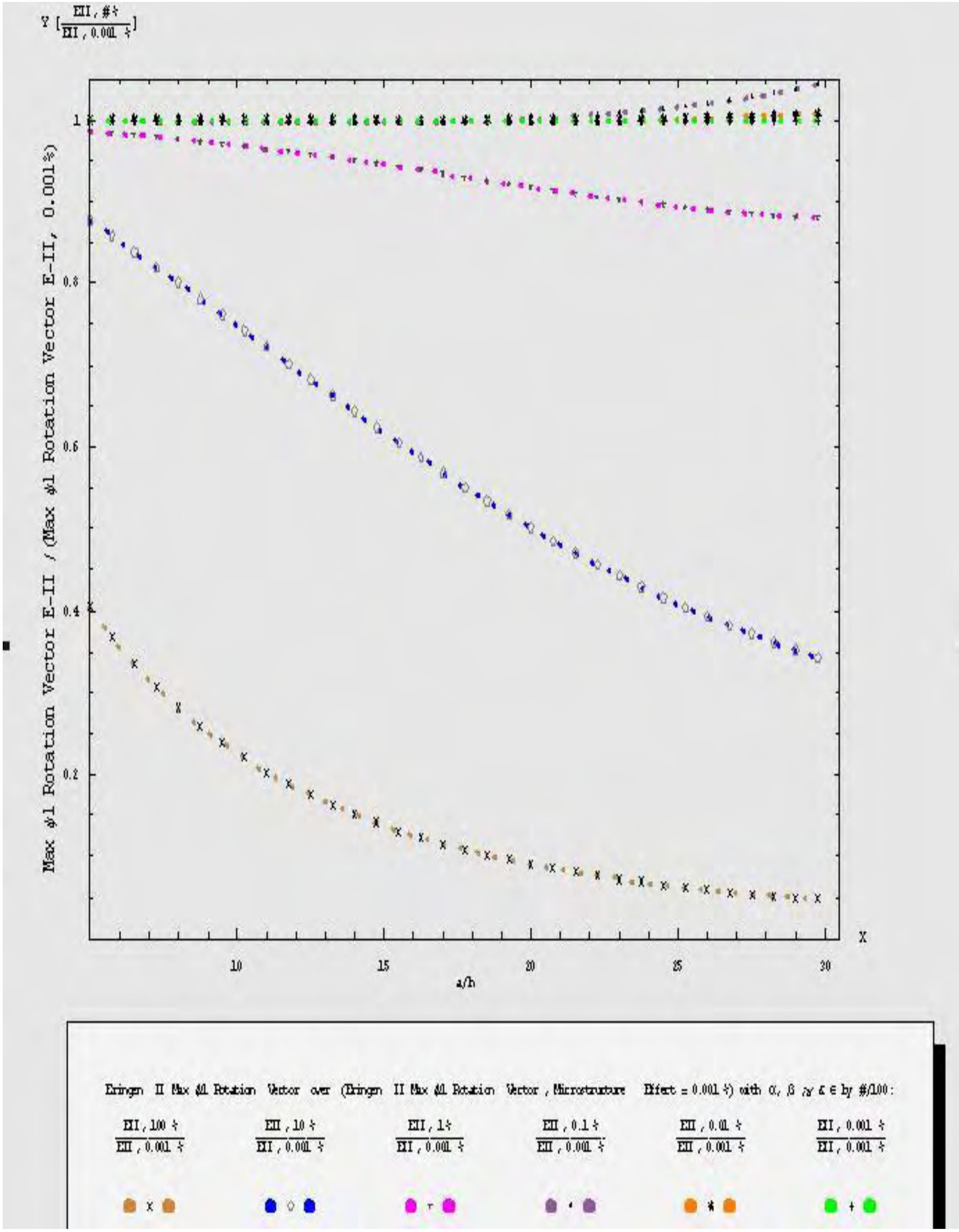

Figure 4.8 Maximum of Ψ_1 Corresponding to Eringen Model II over Maximum of Ψ_1 Corresponding to Eringen Model II with 0.001% (level of Microstructure)

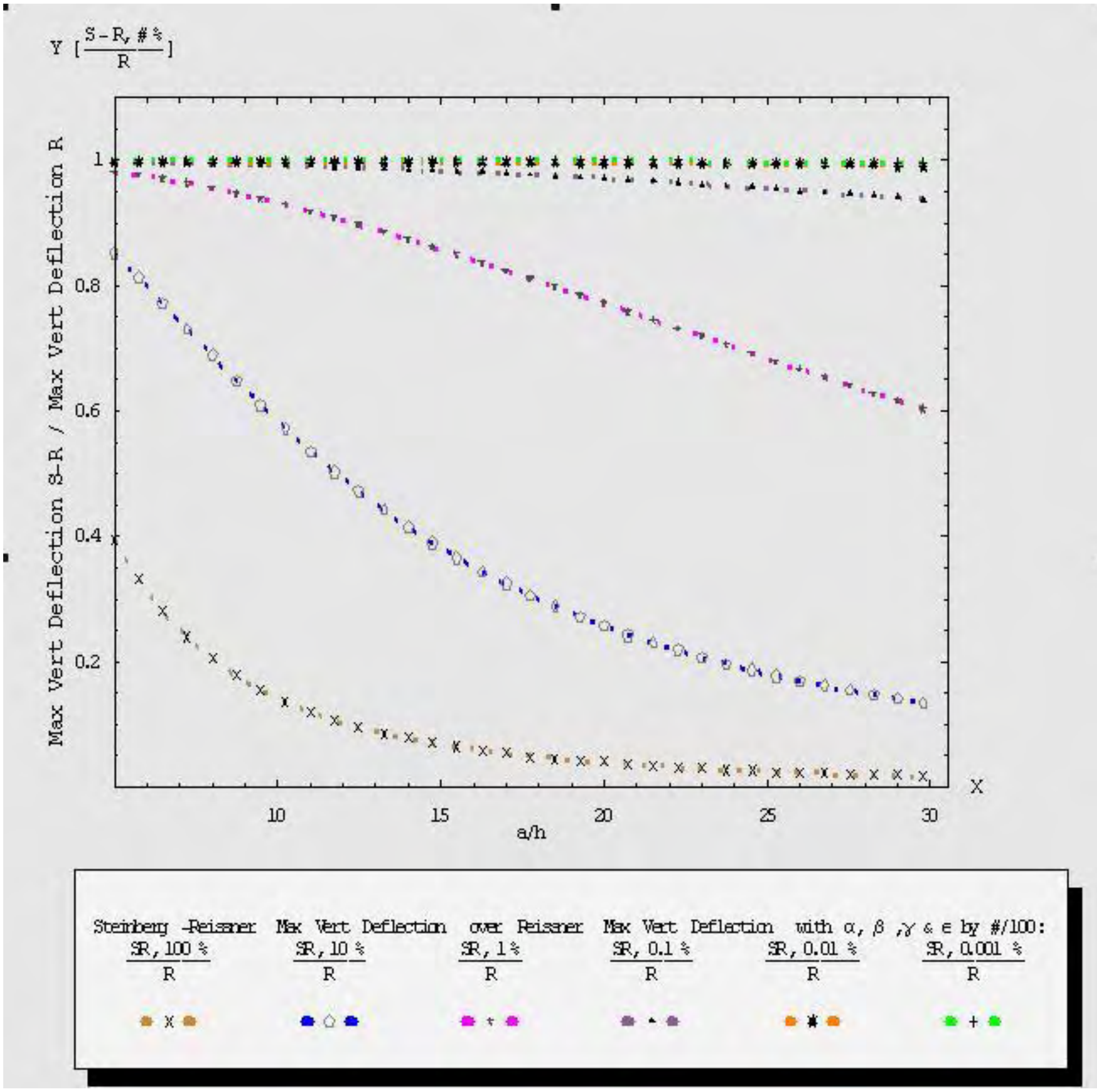

Figure 4.9 Maximum Vertical Deflection of Steinberg-Reissner Model over Maximum Vertical Deflection of Reissner Model

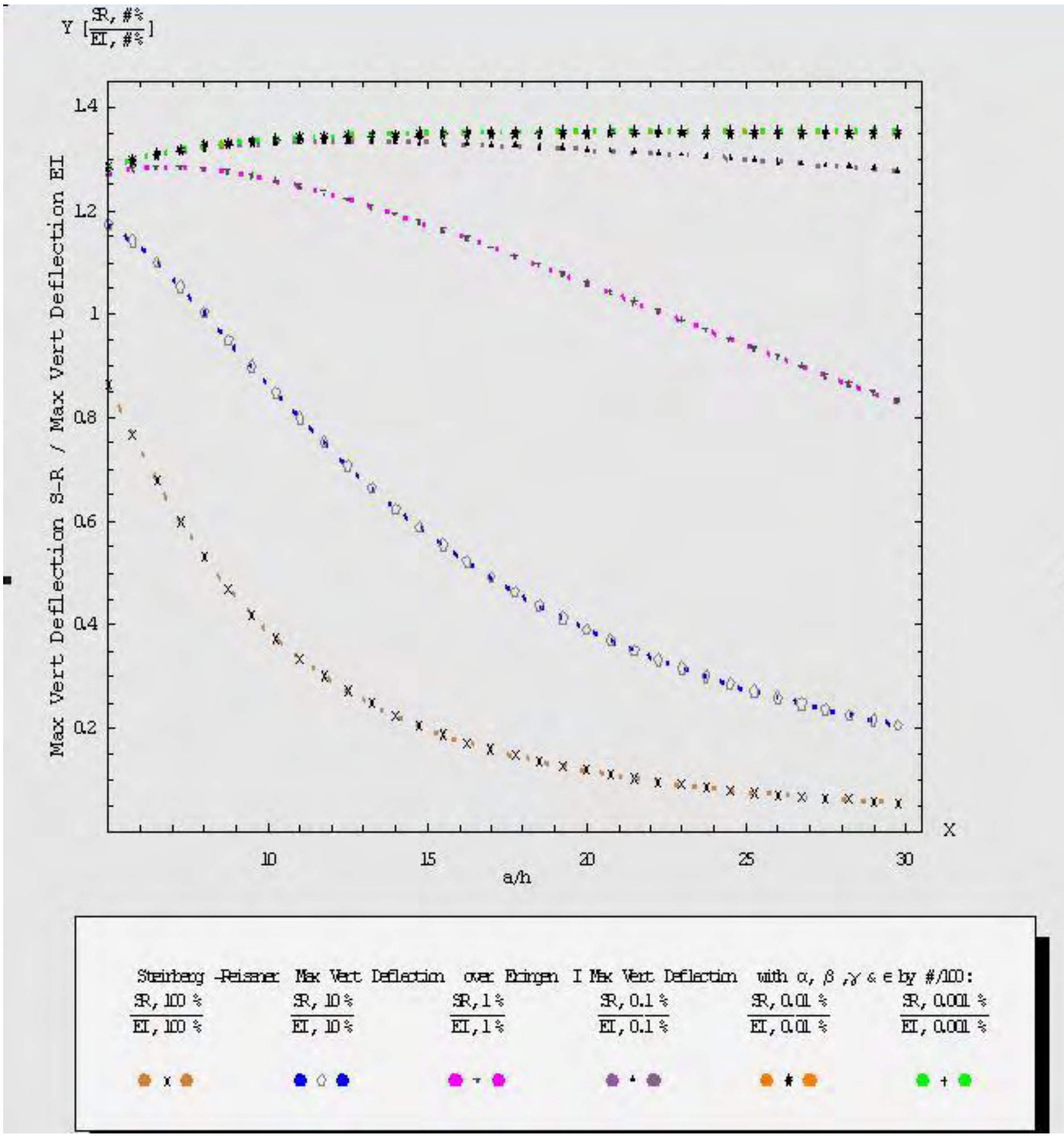

Figure 4.10 Maximum Vertical Deflection of Steinberg-Reissner Model over Maximum Vertical Deflection of Eringen Model I

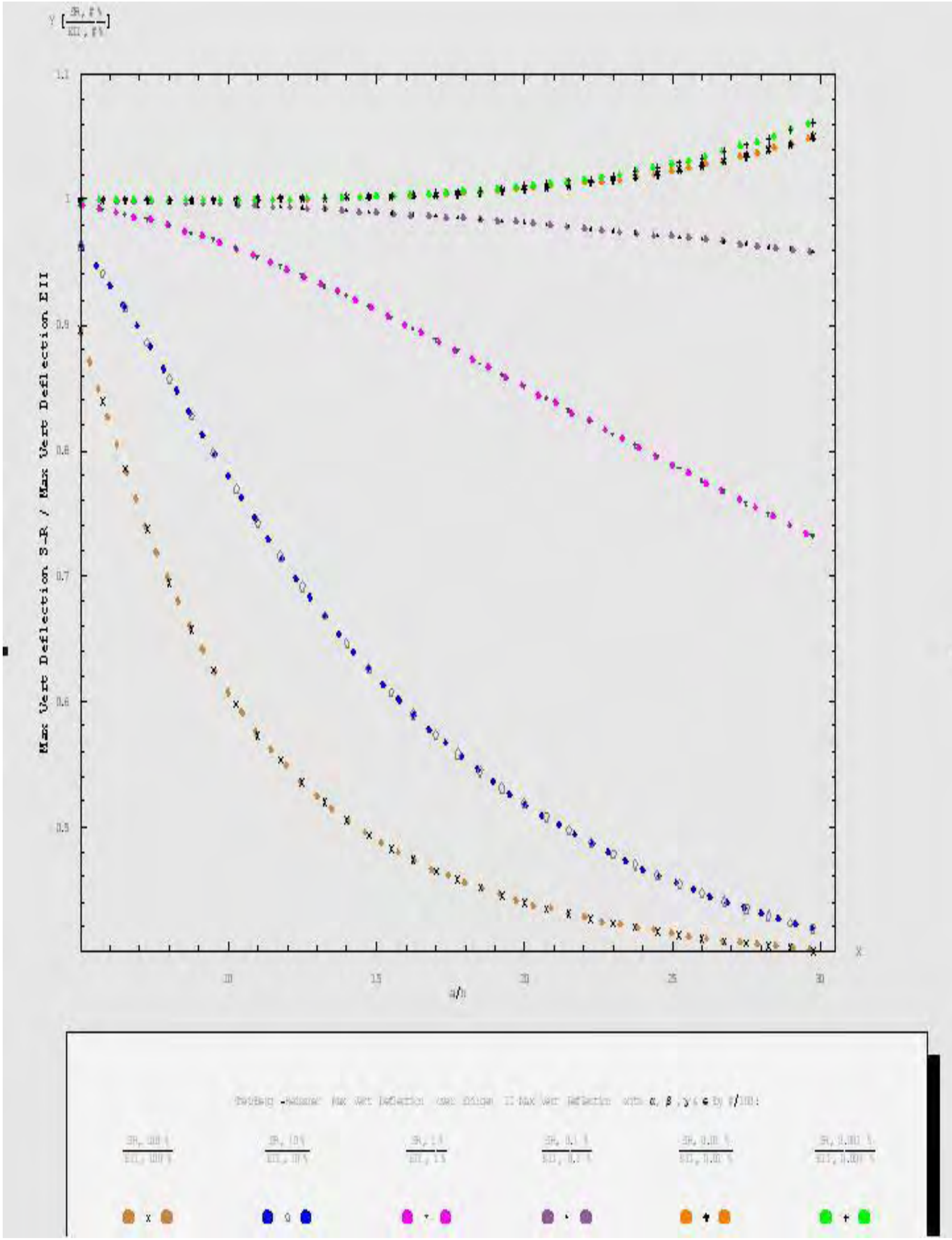

Figure 4.11 Maximum Vertical Deflection of Steinberg-Reissner Model over Maximum Vertical Deflection of Eringen Model II

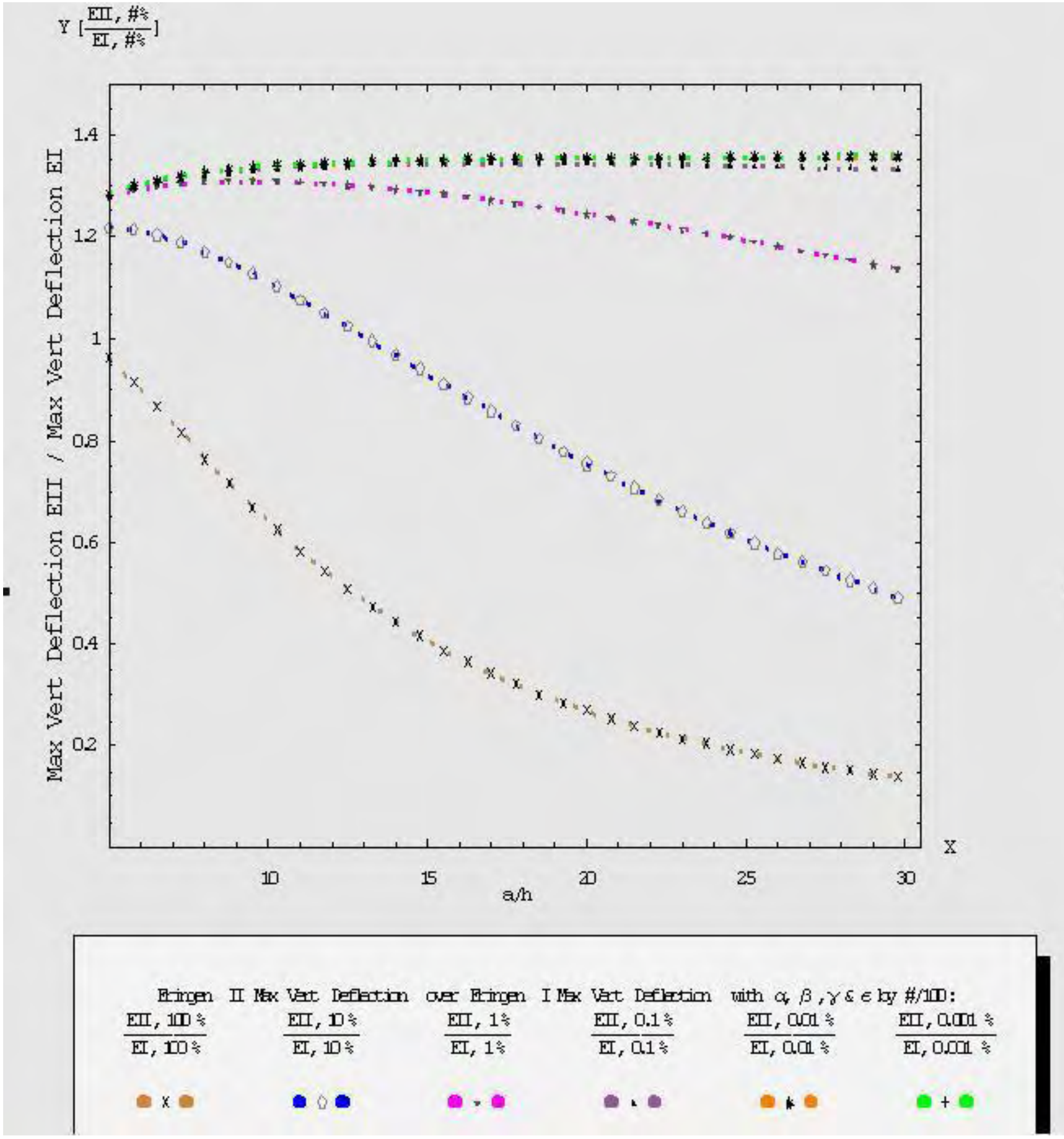

Figure 4.12 Maximum Vertical Deflection of Eringen Model II over Maximum Vertical Deflection of Eringen Model I

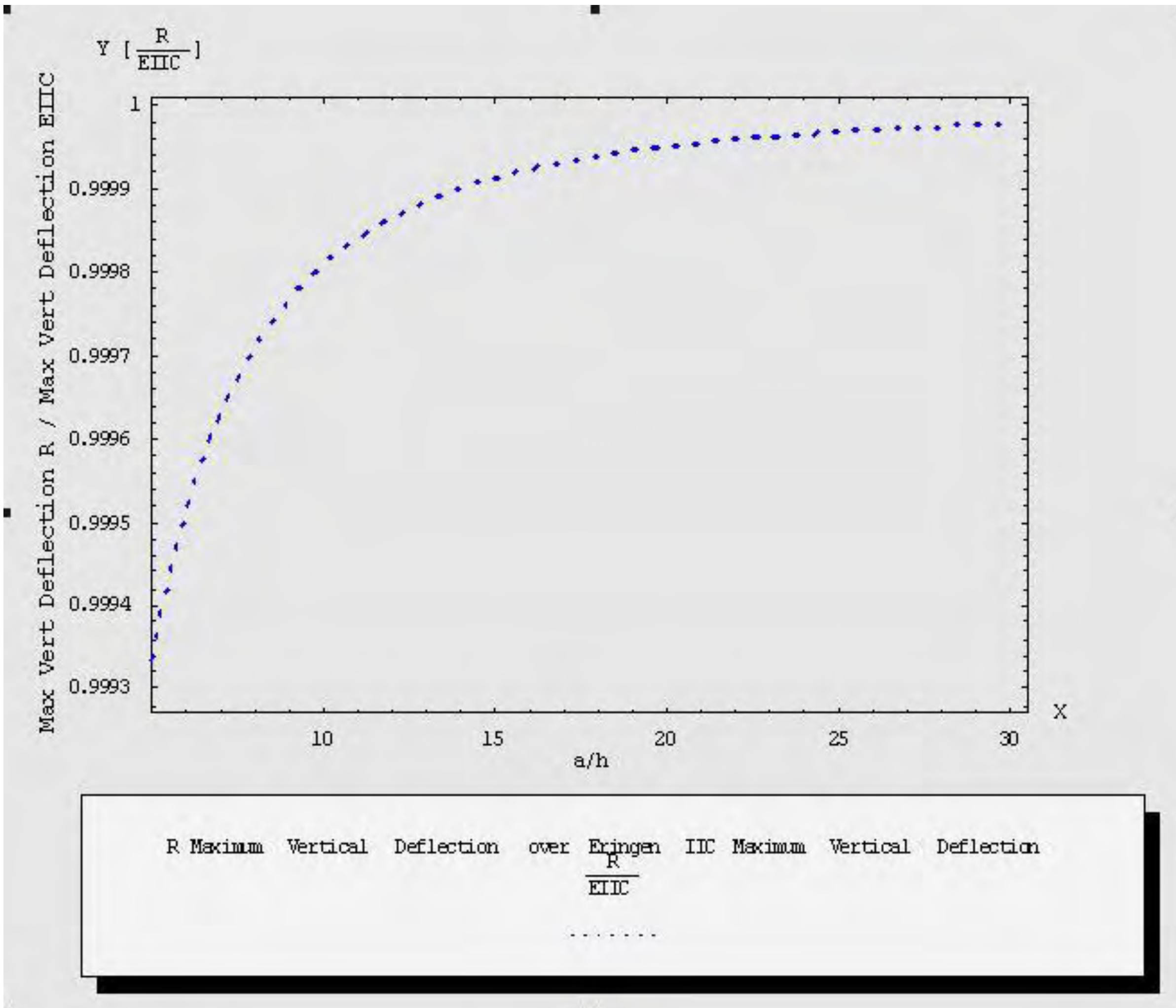

Figure 4.13 Maximum Vertical Deflection of Reissner Model over Maximum Vertical Deflection of Eringen Model II for the Elastic (Classic) Materials

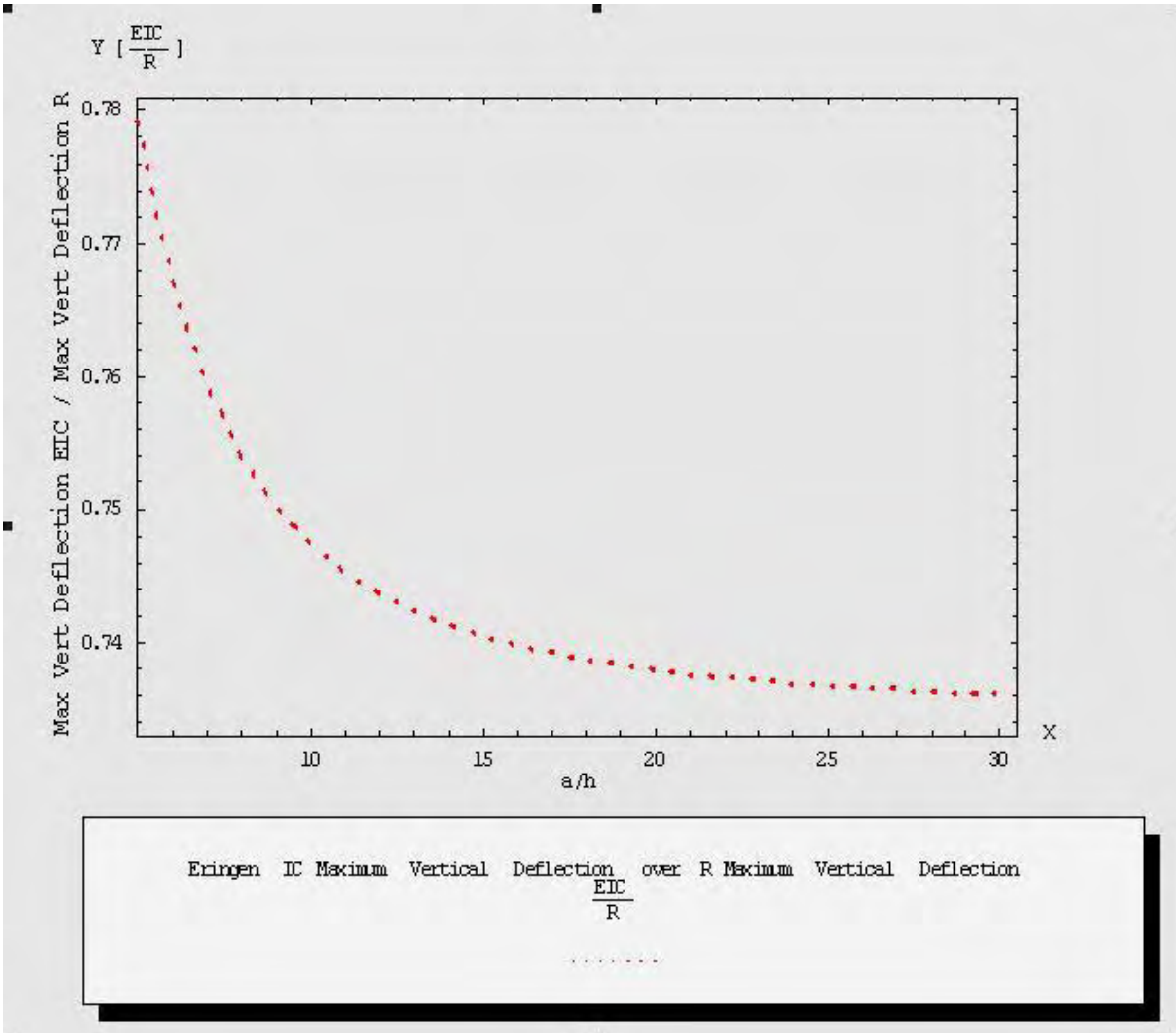

Figure 4.14 Maximum Vertical Deflection of Eringen Model I for the Elastic (Classic) Materials over Maximum Vertical Deflection of Reissner Model

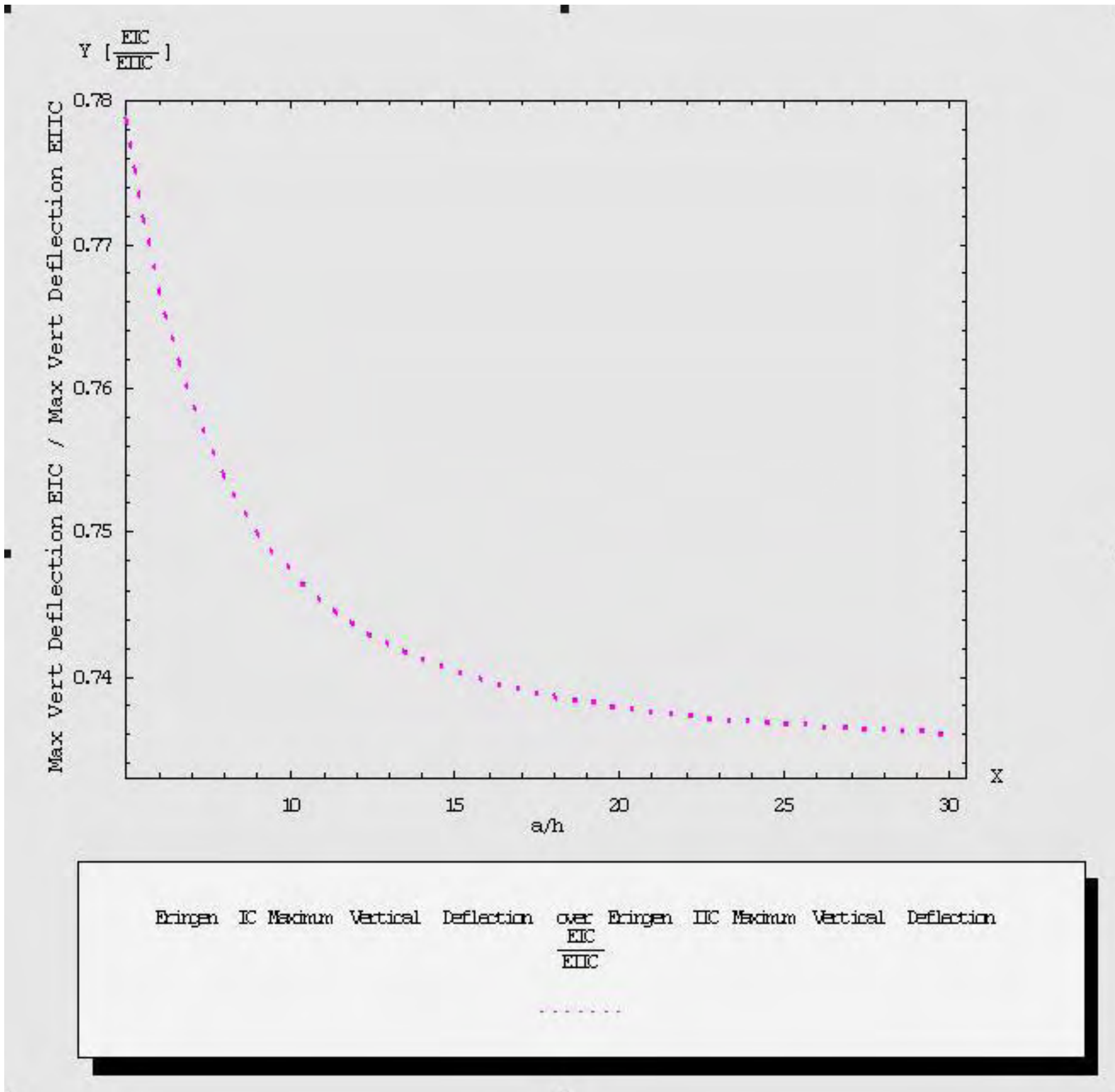

Figure 4.15 Maximum Vertical Deflection of Eringen Model I for the Elastic (Classic) Materials over Maximum Vertical Deflection of Eringen Model II for the Elastic (Classic) Materials

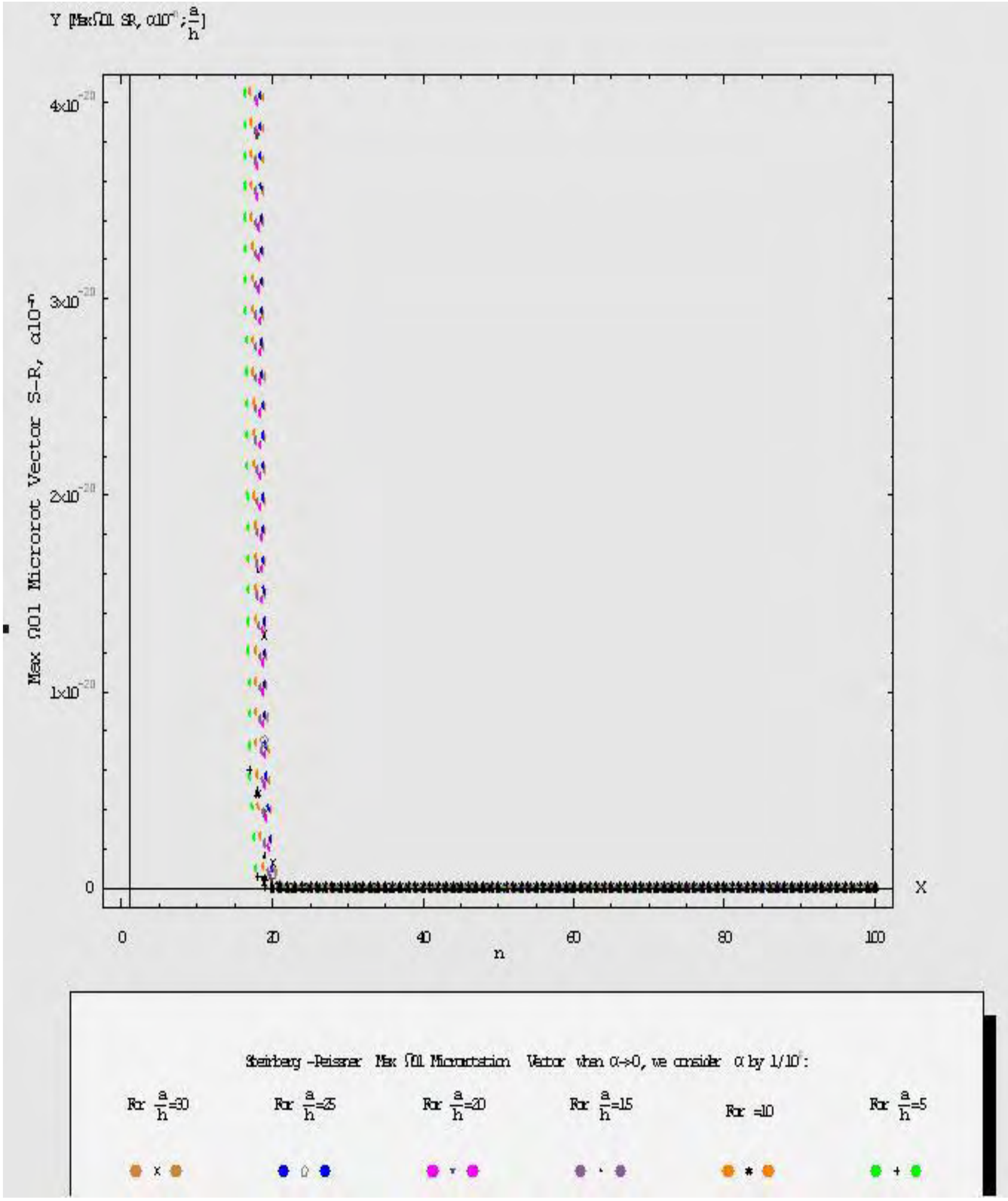

Figure 4.16 The Maximum of Micro-rotation Vector Ω_1^0 for Steinberg-Reissner Model Converges to Zero

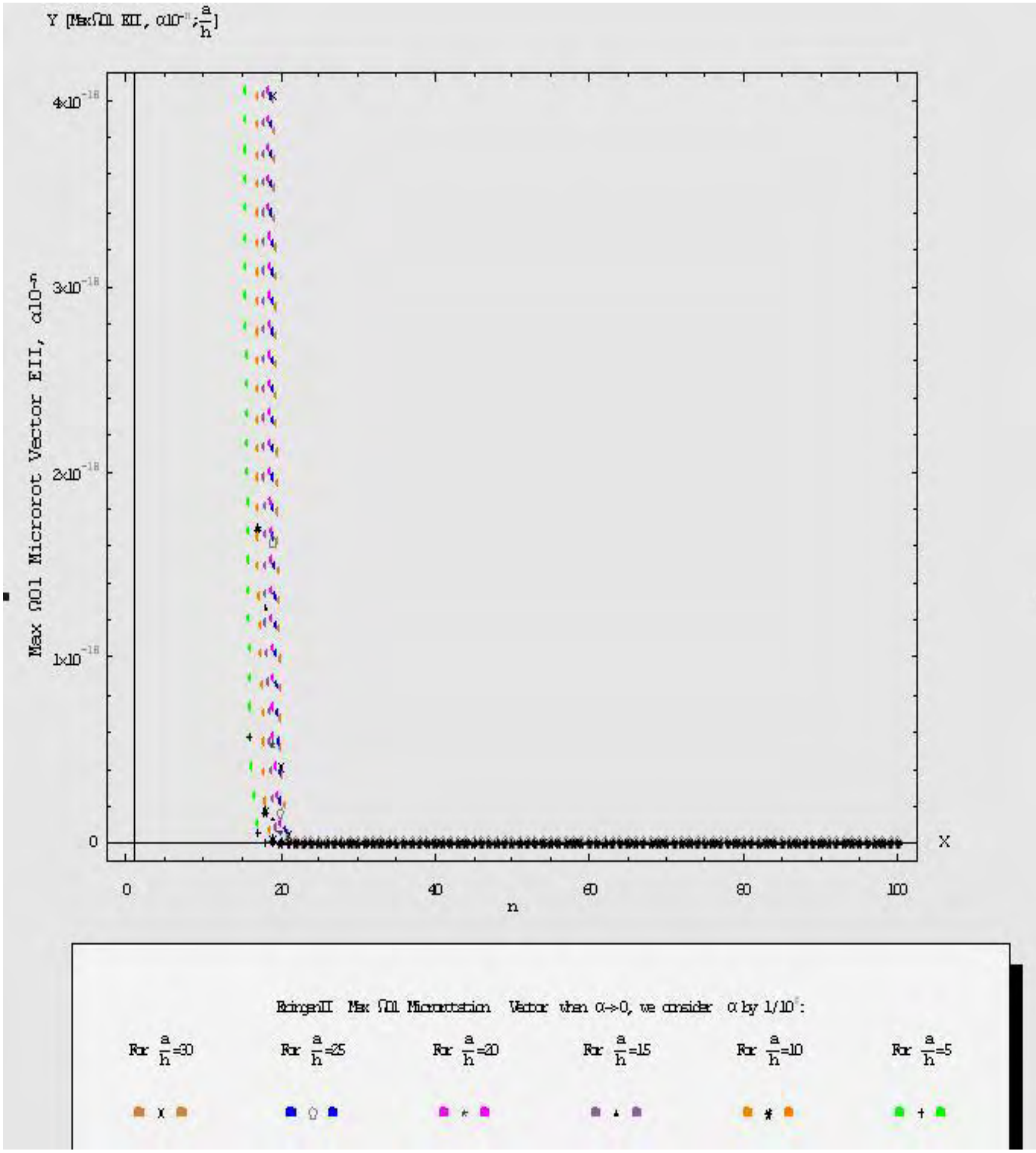

Figure 4.17 The Maximum of Micro-rotation Vector Ω_1^0 for Eringen Model II Converges to Zero

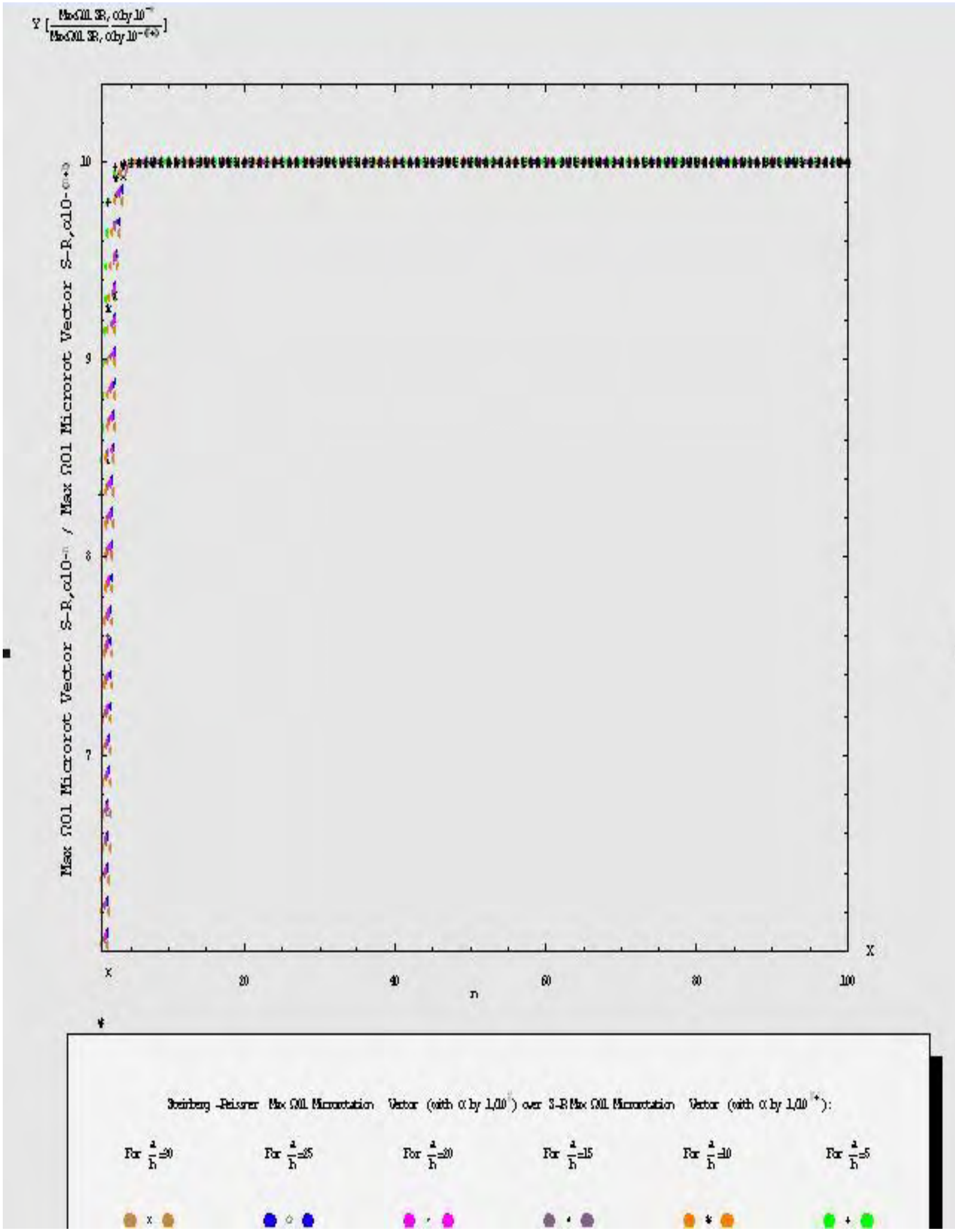

Figure 4.18 The Behavior of the Maximum of Micro-rotation Vector Ω_1^0 for Steinberg-Reissner Model as a function of α in a Neighborhood of Zero

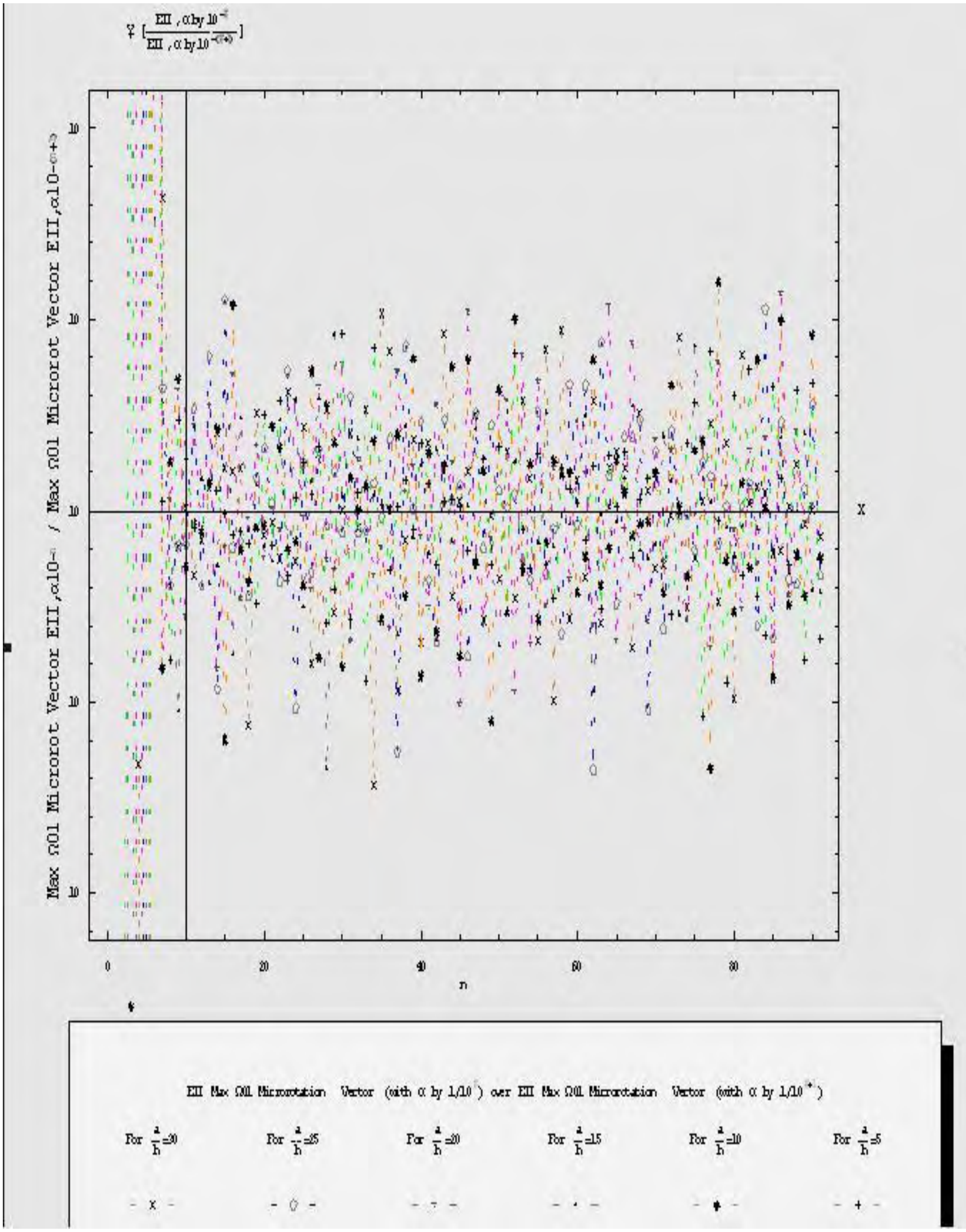

Figure 4.19 The Behavior of the Maximum of Micro-rotation Vector Ω_1^0 for Erigen Model II as a function of α in a Neighborhood of Zero

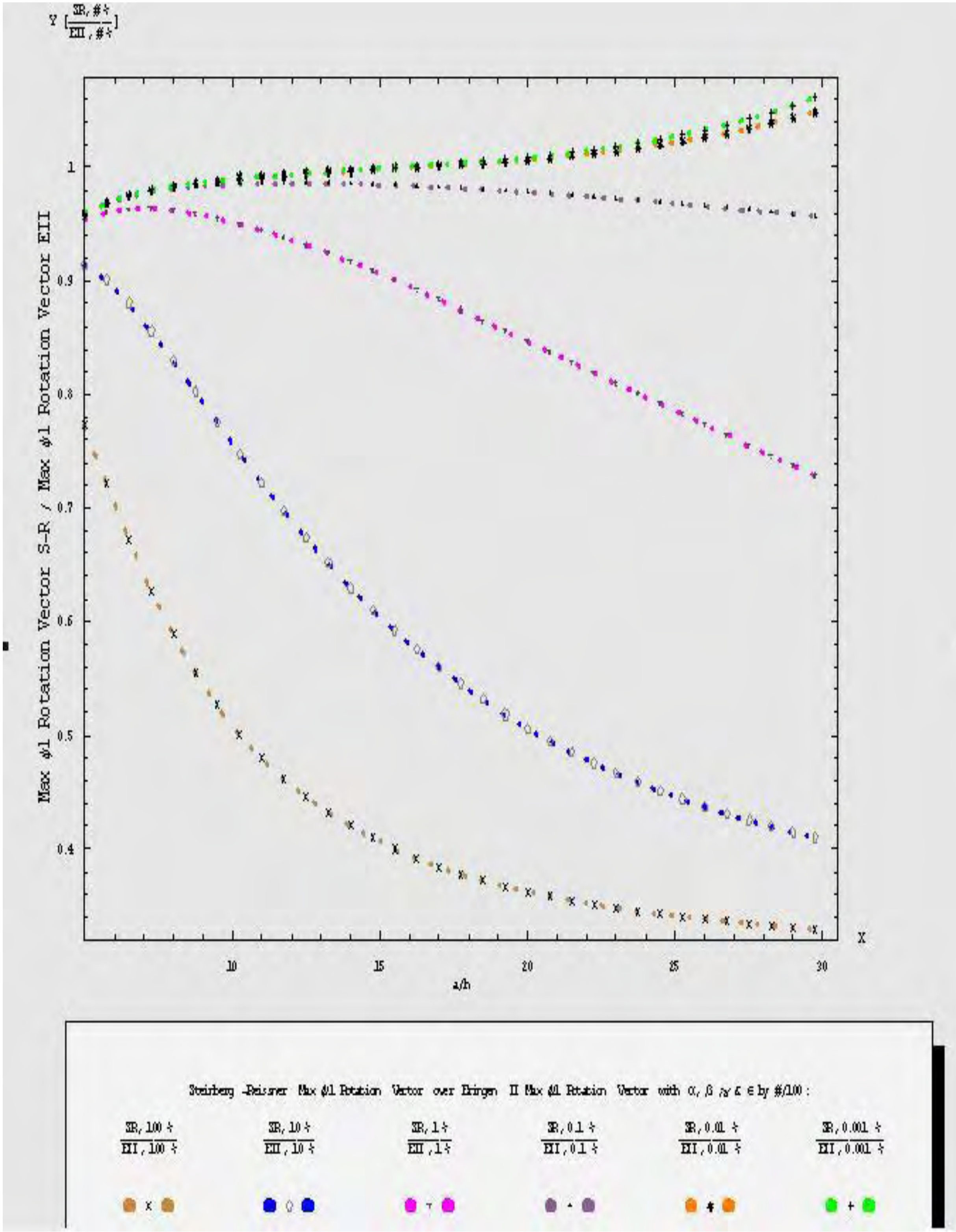

Figure 4.20 Maximum of Rotation Vector Ψ_1 of Steinberg-Reissner Model over Maximum of Rotation Vector Ψ_1 of Eringen Model II

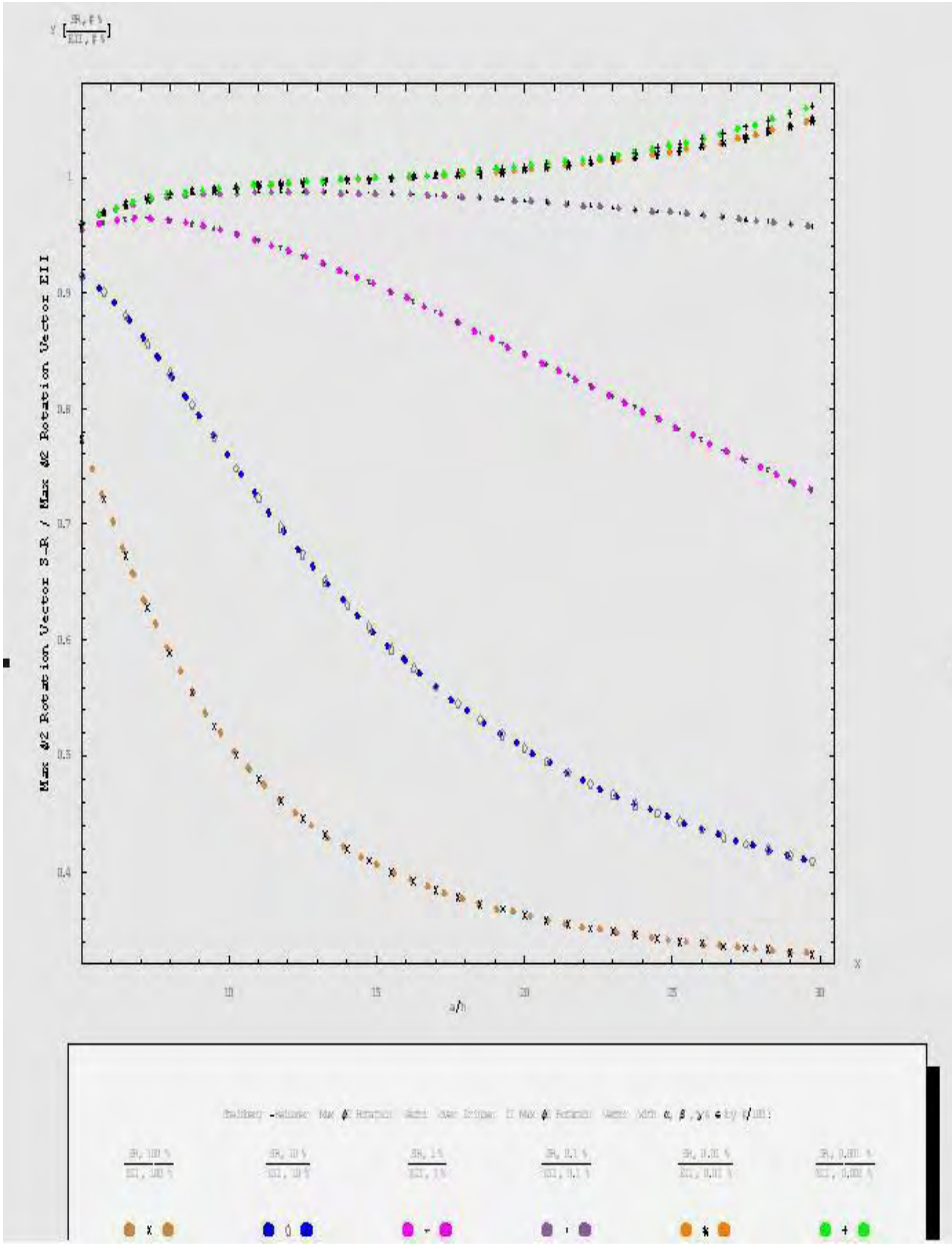

Figure 4.21 Maximum of Rotation Vector Ψ_2 of Steinberg-Reissner Model over Maximum of Rotation Vector Ψ_2 of Eringen Model II

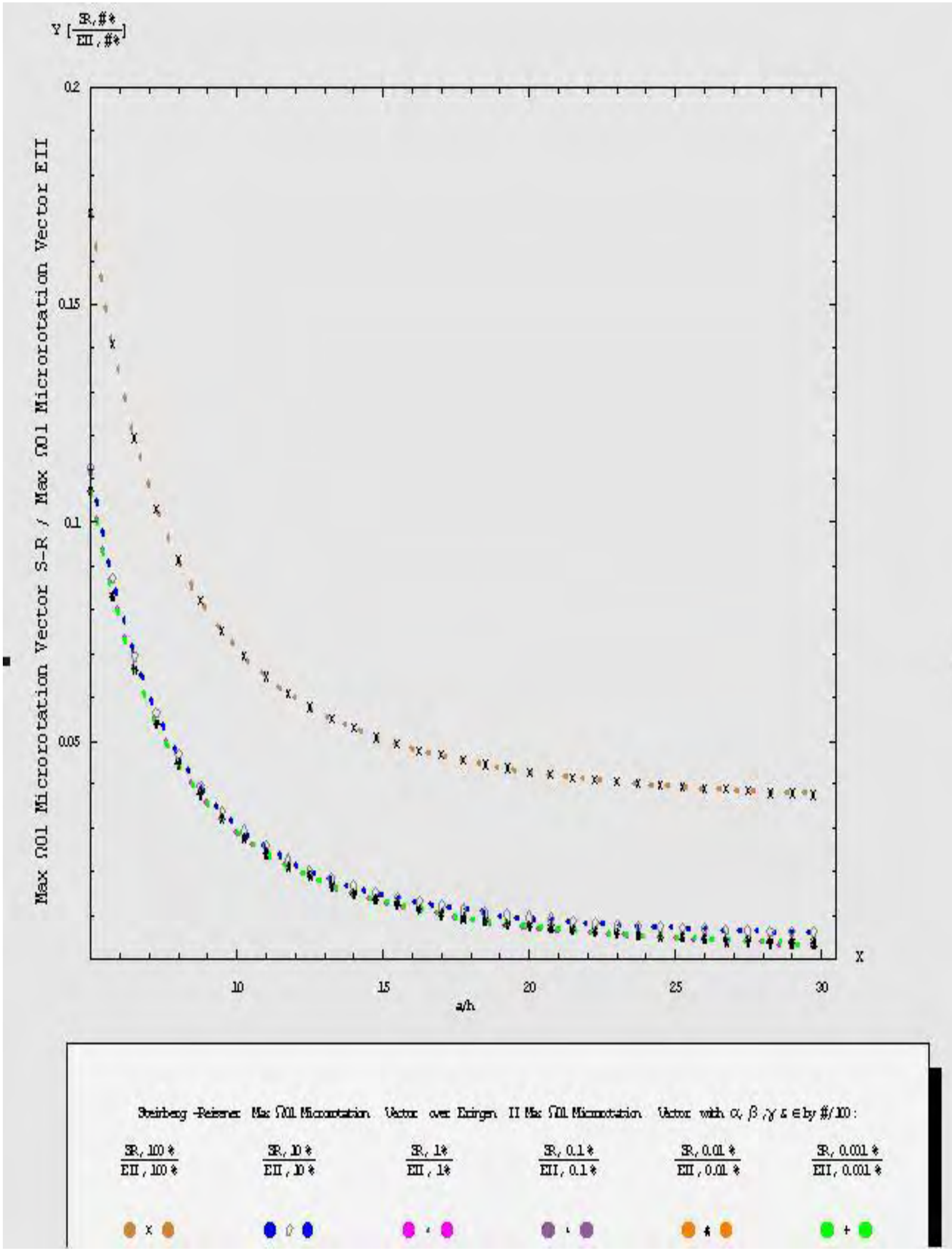

Figure 4.22 Maximum of Micro-rotation Vector Ω_1^0 of Steinberg-Reissner Model over Maximum of Micro-rotation Vector Ω_1^0 of Eringen Model II

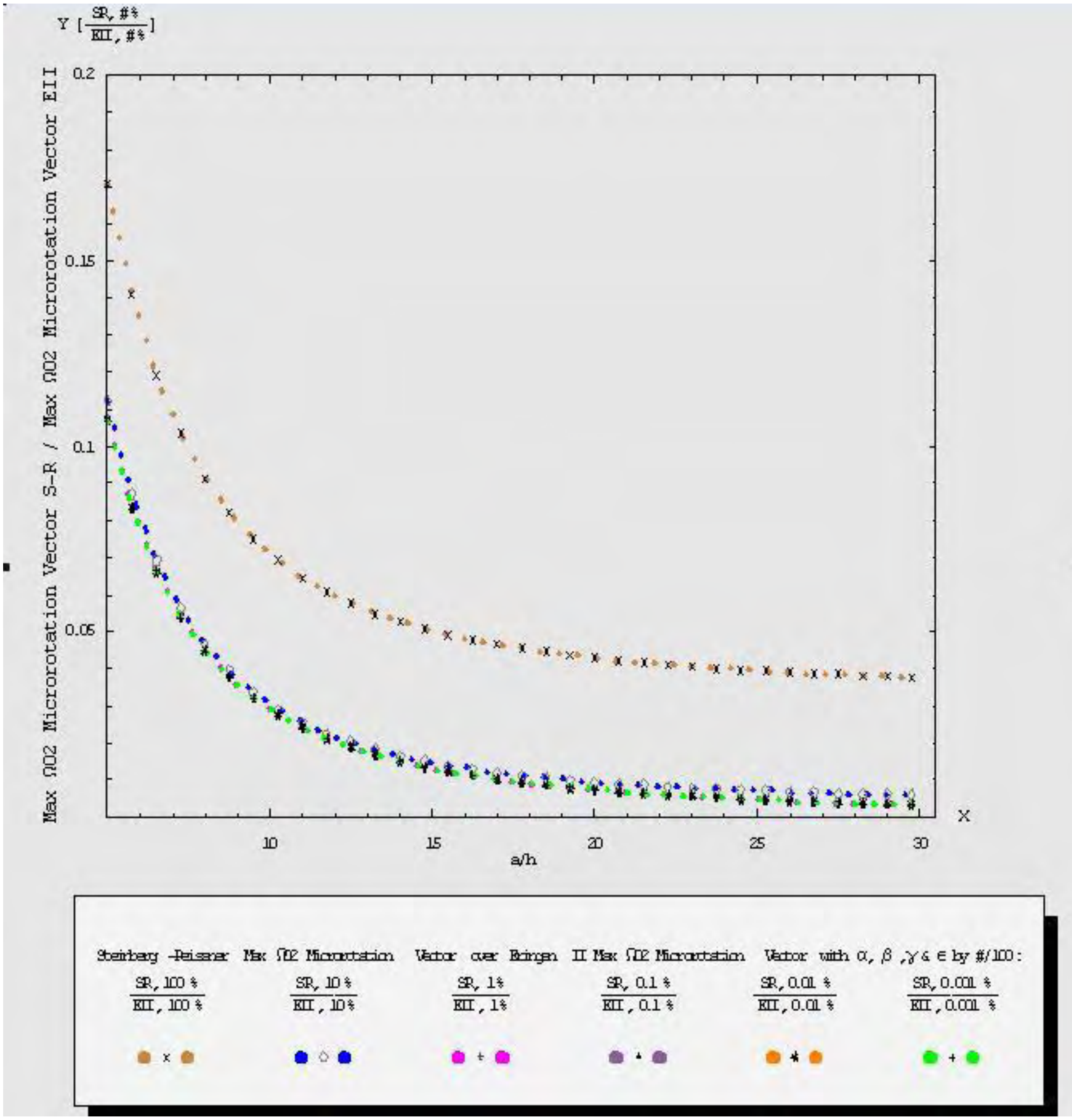

Figure 4.23 Maximum of Micro-rotation Vector Ω_2^0 of Steinberg-Reissner Model over Maximum of Micro-rotation Vector Ω_2^0 of Eringen Model II

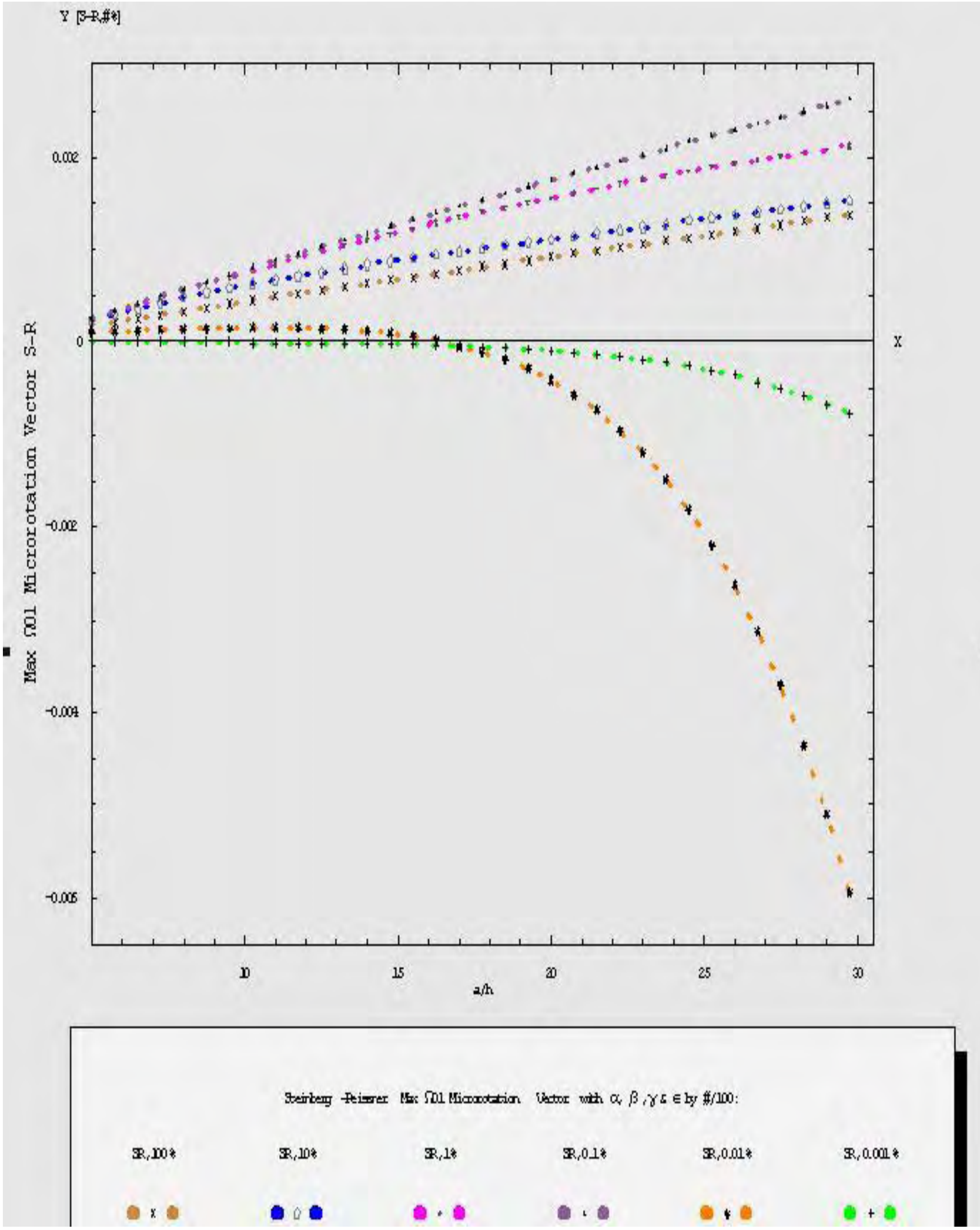

Figure 4.24 Maximum of Micro-rotation Vector Ω_1^0 of Steinberg-Reissner Model

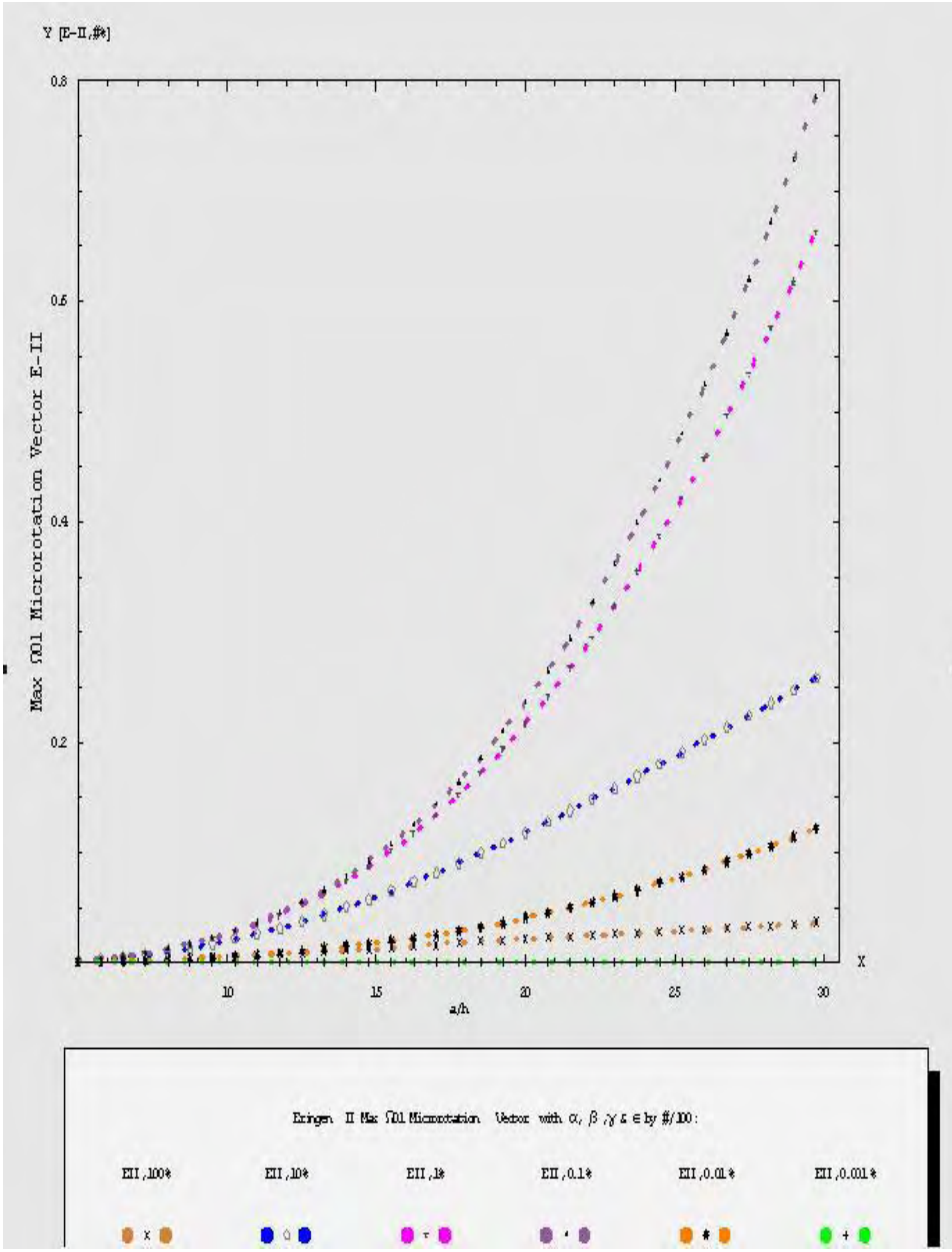

Figure 4.25 Maximum of Micro-rotation Vector Ω_1^0 of Eringen Model II

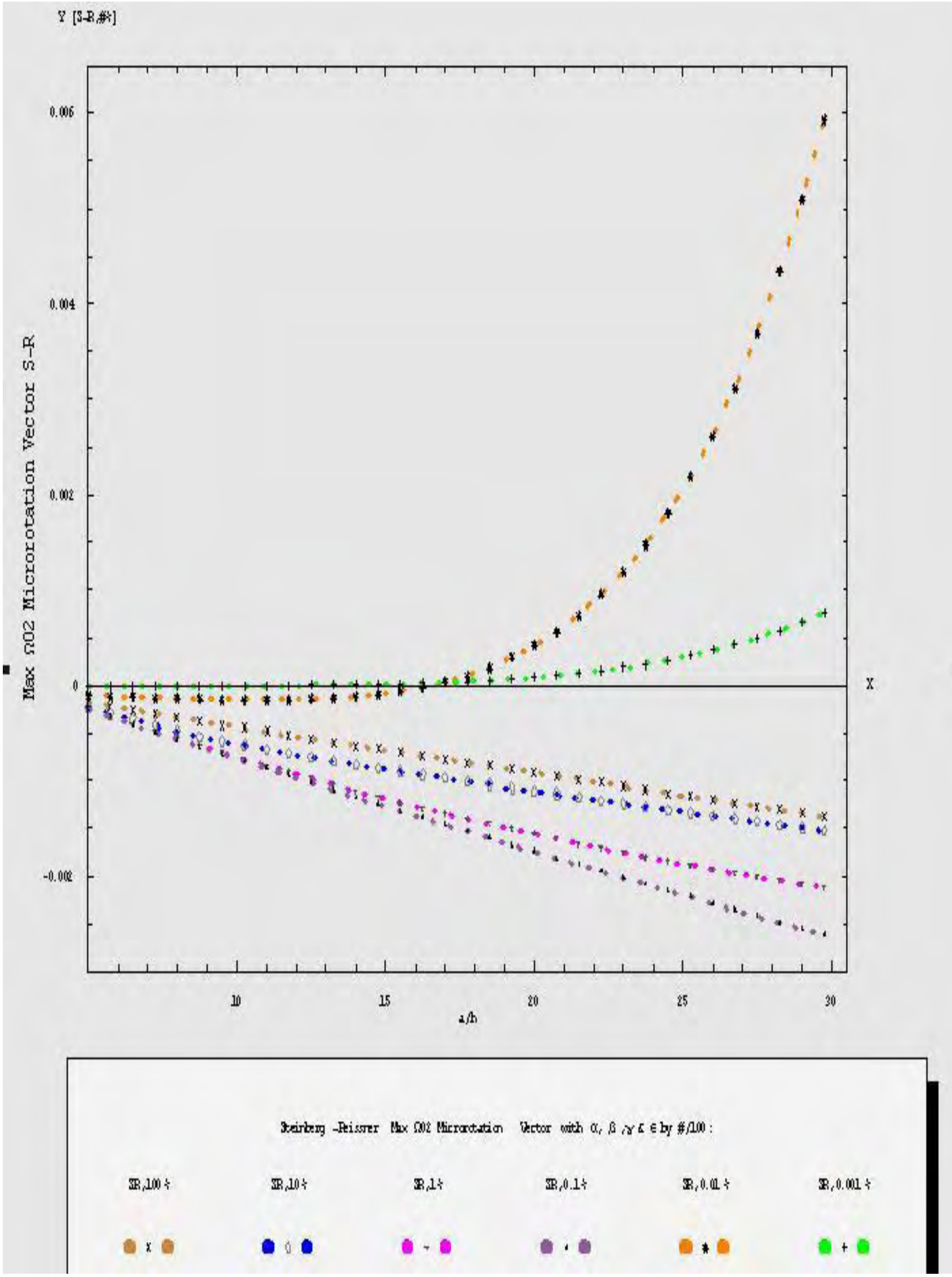

Figure 4.26 Maximum of Micro-rotation Vector Ω_2^0 of Steinberg-Reissner Model

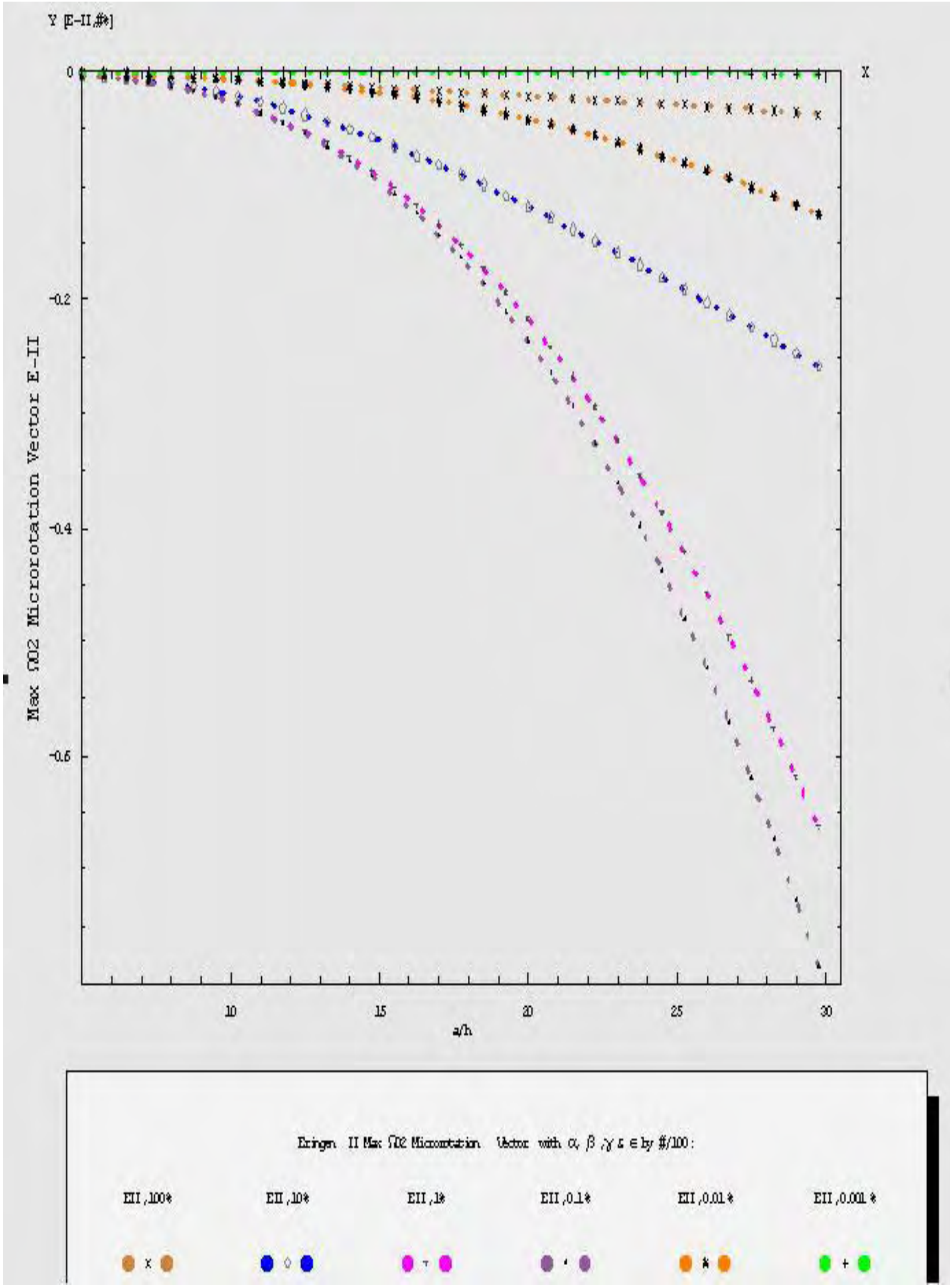

Figure 4.27 Maximum of Micro-rotation Vector Ω_2^0 of Eringen Model II

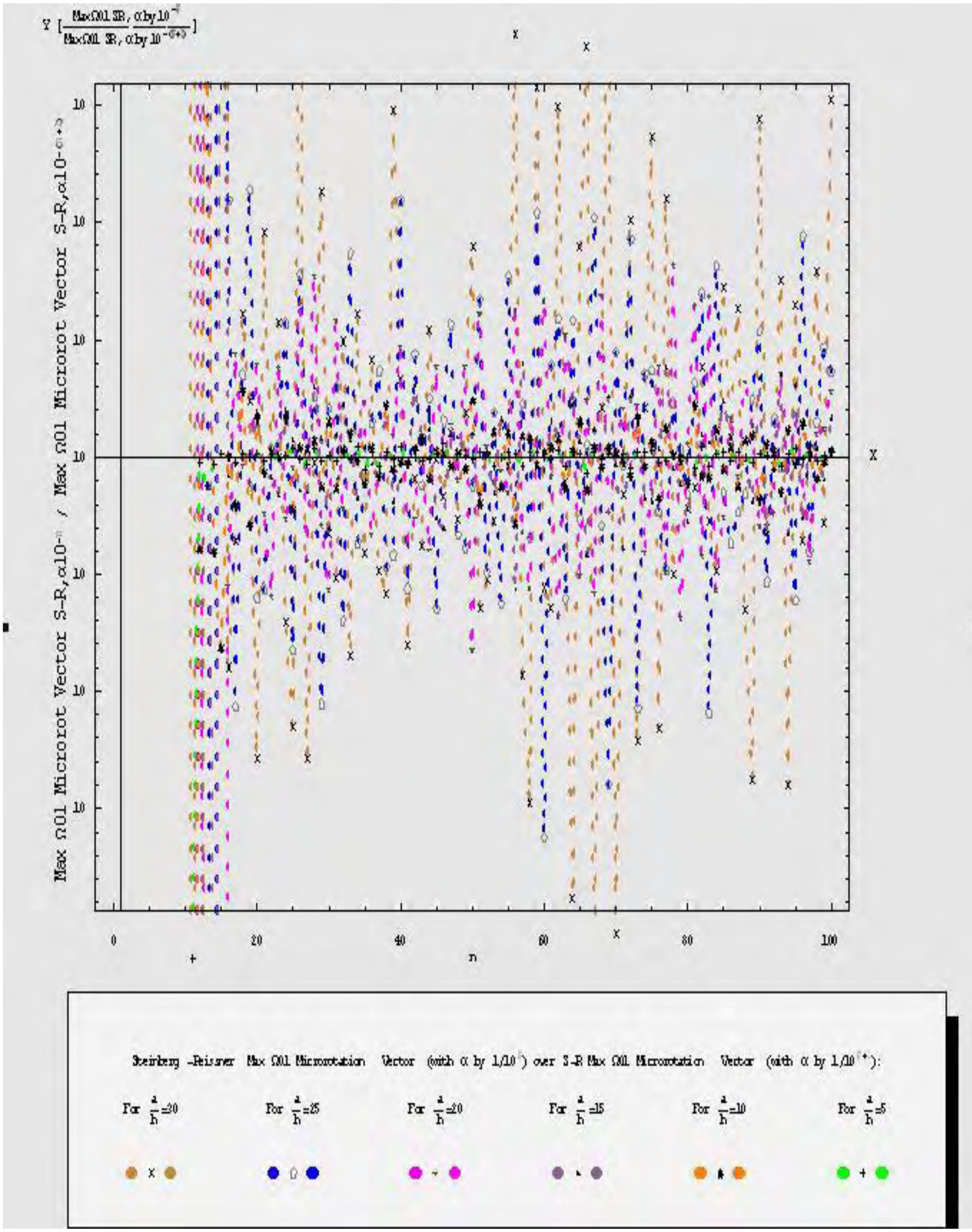

Figure 4.28 The Behavior of the Maximum of Micro-rotation Vector Ω_1^0 for Steinberg-Reissner Model as a function of α in a Neighborhood of Zero (see Figure 4.18)

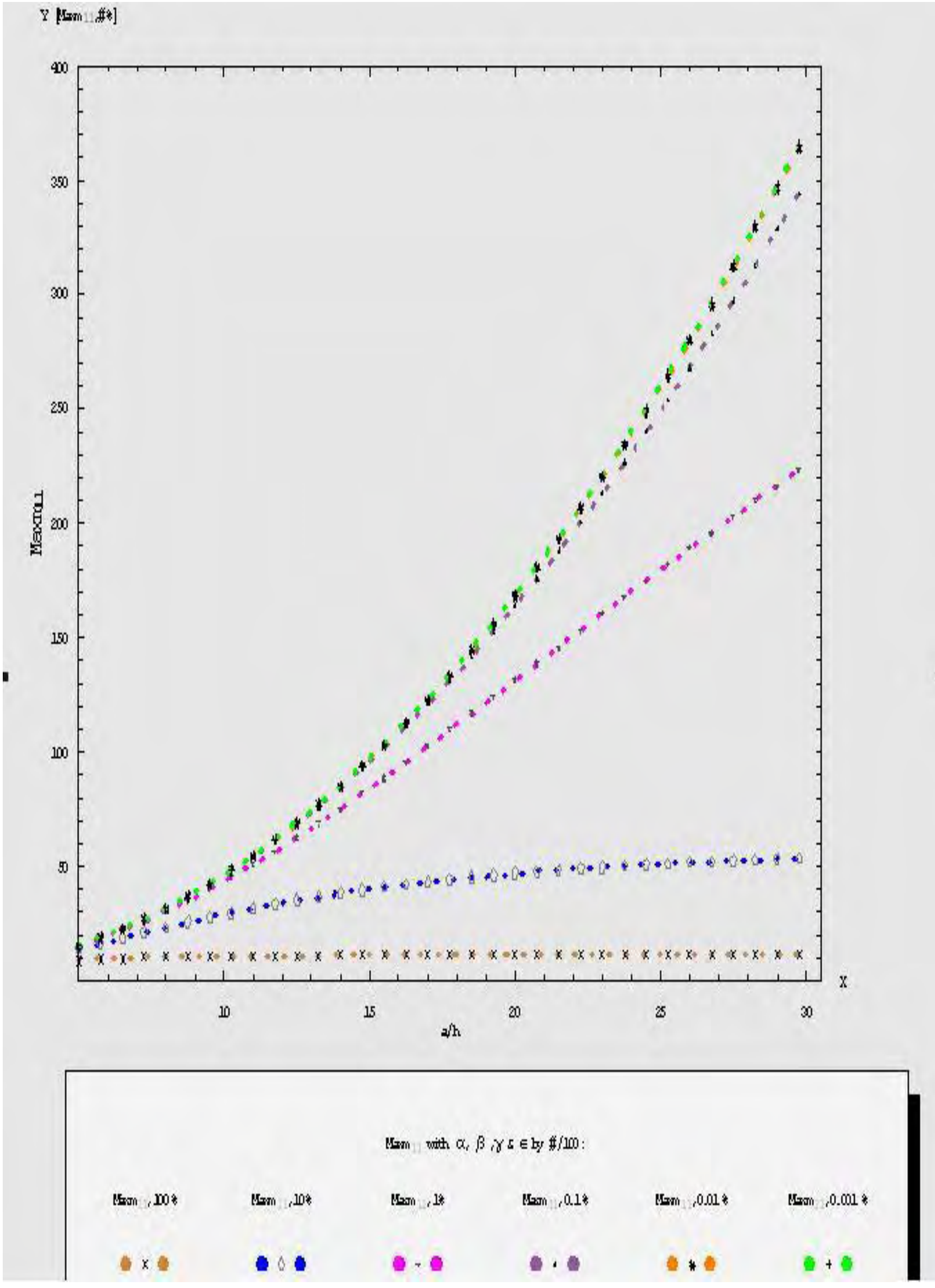

Figure 4.29 Maximum of the stress component m_{11} for Steinberg-Reissner Model

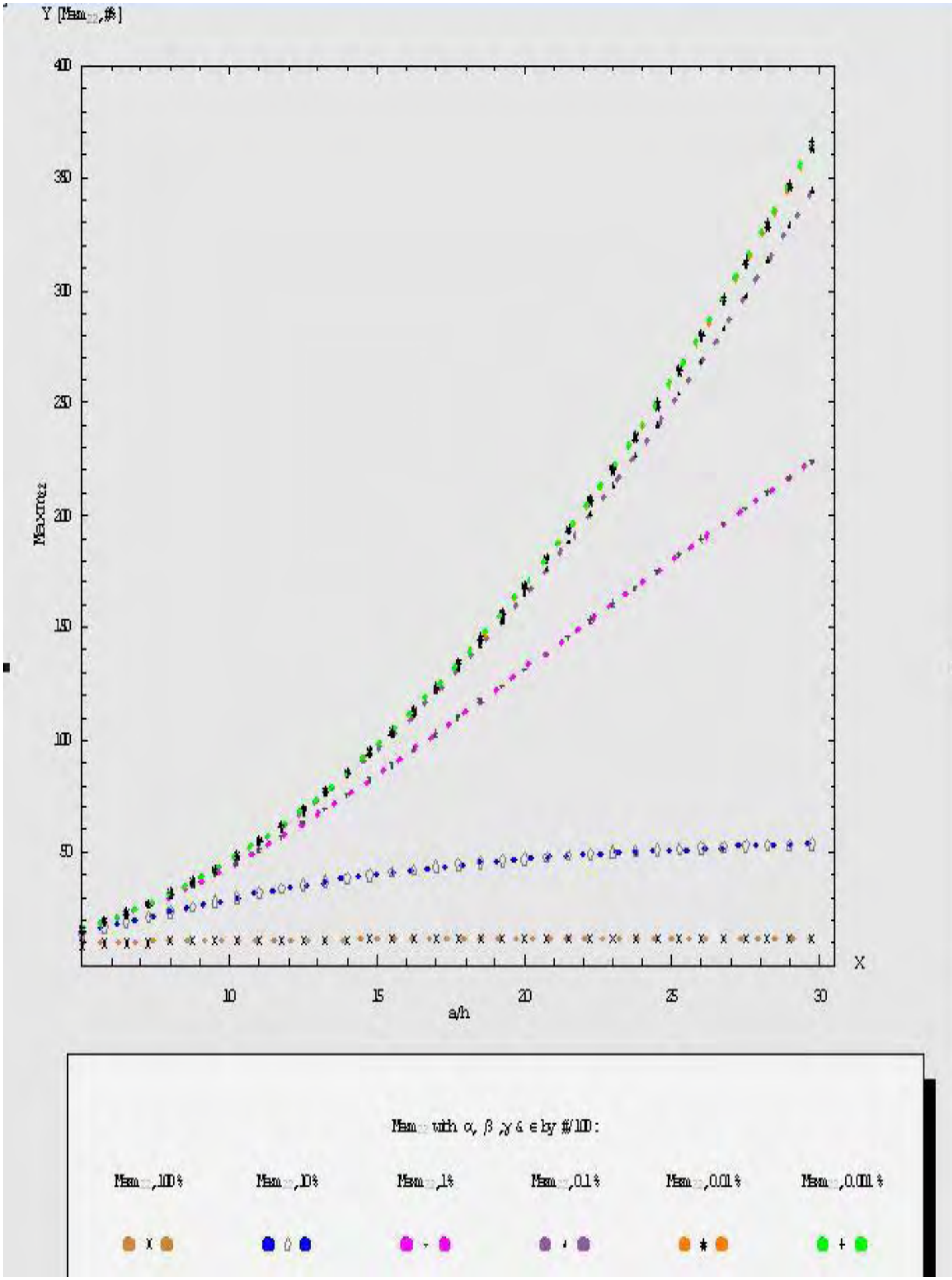

Figure 4.30 Maximum of the stress component m_{22} for Steinberg-Reissner Model

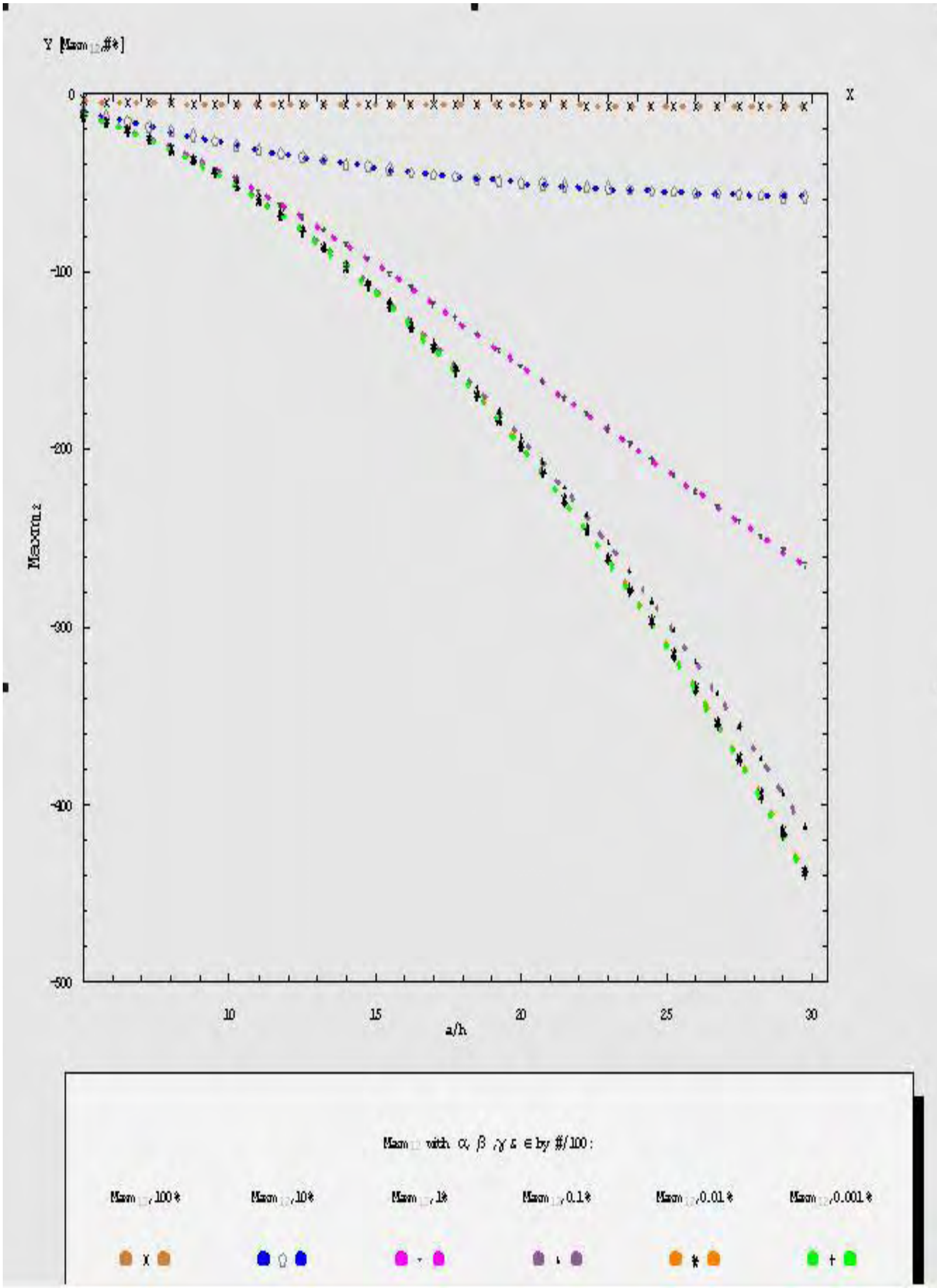

Figure 4.31 Maximum of the stress component m_{12} for Steinberg-Reissner Model

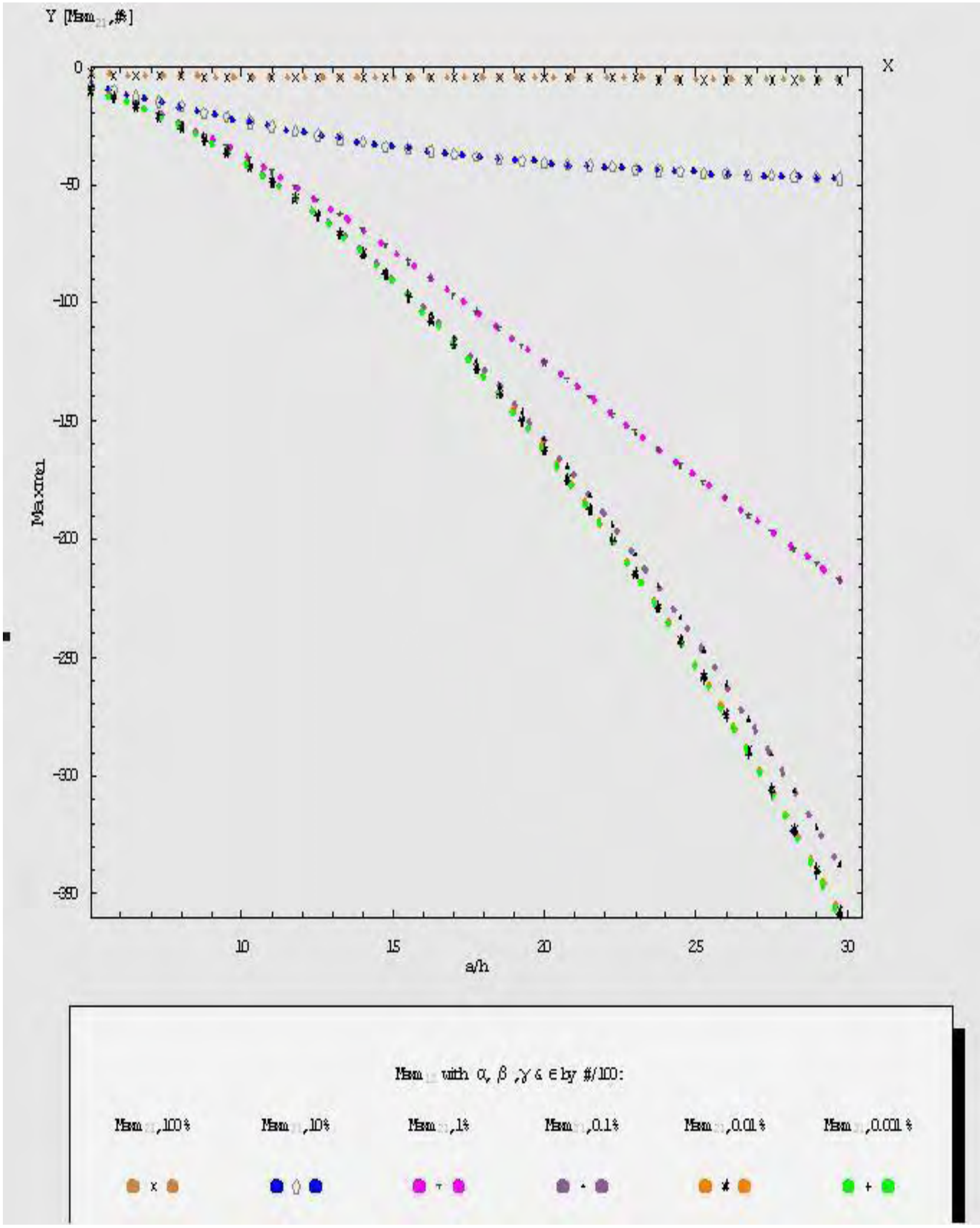

Figure 4.32 Maximum of the stress component m_{21} for Steinberg-Reissner Model

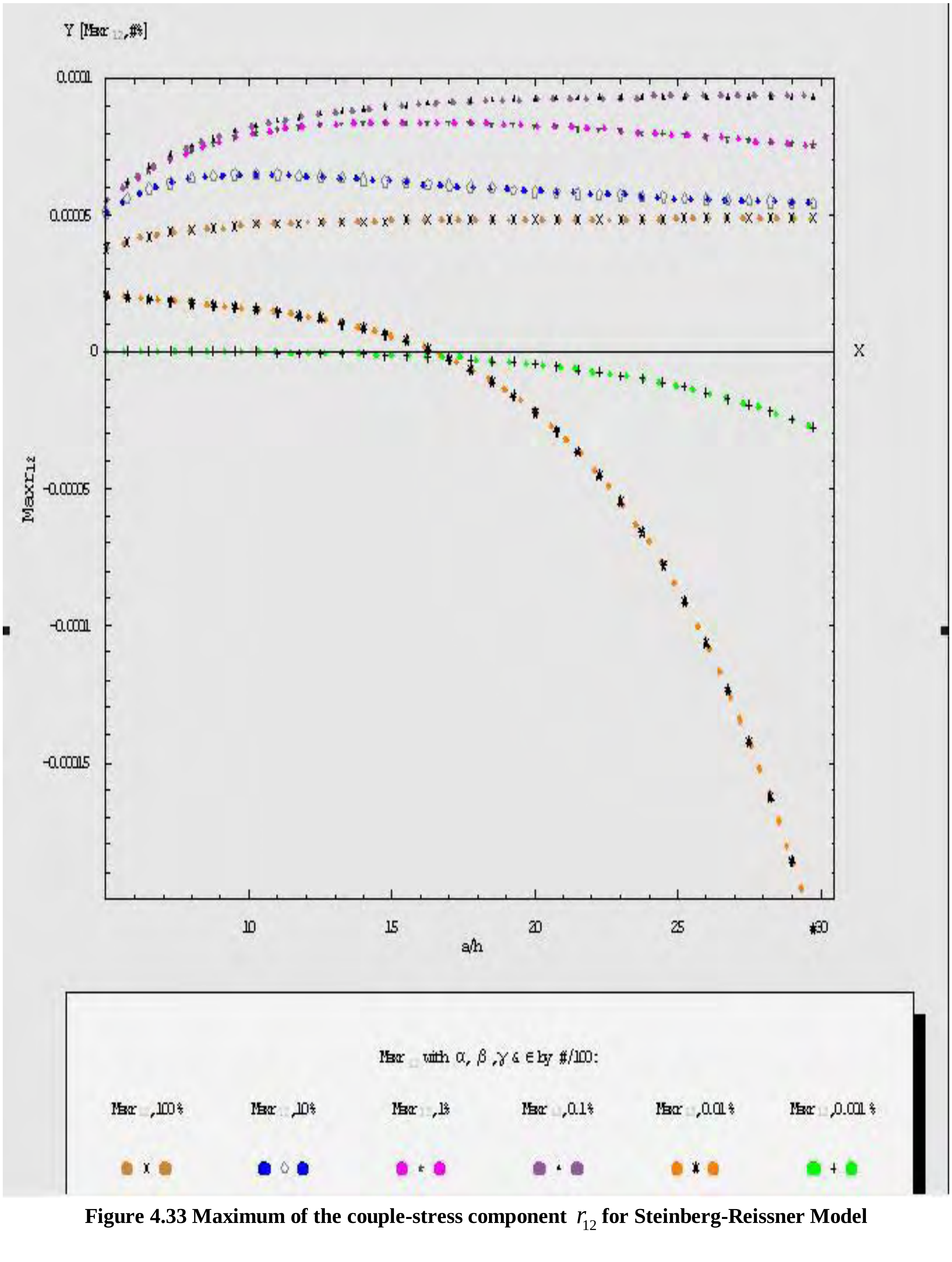

Figure 4.33 Maximum of the couple-stress component r_{12} for Steinberg-Reissner Model

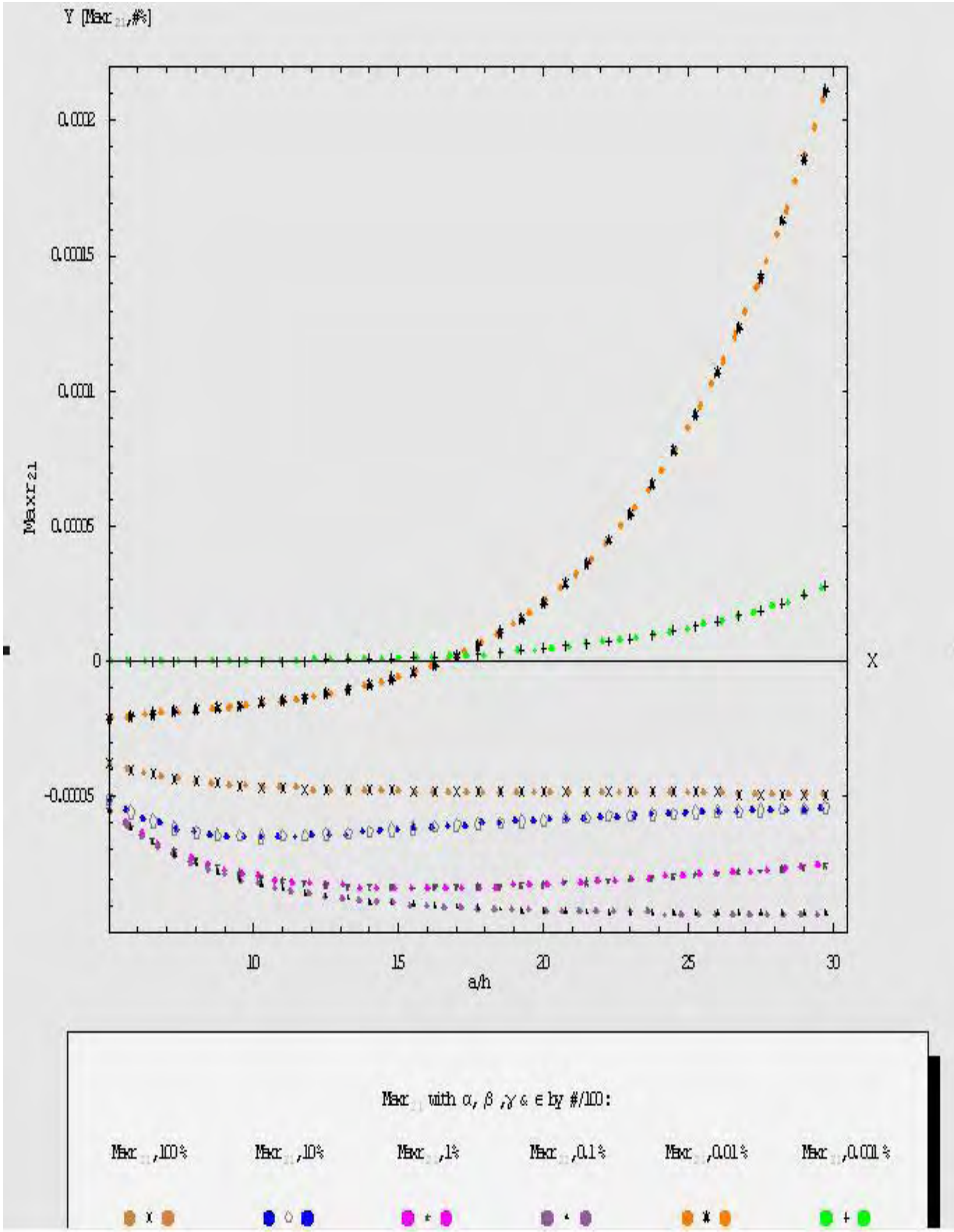

Figure 4.34 Maximum of the couple-stress component r_{21} for Steinberg-Reissner Model

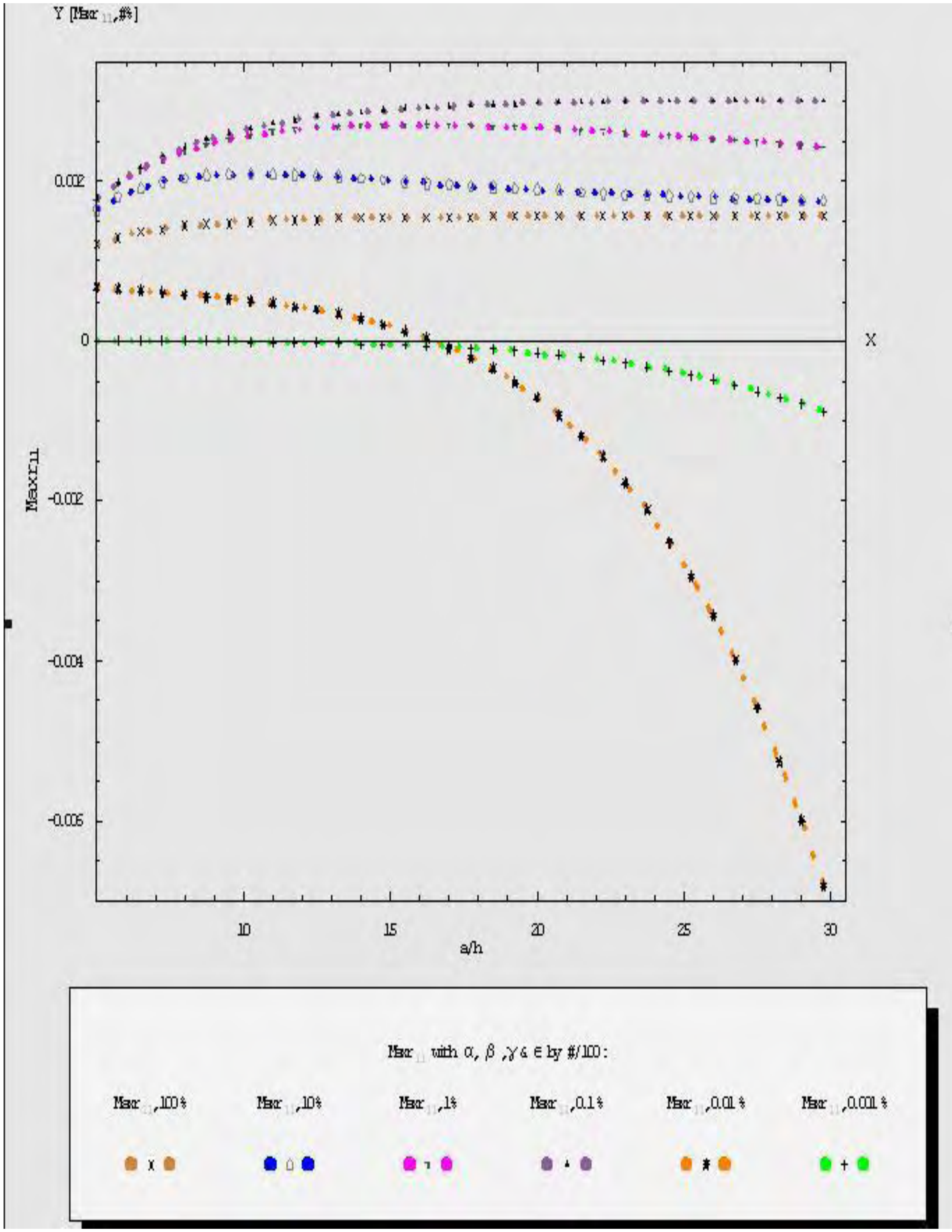

Figure 4.35 Maximum of the couple-stress component r_{11} for Steinberg-Reissner Model

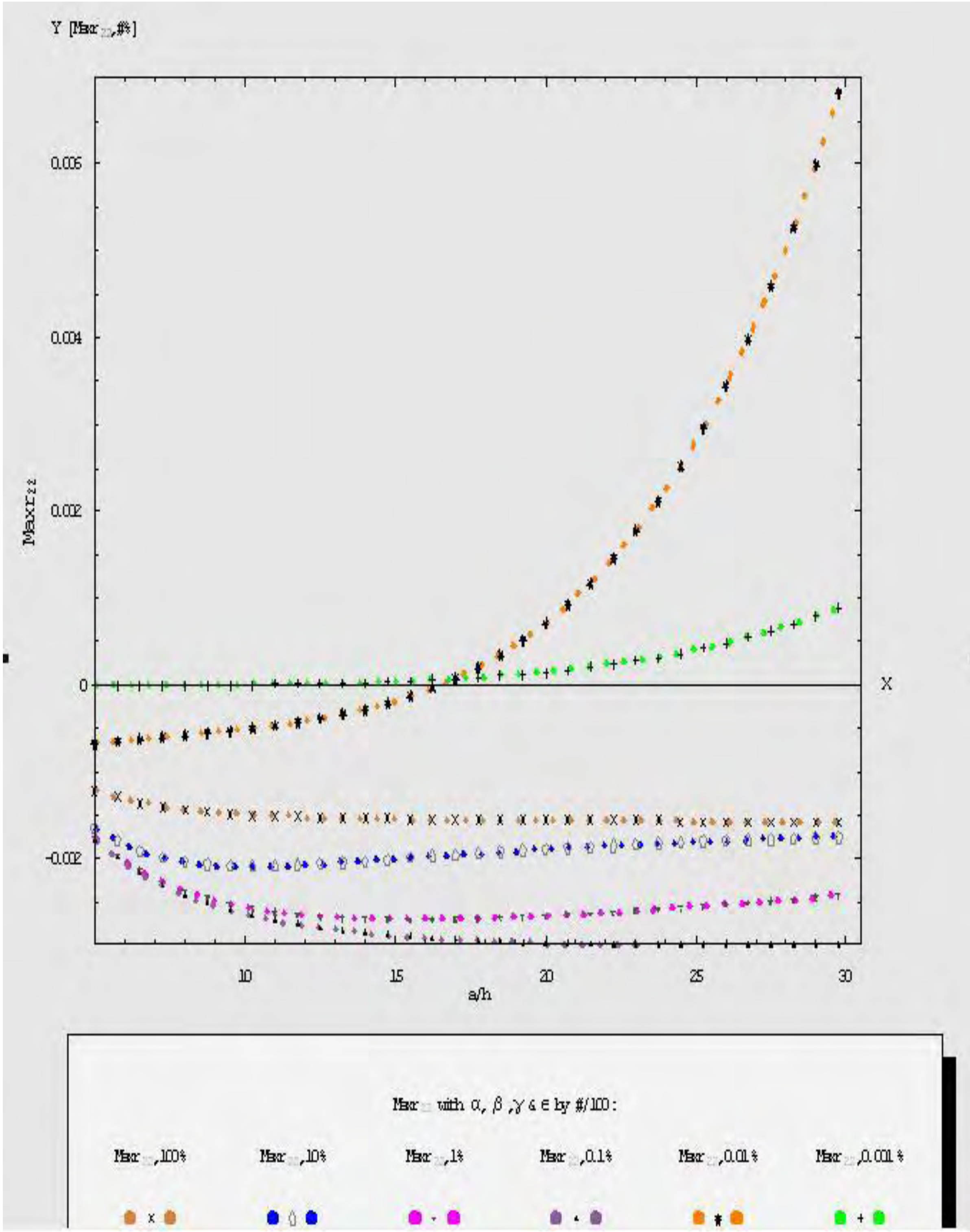

Figure 4.36 Maximum of the couple-stress component r_{22} for Steinberg-Reissner Model

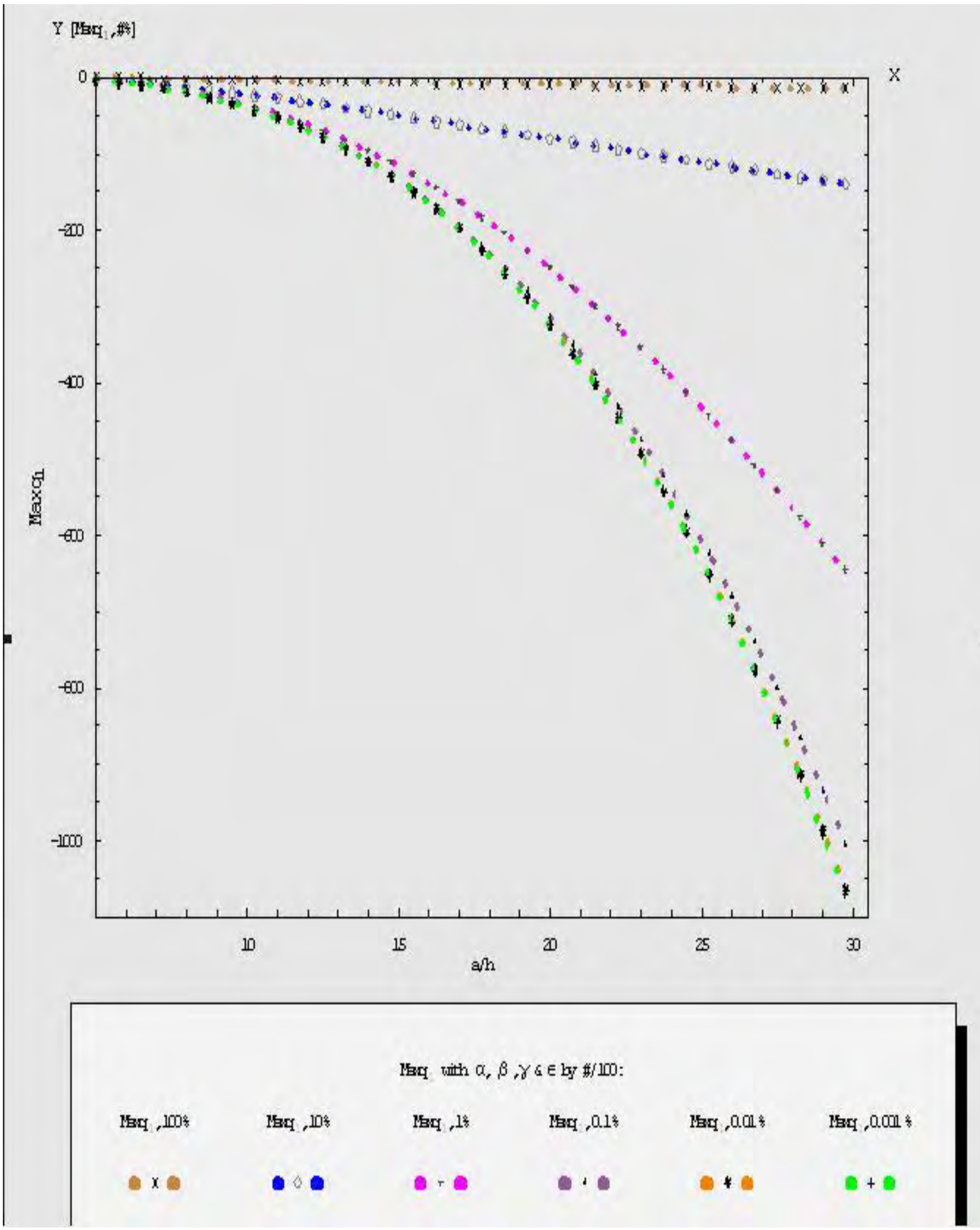

Figure 4.37 Maximum of the stress component q_1 for Steinberg-Reissner Model

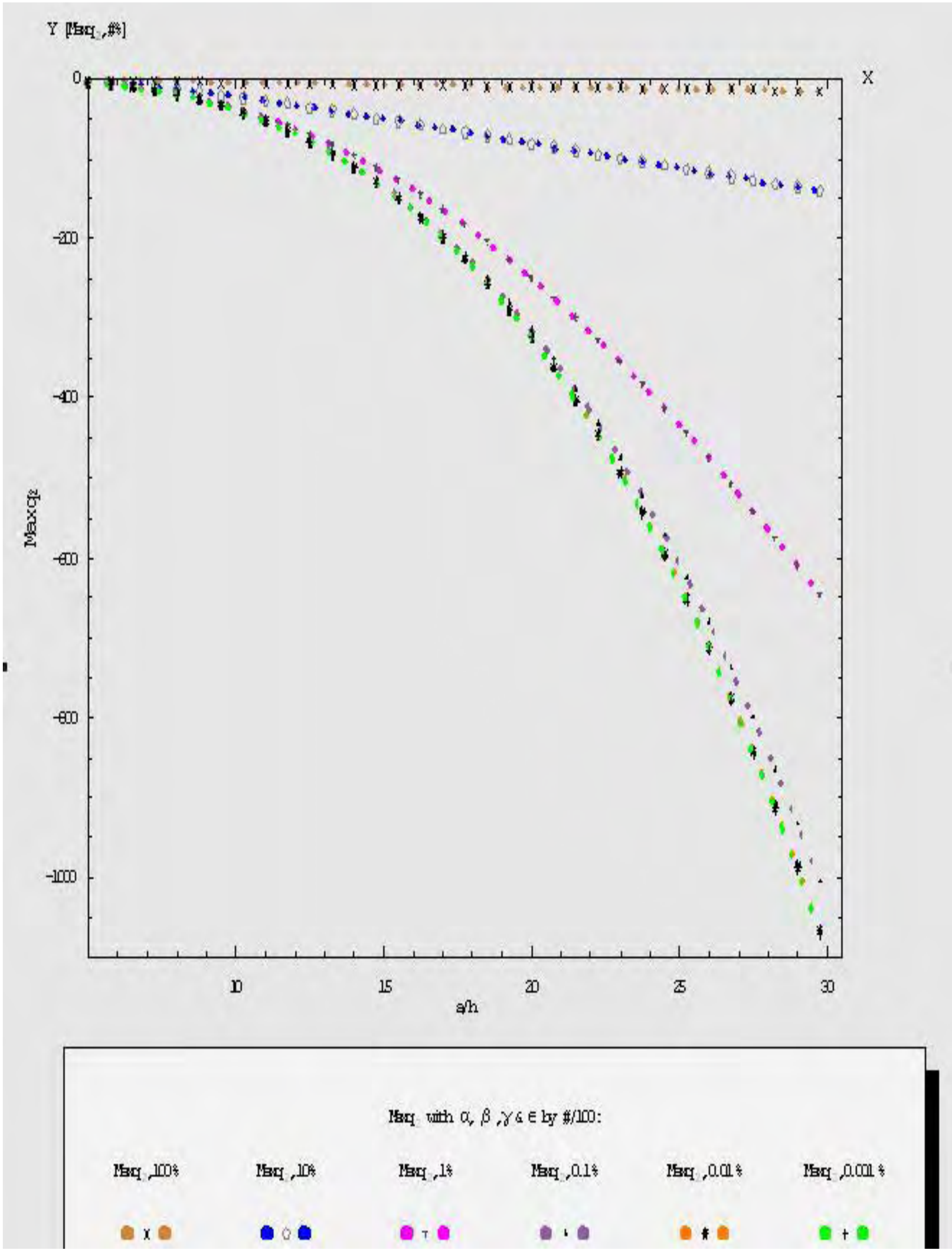

Figure 4.38 Maximum of the stress component q_2 for Steinberg-Reissner Model

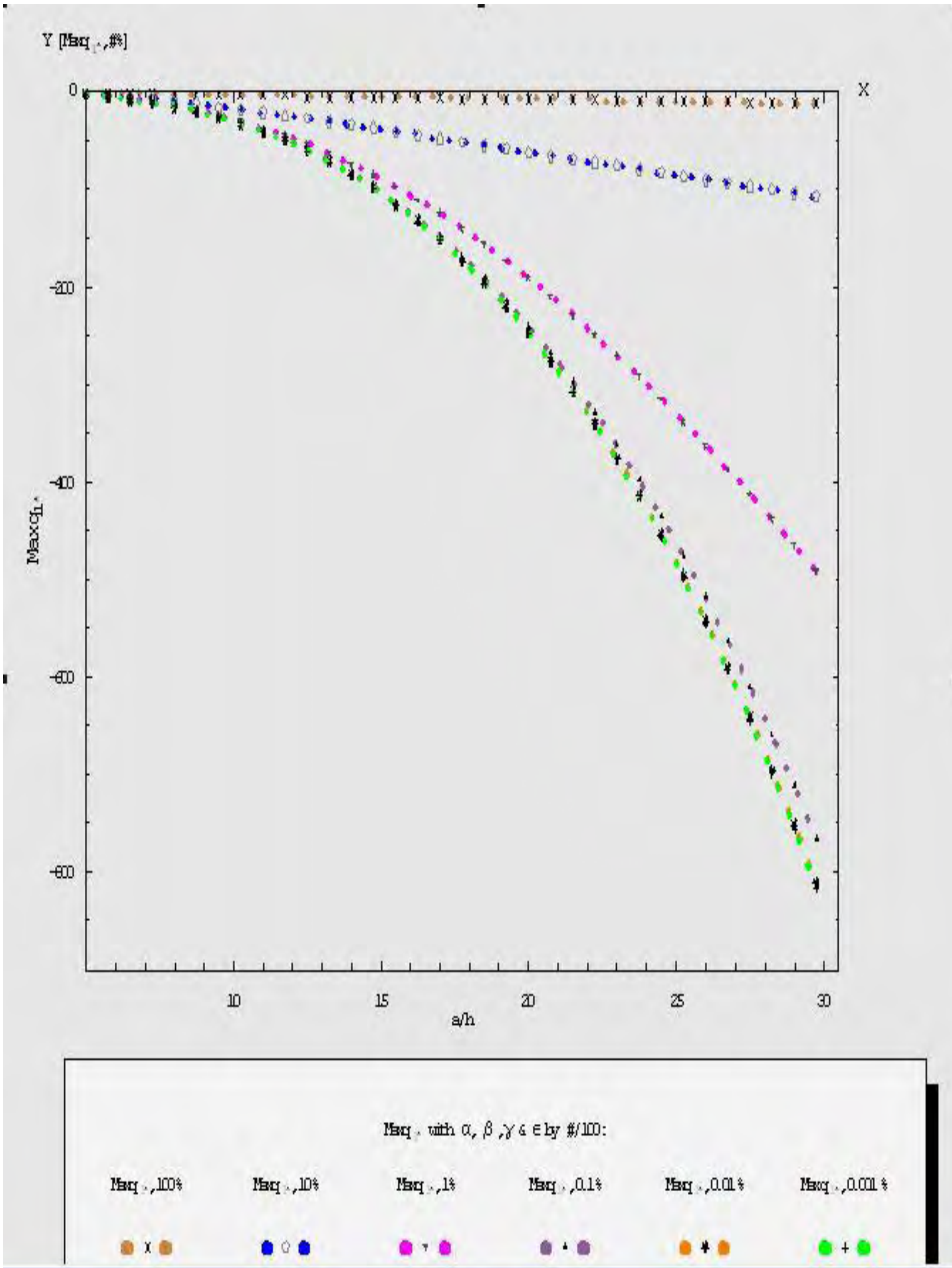

Figure 4.39 Maximum of the stress component q_1^* for Steinberg-Reissner Model

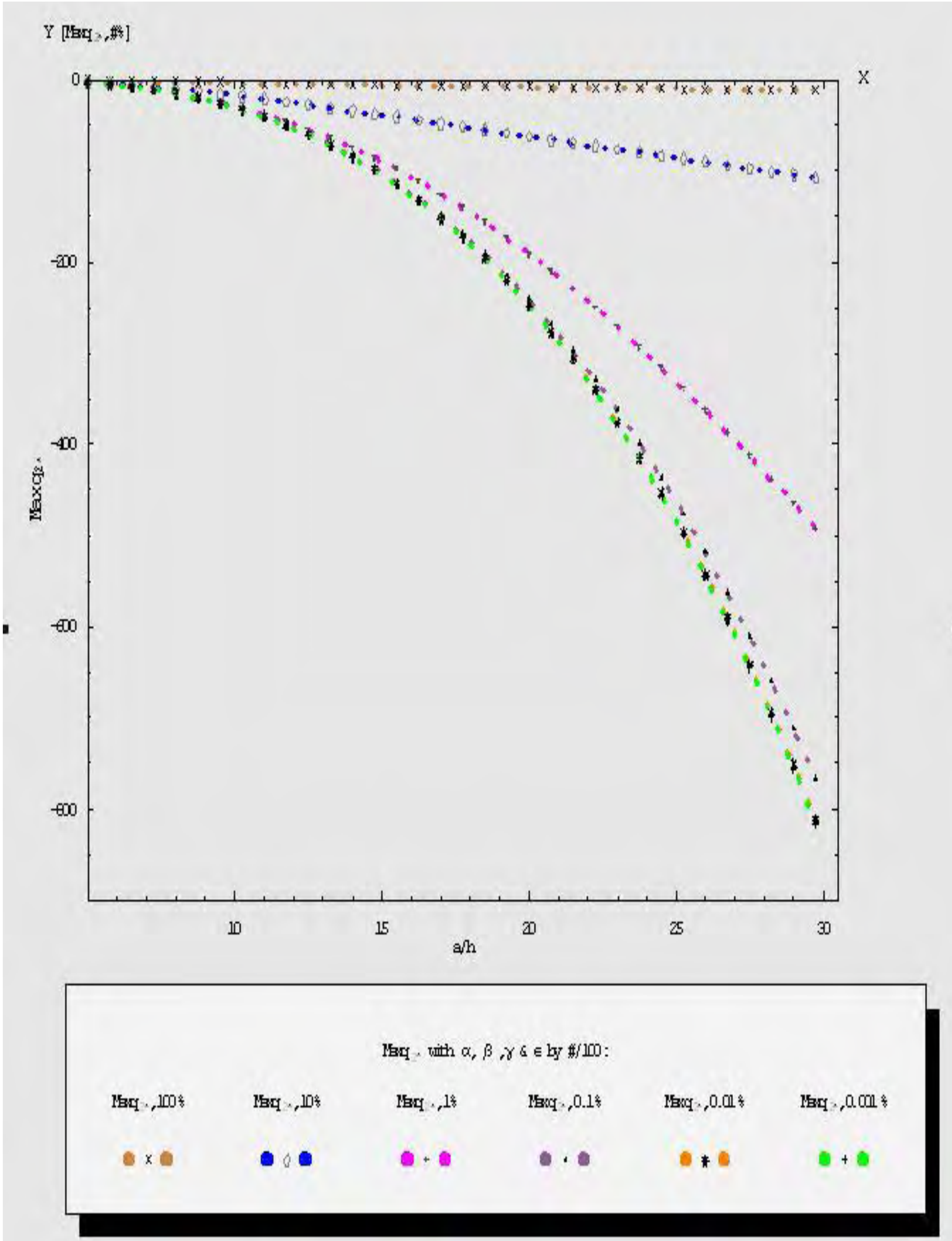

Figure 4.40 Maximum of the stress component q_2^* for Steinberg-Reissner Model

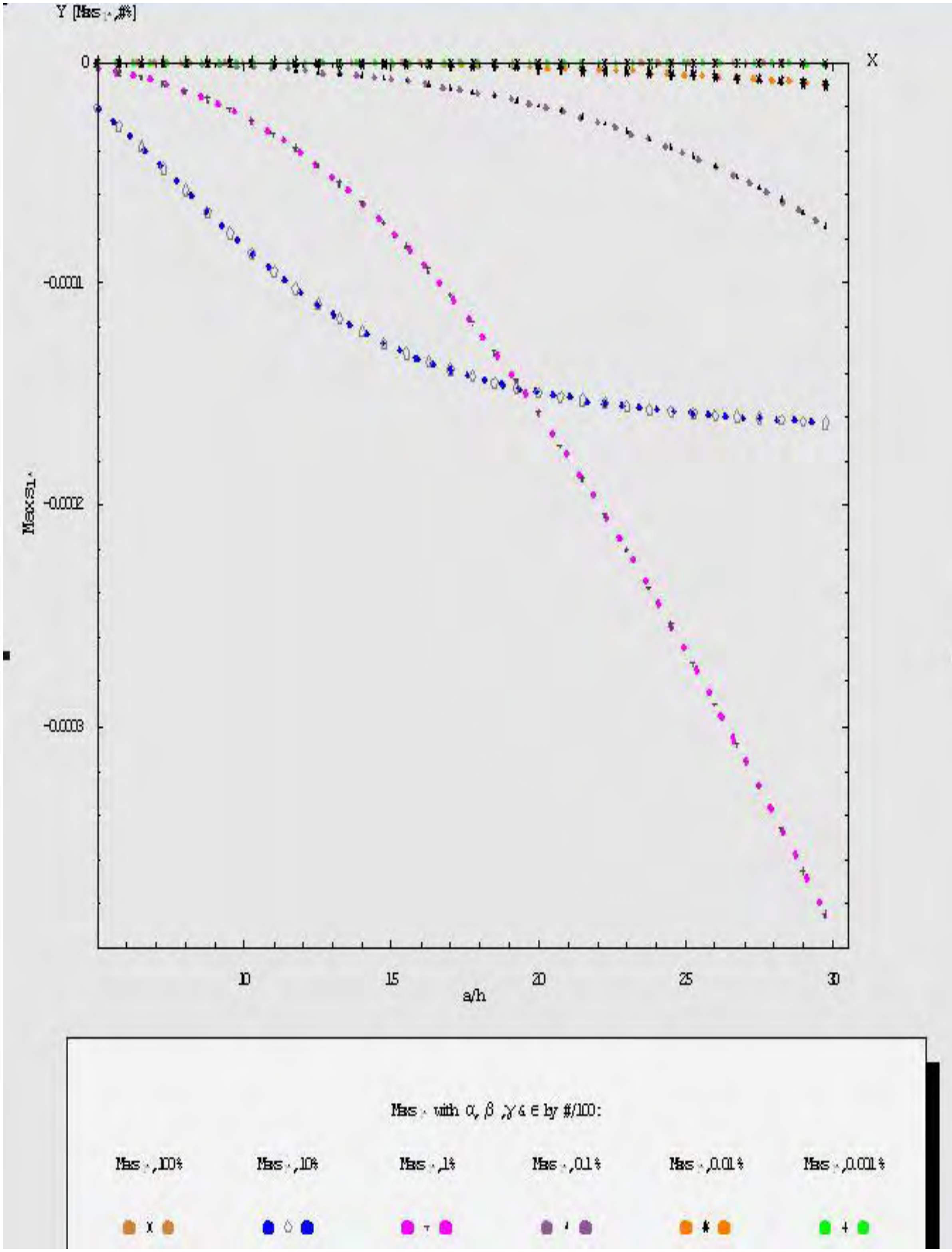

Figure 4.41 Maximum of the couple-stress component s_1^* for Steinberg-Reissner Model

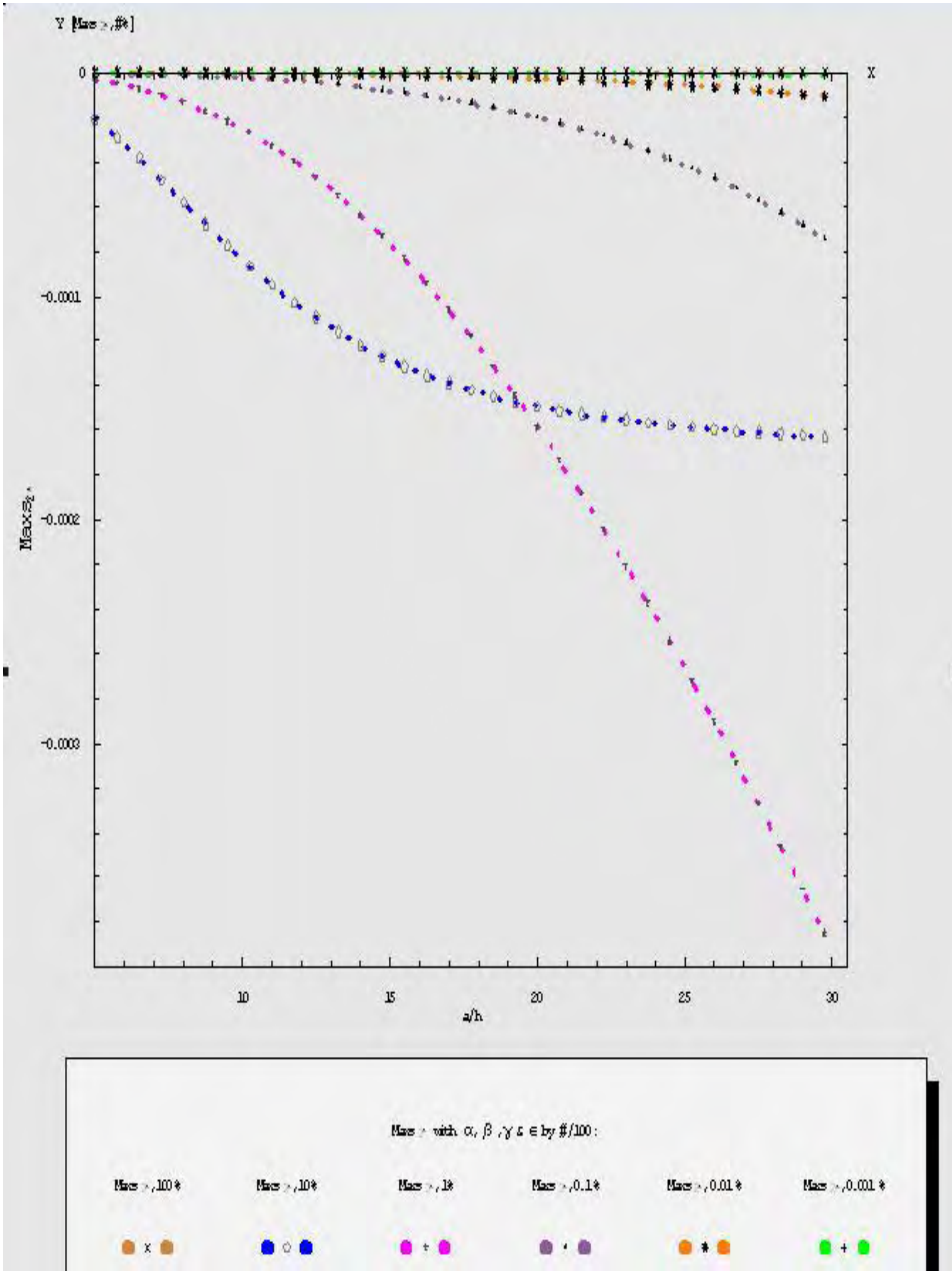

Figure 4.42 Maximum of the couple-stress component s_2^* for Steinberg-Reissner Model

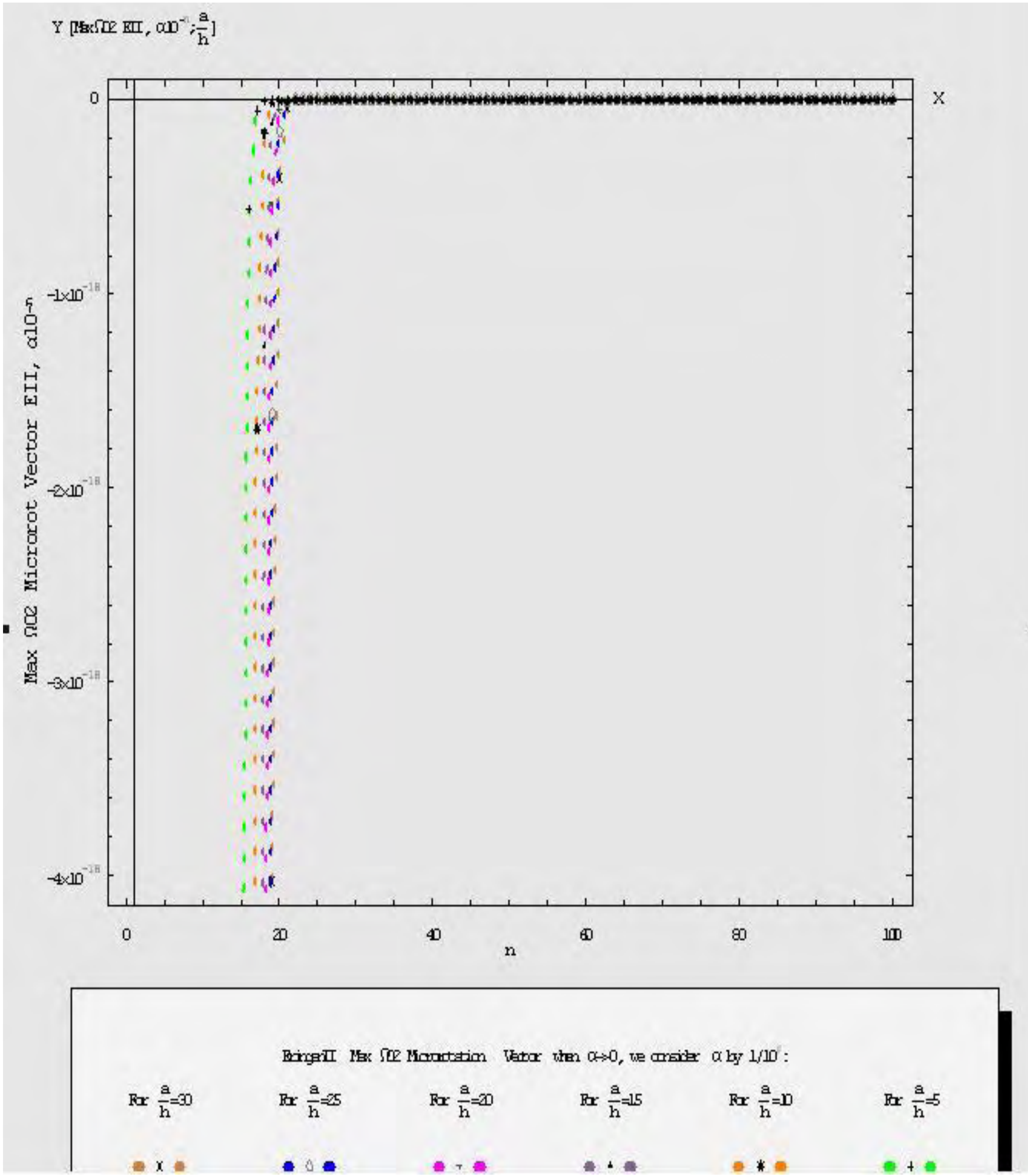

Figure 4.43 The Maximum of Micro-rotation Vector Ω_2^0 for Eringen Model II Converges to Zero

4.3 CONCLUSIONS

1. The Maximum Vertical Deflection of Steinberg-Reissner Model is always less than the Maximum Vertical Deflection of Reissner Model as we show in the figure 4.9, but according as the size of the side of the square "a" is smaller (close to $a = 0.5$) with respect to the thickness of the square "h"(for this experiment we keep constant $h = 0.1$). The Maximum Vertical Deflection of both models are near to mach (not only for $a = 0.5$, if not also for other values of "a" from 0 .5 to 3) while the effect of the microstructure decreases gradually from 100% to get to be 1%, almost disappear.

2. The Maximum Vertical Deflection calculated using the Eringen Model II is greater than calculated using the Reissner Model, but are close when it loses the effect of the microstructure from 100% to 1% and the size of the side of the square "a" go see becoming smaller (close to $a = 0.5$) with respect to its thickness "h"(for this experiment we keep constant the value of $h = 0.1$), but they do not become equal by a sufficiently small error equal to $\left|3.63 * 10^{-6}\right|$ (we obtain $\left|4.678 * 10^{-7}\right|$ when the effect of the microstructure is .001%). Then by transitivity the Eringen Model II and our model (the Steinberg-Reissner Model) are also close but they not converge. We show this behavior in the figure 4.11.

3. The modules of micro-rotation vector in the middle plane Ω_1^0 and Ω_2^0 converge to zero as the effect of microstructure disappears for the case of the models Steinberg-Reissner and Eringen II. This is shown in the Figure 4.16 that the Maximum of Ω_1^0 for Steinberg-Reissner Model Converges to Zero and the Figure 4.17 we show that the Maximum of Ω_1^0 for Eringen Model II Converges to Zero, for the case of Ω_1^0. Also the same applies to the case of Ω_2^0 for both models.

4. In the Figure 4.18 and Figure 4.19 the Behavior of the Maximum of Ω_1^0 as a function of α (with α in a Neighborhood of Zero) for Steinberg-Reissner Model and Eringen Model II, respectively are examples in the which $\Omega_1^0[\alpha]$ behaves as a function of α, with the property that $\Omega_1^0[\alpha_0 10^{-n}] = 10\Omega_1^0[\alpha_0 10^{-(n+1)}]$ for some "n" large enough and some number α_0. This is even more general: $\Omega_1^0[\alpha C] = C\Omega_1^0[\alpha]$, when α is in a Neighborhood of Zero and for any real constant C such that α and αC both also are in a Neighborhood of Zero. In other words we have that "The function Ω_1^0 satisfy the homogeneity property (corresponding to the definition of linear transformation) in a Neighborhood of Zero". This interesting property is true for Ω_2^0 in both models, the Steinberg-Reissner Model and the Eringen Model II.

APPENDICES

APPENDIX A

ELLIPTICITY OF THE FIELD EQUATIONS OF THE STEINBERG-REISSNER MODEL

The principal part of the system of the ***Field Equations*** **(2.48) (**seen as **(C.1))** is given by the following matrix**:**

$$P(x,\varepsilon)=\begin{pmatrix} k_1\varepsilon_1^2+k_2\varepsilon_2^2 & k_{10}\varepsilon_1\varepsilon_2 & 0 & 0 & 0 & 0 \\ k_{10}\varepsilon_1\varepsilon_2 & k_2\varepsilon_1^2+k_1\varepsilon_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_4(\varepsilon_1^2+\varepsilon_2^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & k_5(\varepsilon_1^2+\varepsilon_2^2) & 0 & 0 \\ 0 & 0 & 0 & 0 & k_7\varepsilon_1^2+k_8\varepsilon_2^2 & k_{14}\varepsilon_1\varepsilon_2 \\ 0 & 0 & 0 & 0 & k_{14}\varepsilon_1\varepsilon_2 & k_7\varepsilon_1^2+k_8\varepsilon_2^2 \end{pmatrix}$$

This matrix has determinant different of zero for all non-zero vectors $\varepsilon=(\varepsilon_1,\varepsilon_2)$ in R^2, i.e. the system of the ***Field Equations*** **(2.48)** is **Elliptic.**

Let $\varepsilon=(\varepsilon_1,\varepsilon_2)\neq 0$ any vector in R^2,

$$\begin{aligned}
&\det[P(x,\varepsilon)]=((k_1\varepsilon_1^2+k_2\varepsilon_2^2)(k_2\varepsilon_1^2+k_1\varepsilon_2^2)-k_{10}^2\varepsilon_1^2\varepsilon_2^2)\\
&k_4(\varepsilon_1^2+\varepsilon_2^2)k_5(\varepsilon_1^2+\varepsilon_2^2)((k_7\varepsilon_1^2+k_8\varepsilon_2^2)(k_7\varepsilon_2^2+k_8\varepsilon_1^2)-k_{14}^2\varepsilon_1^2\varepsilon_2^2)\\
&=(k_1k_2(\varepsilon_1^4+\varepsilon_2^4)+(k_1^2+k_2^2-k_{10}^2)\varepsilon_1^2\varepsilon_2^2)(k_7k_8(\varepsilon_1^4+\varepsilon_2^4)+\\
&(k_7^2+k_8^2-k_{14}^2)\varepsilon_1^2\varepsilon_2^2)k_4k_5(\varepsilon_1^2+\varepsilon_2^2)^2\\
&=[k_1k_2(\varepsilon_1^4+\varepsilon_2^4)+(k_1^2+k_2^2-k_{10}^2)\varepsilon_1^2\varepsilon_2^2](\varepsilon_1^2+\varepsilon_2^2)^2k_7k_8k_4k_5(\varepsilon_1^2+\varepsilon_2^2)^2\\
&=k_1k_2[(\varepsilon_1^2-\varepsilon_2^2)+(2+\frac{k_1^2+k_2^2-k_{10}^2}{k_1k_2})\varepsilon_1^2\varepsilon_2^2](\varepsilon_1^2+\varepsilon_2^2)^4k_7k_8k_4k_5\\
&=k_1k_2[(\varepsilon_1^2-\varepsilon_2^2)^2+\frac{((k_1+k_2)^2-k_{10}^2)}{k_1k_2}\varepsilon_1^2\varepsilon_2^2](\varepsilon_1^2+\varepsilon_2^2)^4k_7k_8k_4k_5\\
&=[k_1k_2(\varepsilon_1^2-\varepsilon_2^2)^2+2k_2(k_1+k_2+k_{10})\varepsilon_1^2\varepsilon_2^2](\varepsilon_1^2+\varepsilon_2^2)^4k_7k_8k_4k_5>0
\end{aligned}$$

Bellow we prove the following inequalities:

$$k_1,\ k_2,\ k_1+k_2+k_{10},\ \text{and}\ k_7k_8k_4k_5>0.$$

We have:

$$(\varepsilon_1^2 - \varepsilon_2^2)^2 \geq 0,$$

We have the following definitions:

$$k_1 := D(1-N^2),$$
$$k_2 := \frac{D(1-\nu)}{2},$$
$$k_{10} := \frac{D(1+\nu-2N^2)}{2}.$$

By the coercivity conditions **(1.12)** and Cosserat elastic energy we have $\mu+\alpha>0,$

From $3\lambda+2\mu>0$ and $\mu>0,$ we obtain:

$$\lambda > \frac{-2\mu}{3},$$
$$\lambda+2\mu > \lambda+\mu > \mu - \frac{2\mu}{3} > \frac{\mu}{3} > 0.$$

And then:

$$D = \frac{\mu h^3(\lambda+\mu)}{3(\lambda+2\mu)} > 0,$$

(a) $k_1+k_2+k_{10},\ k_2$ and $k_1>0$:

$$k_1+k_2+k_{10} = 2k_1 = 2D[1-N^2] = \frac{2\mu^2 h^3(\lambda+\mu)}{3(\lambda+2\mu)(\mu+\alpha)} > 0,$$

$$2k_2 = k_1+k_2-k_{10} = \frac{D}{2}(1-\nu)$$

$$= \frac{D(\lambda+2\mu)}{2(\lambda+\mu)} > 0,$$

and also $k_2>0.$

(b) $k_7 k_8 k_4 k_5 > 0$:

$$k_4 := \frac{5Gh}{6} = \frac{5\mu h}{6} > 0,$$

From $\gamma > 0$ & $3\beta + 2\gamma > 0$, $\beta > \frac{-2\gamma}{3}$. Then:

$$\beta + \gamma > \frac{-2\gamma}{3} + \gamma = \frac{\gamma}{3} > 0, \text{ and}$$
$$\beta + 2\gamma > \frac{-2\gamma}{3} + 2\gamma = \frac{4\gamma}{3} > 0.$$

And then:

$$2 - \Psi = 2 - \frac{2\gamma}{(\beta + 2\gamma)} = \frac{2(\beta + \gamma)}{\beta + 2\gamma} > 0.$$

$$l_t^2 = \frac{\gamma}{\mu} > 0,$$
$$(4l_b^2 - l_t^2) = 4\frac{(\gamma + \varepsilon)}{4\mu} - \frac{\gamma}{\mu} = \frac{\varepsilon}{\mu} > 0,$$
$$l_b^2 = \frac{\gamma + \varepsilon}{4\mu} > 0.$$

And then:

$$k_5 = \frac{[2k_2 l_t^2 (4l_b^2 - l_t^2)\frac{k_1}{D}]}{2l_b^2} > 0,$$

$$k_7 = 5h(1 - N^2) G l_t^2 \frac{(2 - \Psi)}{3} = 05h\frac{k_1}{D}\mu l_t^2 \frac{(2 - \Psi)}{3} > 0,$$

$$k_8 = 10h(1 - N^2)\frac{G l_b^2}{3} = 10h\frac{k_1}{D}\frac{\mu l_b^2}{3} > 0.$$

APPENDIX B

NUMERICAL SOLUTIONS AND PROGRAMS IN MATHEMATICA FOR PLATE THEORIES

In this appendix we show additional tables related to subchapter 4.2 (look at this subchapter). Also the corresponding files for some figures.

Input Data

We consider the quadratic plate with the following technical constants for our material in consideration **[24]**:

$$h = 0.1\,\mathbf{m},\ E = 2758\ \mathbf{Mpa},\ G = 1029.1\ \mathbf{Mpa},\ \nu = 0.34,$$

$$l_t = 0.065\ \mathbf{mm},\ l_b = 0.033,\ N^2 = 0.1,\ \Psi = 1.5$$

The following values of the elastic material constants are in correspondence with the technical constants ($h, E, G, \nu, l_t, l_b, N$ and Ψ):

$$\alpha = 114, \quad \lambda = 2186, \quad \beta = -2.898, \quad \gamma = 4.3, \quad \varepsilon = 0.135, \quad \mu = 1029.1.$$

Following, some programs in MATHEMATICA 4.1, used for comparison between solutions of the plate models.

Constants Used to Define the System of Linear Equations Corresponding to Steinberg-Reissner Model

$$\text{k1}[\{\alpha\_,\ \lambda\_,\ \mu\_,\ \text{h}\_\}] := \frac{\text{h}^3\,\mu^2\,(\lambda+\mu)\,(3\,\lambda+2\,\mu)}{12\,(\alpha+\mu)\,(0.75`\,\lambda^2+2\,\lambda\,\mu+\mu^2)}$$

$$\text{k2}[\{\lambda\_,\ \mu\_,\ \text{h}\_\}] := \frac{0.0417\,\text{h}^3\,\mu\,(0.5`\,\lambda+\mu)\,(3\,\lambda+2\,\mu)}{0.75`\,\lambda^2+2\,\lambda\,\mu+\mu^2}$$

$$\text{k3}[\{\lambda\_,\ \mu\_,\ \text{h}\_\}] := -\frac{0.4167\,\text{h}\,\mu\,(3\,\lambda+2\,\mu)}{1.5`\,\lambda+\mu}$$

$$\text{k5}[\{\alpha\_,\ \lambda\_,\ \gamma\_,\ \epsilon\_,\ \mu\_,\ \text{h}\_\}] := \frac{0.1667\,\text{h}^3\,\gamma\,(0.`\,\gamma+1.`\,\epsilon)\,\mu\,(0.5`\,\lambda+\mu)\,(3\,\lambda+2\,\mu)}{(\gamma+\epsilon)\,(\alpha+\mu)\,(0.75`\,\lambda^2+2\,\lambda\,\mu+\mu^2)}$$

$$\text{k6}[\{\alpha\_,\ \lambda\_,\ \mu\_,\ \text{h}\_\}] := \frac{\text{h}^3\,\alpha\,\mu\,(0.5`\,\lambda+\mu)\,(3\,\lambda+2\,\mu)}{6\,(\alpha+\mu)\,(0.75`\,\lambda^2+2\,\lambda\,\mu+\mu^2)}$$

$$\text{k7}[\{\alpha\_,\ \lambda\_,\ \beta\_,\ \gamma\_,\ \mu\_,\ \text{h}\_\}] := \frac{1.6667\,\text{h}\,\gamma\,(\beta+\gamma)\,\mu\,(3\,\lambda+2\,\mu)}{(\beta+2\,\gamma)\,(\alpha+\mu)\,(1.5`\,\lambda+\mu)}$$

$$\text{k8}[\{\alpha\_,\ \lambda\_,\ \gamma\_,\ \epsilon\_,\ \mu\_,\ \text{h}\_\}] := \frac{0.4167\,\text{h}\,(\gamma+\epsilon)\,\mu\,(3\,\lambda+2\,\mu)}{(\alpha+\mu)\,(1.5`\,\lambda+\mu)}$$

$$\text{k9}[\{\alpha\_,\ \lambda\_,\ \mu\_,\ \text{h}\_\}] := \frac{1.6667\,\text{h}\,\alpha\,\mu\,(3\,\lambda+2\,\mu)}{(\alpha+\mu)\,(1.5`\,\lambda+\mu)}$$

$$\text{k11}[\{\alpha\_,\ \lambda\_,\ \mu\_,\ \text{h}\_\}] := -\frac{0.4167\,\text{h}\,\mu\,(-\alpha+\mu)\,(3\,\lambda+2\,\mu)}{(\alpha+\mu)\,(1.5`\,\lambda+\mu)}$$

Matrix of the System of Linear Equations Corresponding to Steinberg-Reissner Model (SteinbergReissnerMatrix)

and

Vector of Constants (SteinbergReissnerSourc)

$$
\begin{aligned}
&\text{SteinbergReissnerMatrix}[\{\alpha\_,\lambda\_,\beta\_,\gamma\_,\epsilon\_,\mu\_,h\_,a\_,b\_,ph\_\}] :=\\
&\Big\{\Big\{-\frac{ph^2\,k1[\{\alpha,\lambda,\mu,h\}]}{a^2}-\frac{ph^2\,k2[\{\lambda,\mu,h\}]}{b^2}+k3[\{\lambda,\mu,h\}],\ -\frac{ph^2\,(k1[\{\alpha,\lambda,\mu,h\}]-k2[\{\lambda,\mu,h\}])}{a\,b},\\
&\quad \frac{ph\,k11[\{\alpha,\lambda,\mu,h\}]}{a},\ -\frac{0.5\text{`}\,ph\,k6[\{\alpha,\lambda,\mu,h\}]}{b},\ 0,\ 0.5\text{`}\,k9[\{\alpha,\lambda,\mu,h\}]\Big\},\\
&\quad \Big\{-\frac{ph^2\,(k1[\{\alpha,\lambda,\mu,h\}]-k2[\{\lambda,\mu,h\}])}{a\,b},\ -\frac{ph^2\,k1[\{\alpha,\lambda,\mu,h\}]}{b^2}-\frac{ph^2\,k2[\{\lambda,\mu,h\}]}{a^2}+k3[\{\lambda,\mu,h\}],\\
&\quad \frac{ph\,k11[\{\alpha,\lambda,\mu,h\}]}{b},\ -\frac{0.5\text{`}\,ph\,k6[\{\alpha,\lambda,\mu,h\}]}{a},\ -0.5\text{`}\,k9[\{\alpha,\lambda,\mu,h\}],\ 0\Big\},\\
&\quad \Big\{\frac{ph\,k11[\{\alpha,\lambda,\mu,h\}]}{a},\ \frac{ph\,k11[\{\alpha,\lambda,\mu,h\}]}{b},\ \left(\frac{ph^2}{a^2}+\frac{ph^2}{b^2}\right)k3[\{\lambda,\mu,h\}],\ 0,\ \frac{0.5\text{`}\,ph\,k9[\{\alpha,\lambda,\mu,h\}]}{b},\\
&\quad -\frac{0.5\text{`}\,ph\,k9[\{\alpha,\lambda,\mu,h\}]}{a}\Big\},\ \Big\{-\frac{0.5\text{`}\,ph\,k6[\{\alpha,\lambda,\mu,h\}]}{b},\ \frac{0.5\text{`}\,ph\,k6[\{\alpha,\lambda,\mu,h\}]}{a},\ 0,\\
&\quad \left(-\frac{ph^2}{a^2}-\frac{ph^2}{b^2}\right)k5[\{\alpha,\lambda,\gamma,\epsilon,\mu,h\}]-k6[\{\alpha,\lambda,\mu,h\}],\ 0,\ 0\Big\},\\
&\quad \Big\{0,\ 0.5\text{`}\,k9[\{\alpha,\lambda,\mu,h\}],\ \frac{0.5\text{`}\,ph\,k9[\{\alpha,\lambda,\mu,h\}]}{b},\ 0,\\
&\quad -\frac{ph^2\,k7[\{\alpha,\lambda,\beta,\gamma,\mu,h\}]}{a^2}-\frac{ph^2\,k8[\{\alpha,\lambda,\gamma,\epsilon,\mu,h\}]}{b^2}-k9[\{\alpha,\lambda,\mu,h\}],\\
&\quad \frac{ph^2\,(-k7[\{\alpha,\lambda,\beta,\gamma,\mu,h\}]+k8[\{\alpha,\lambda,\gamma,\epsilon,\mu,h\}])}{a\,b}\Big\},\\
&\quad \Big\{0.5\text{`}\,k9[\{\alpha,\lambda,\mu,h\}],\ 0,\ \frac{0.5\text{`}\,ph\,k9[\{\alpha,\lambda,\mu,h\}]}{a},\ 0,\ \frac{ph^2\,(k7[\{\alpha,\lambda,\beta,\gamma,\mu,h\}]-k8[\{\alpha,\lambda,\gamma,\epsilon,\mu,h\}])}{a\,b},\\
&\quad \frac{ph^2\,k7[\{\alpha,\lambda,\beta,\gamma,\mu,h\}]}{b^2}+\frac{ph^2\,k8[\{\alpha,\lambda,\gamma,\epsilon,\mu,h\}]}{a^2}+k9[\{\alpha,\lambda,\mu,h\}]\Big\}\Big\}
\end{aligned}
$$

$$
\text{SteinbergReissnerSourc}[\{\alpha\_,\lambda\_,\mu\_,h\_,a\_,b\_\}] := \Big\{-\frac{0.1571\,h^2\,\lambda\,\mu}{a\,(\alpha+\mu)\,(0.5\text{`}\,\lambda+\mu)},\ -\frac{0.1571\,h^2\,\lambda\,\mu}{b\,(\alpha+\mu)\,(0.5\text{`}\,\lambda+\mu)},\ -\frac{\mu}{\alpha+\mu},\ 0,\ 0,\ 0\Big\}
$$

Functions Used to Define the Maximum of Each Unknown Function of the System of Linear Equations Corresponding to Steinberg-Reissner Model

(The Tikhonov Regularization Method has been applied)

```
Maxψ1RotationVectorSteinbergReissnerTikhonovRegularization[
  {α_, λ_, β_, γ_, ϵ_, μ_, h_, a_, b_, ph_, t_, FactorTikhonovRegularizationEffect_}] :=
 {Inverse[Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
      SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}] +
     FactorTikhonovRegularizationEffect^2 * IdentityMatrix[6]].
   Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
   SteinbergReissnerSourc[{α/t, λ, μ, h, a*0.1, b*0.1}]}[[1, 1]]
```

```
Maxψ2RotationVectorSteinbergReissnerTikhonovRegularization[
  {α_, λ_, β_, γ_, ϵ_, μ_, h_, a_, b_, ph_, t_, FactorTikhonovRegularizationEffect_}] :=
 {Inverse[Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
      SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}] +
     FactorTikhonovRegularizationEffect^2 * IdentityMatrix[6]].
   Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
   SteinbergReissnerSourc[{α/t, λ, μ, h, a*0.1, b*0.1}]}[[1, 2]]
```

```
MaxwVerticalDeflectionSteinbergReissnerTikhonovRegularization[
  {α_, λ_, β_, γ_, ϵ_, μ_, h_, a_, b_, ph_, t_, FactorTikhonovRegularizationEffect_}] :=
 {Inverse[Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
      SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}] +
     FactorTikhonovRegularizationEffect^2 * IdentityMatrix[6]].
   Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
   SteinbergReissnerSourc[{α/t, λ, μ, h, a*0.1, b*0.1}]}[[1, 3]]
```

```
MaxΩ3DerivativeMicrorotationSteinbergReissnerTikhonovRegularization[
  {α_, λ_, β_, γ_, ϵ_, μ_, h_, a_, b_, ph_, t_, FactorTikhonovRegularizationEffect_}] :=
 {Inverse[Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
      SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}] +
     FactorTikhonovRegularizationEffect^2 * IdentityMatrix[6]].
   Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
   SteinbergReissnerSourc[{α/t, λ, μ, h, a*0.1, b*0.1}]}[[1, 4]]
```

```
MaxΩ01MicrorotationVectorSteinbergReissnerTikhonovRegularization[
  {α_, λ_, β_, γ_, ϵ_, μ_, h_, a_, b_, ph_, t_, FactorTikhonovRegularizationEffect_}] :=
 {Inverse[Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
      SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}] +
     FactorTikhonovRegularizationEffect^2 * IdentityMatrix[6]].
   Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
   SteinbergReissnerSourc[{α/t, λ, μ, h, a*0.1, b*0.1}]}[[1, 5]]
```

```
MaxΩ02MicrorotationVectorSteinbergReissnerTikhonovRegularization[
  {α_, λ_, β_, γ_, ϵ_, μ_, h_, a_, b_, ph_, t_, FactorTikhonovRegularizationEffect_}] :=
 {Inverse[Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
      SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}] +
     FactorTikhonovRegularizationEffect^2 * IdentityMatrix[6]].
   Transpose[SteinbergReissnerMatrix[{α/t, λ, β/t, γ/t, ϵ/t, μ, h, a*0.1, b*0.1, ph}]].
   SteinbergReissnerSourc[{α/t, λ, μ, h, a*0.1, b*0.1}]}[[1, 6]]
```

Formula Used to EringenIIMatrix (Corresponding to the Matrix of the System of Linear Equations of Eringen II Model) and Formula to EringenIISourc (Corresponding to the Vector of Constants of the Linear System of Eringen II Model)

```
EringenIIMatrix[{α_, λ_, β_, γ_, μ_, h_, a_, b_, ph_, m_, n_}] :=
```

$$\Big\{\Big\{-h\,(\alpha+\mu)+\frac{1}{12}\,h^3\left(-\frac{m^2\,ph^2}{b^2}-\frac{n^2\,ph^2}{a^2}\right)(\alpha+\mu)-\frac{h^3\,n^2\,ph^2\left(-\alpha+\frac{3\,\lambda\,\mu+2\,\mu^2}{\lambda+2\,\mu}\right)}{12\,a^2},\ -\frac{h^3\,m\,n\,ph^2\left(-\alpha+\frac{3\,\lambda\,\mu+2\,\mu^2}{\lambda+2\,\mu}\right)}{12\,a\,b},$$
$$-\frac{h\,n\,ph\,(-\alpha+\mu)}{a},\ 0,\ 2\,h\,\alpha\Big\},$$
$$\Big\{-\frac{h^3\,m\,n\,ph^2\left(-\alpha+\frac{3\,\lambda\,\mu+2\,\mu^2}{\lambda+2\,\mu}\right)}{12\,a\,b},\ -h\,(\alpha+\mu)+\frac{1}{12}\,h^3\left(-\frac{m^2\,ph^2}{b^2}-\frac{n^2\,ph^2}{a^2}\right)(\alpha+\mu)-\frac{h^3\,m^2\,ph^2\left(-\alpha+\frac{3\,\lambda\,\mu+2\,\mu^2}{\lambda+2\,\mu}\right)}{12\,b^2},$$
$$-\frac{h\,m\,ph\,(-\alpha+\mu)}{b},\ -2\,h\,\alpha,\ 0\Big\},\ \Big\{-\frac{n\,ph\,(-\alpha+\mu)}{a},\ -\frac{m\,ph\,(-\alpha+\mu)}{b},\ \left(-\frac{m^2\,ph^2}{b^2}-\frac{n^2\,ph^2}{a^2}\right)(\alpha+\mu),\ \frac{2\,m\,ph\,\alpha}{b},\ 0\Big\},$$
$$\Big\{0,\ -2\,\alpha,\ \frac{2\,m\,ph\,\alpha}{b},\ -4\,\alpha-\frac{n^2\,ph^2\,(\alpha+\beta)}{a^2}+\left(-\frac{m^2\,ph^2}{b^2}-\frac{n^2\,ph^2}{a^2}\right)\gamma,\ -\frac{m\,n\,ph^2\,(\alpha+\beta)}{a\,b}\Big\},$$
$$\Big\{2\,\alpha,\ 0,\ -\frac{2\,n\,ph\,\alpha}{a},\ -\frac{m\,n\,ph^2\,(\alpha+\beta)}{a\,b},\ -4\,\alpha-\frac{m^2\,ph^2\,(\alpha+\beta)}{b^2}+\left(-\frac{m^2\,ph^2}{b^2}-\frac{n^2\,ph^2}{a^2}\right)\gamma\Big\}\Big\}$$

```
EringenIISourc := {0, 0, -10, 0, 0}
```

Following, we write the formulas used to the Maximum of each unknown function of the linear system of equations corresponding to Eringen II Model. The Tikhonov Regularization Method has been applied.

```
Maxφ1RotationVectorEringenIITikhonovRegularization[
  {α_, λ_, β_, γ_, μ_, h_, a_, b_, ph_, m_, n_, t_, FactorTikhonovRegularizationEffect_}] :=
 {Inverse[Transpose[EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}]].
       EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}] +
      FactorTikhonovRegularizationEffect^2 * IdentityMatrix[5]].
    Transpose[EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}]].EringenIISourc}[[1, 1]]

Maxφ2RotationVectorEringenIITikhonovRegularization[
  {α_, λ_, β_, γ_, μ_, h_, a_, b_, ph_, m_, n_, t_, FactorTikhonovRegularizationEffect_}] :=
 {Inverse[Transpose[EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}]].
       EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}] +
      FactorTikhonovRegularizationEffect^2 * IdentityMatrix[5]].
    Transpose[EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}]].EringenIISourc}[[1, 2]]

MaxwVerticalDeflectionEringenIITikhonovRegularization[
  {α_, λ_, β_, γ_, μ_, h_, a_, b_, ph_, m_, n_, t_, FactorTikhonovRegularizationEffect_}] :=
 {Inverse[Transpose[EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}]].
       EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}] +
      FactorTikhonovRegularizationEffect^2 * IdentityMatrix[5]].
    Transpose[EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}]].EringenIISourc}[[1, 3]]

MaxΩ01MicrorotationVectorEringenIITikhonovRegularization[
  {α_, λ_, β_, γ_, μ_, h_, a_, b_, ph_, m_, n_, t_, FactorTikhonovRegularizationEffect_}] :=
 {Inverse[Transpose[EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}]].
       EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}] +
      FactorTikhonovRegularizationEffect^2 * IdentityMatrix[5]].
    Transpose[EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}]].EringenIISourc}[[1, 4]]
MaxΩ02MicrorotationVectorEringenIITikhonovRegularization[
  {α_, λ_, β_, γ_, μ_, h_, a_, b_, ph_, m_, n_, t_, FactorTikhonovRegularizationEffect_}] :=
 {Inverse[Transpose[EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}]].
       EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}] +
      FactorTikhonovRegularizationEffect^2 * IdentityMatrix[5]].
    Transpose[EringenIIMatrix[{α/t, λ, β/t, γ/t, μ, h, a*0.1, b*0.1, ph, m, n}]].EringenIISourc}[[1, 5]]
```

Mathematica Packages: Plotting Package

```
<< Graphics`MultipleListPlot`

<< Graphics`Legend`

<< Graphics`Colors`

<< Graphics`FilledPlot`
```

Now, we show some programs used to the graphical representation of the maximum of each component of the vector solution (only the first coefficient of each trigonometric series corresponding to each unknown function) of the linear system corresponding to the Steinberg-Reissner and Eringen II Models and their comparisons.

```
MultipleListPlot[
 Table[
  {a, MaxVerticalDeflectionSteinbergReissnerTikhonovRegularization[
     {114.34494, 2186.84701, -2.89864, \4.34797, 0.13481, 1029.1045, 0.1, a, a, Pi, 1, 0}] /
    MaxVerticalDeflectionReissner[{2186.847, 1029.104478, 0.1, a, a}]}, {a, 5, 30, 0.75}],
 Table[
  {a, MaxVerticalDeflectionSteinbergReissnerTikhonovRegularization[
     {114.3449, 2186.847, -2.8986, 4.347967, 0.13481, 1029.1045, 0.1, a, a, Pi, 10, 0.0102}] /
    MaxVerticalDeflectionReissner[{2186.847, 1029.1045, 0.1, a, a}]}, {a, 5, 30, 0.75}],
 Table[
  {a, MaxVerticalDeflectionSteinbergReissnerTikhonovRegularization[
     {114.3449, 2186.847, -2.898645, 4.347966, 0.1348, 1029.104478, 0.1, a, a, Pi, 100, 0.00489}] /
    MaxVerticalDeflectionReissner[{2186.847, 1029.104478, 0.1, a, a}]}, {a, 5, 30, 0.75}],
 Table[
  {a, MaxVerticalDeflectionSteinbergReissnerTikhonovRegularization[
     {114.3449, 2186.847, -2.8986, 4.34797, 0.13481, 1029.104478, 0.1, a, a, Pi, 1000, 0.000075}] /
    MaxVerticalDeflectionReissner[{2186.847, 1029.104478, 0.1, a, a}]}, {a, 5, 30, 0.75}],
 Table[
  {a, MaxVerticalDeflectionSteinbergReissnerTikhonovRegularization[
     {114.34494, 2186.847, -2.8986, 4.34797, 0.1348, 1029.104478, 0.1, a, a, Pi, 10000, 0.008629}] /
    MaxVerticalDeflectionReissner[{2186.847, 1029.104478, 0.1, a, a}]}, {a, 5, 30, 0.75}],
 Table[
  {a, MaxVerticalDeflectionSteinbergReissnerTikhonovRegularization[
     {114.3449, 2186.847, -2.8986, \4.347966, 0.1348, 1029.1044776, 0.1, a, a, Pi, 100000, 0.010914}] /
    MaxVerticalDeflectionReissner[{2186.847, 1029.1044776, 0.1, a, a}]}, {a, 5, 30, 0.75}],
 PlotStyle → {{Dashing[{Dot}], Peru, Thickness[0.005]}, {Dashing[{Dot}], RGBColor[0, 0, 1], Thickness[0.005]},
   {Dashing[{Dot}], RGBColor[1, 0, 1], Thickness[0.005]}, {Dashing[{Dot}], Violet, Thickness[0.005]},
   {Dashing[{Dot}], Orange, Thickness[0.005]}, {Dashing[{Dot}], RGBColor[0., 1., 0.], Thickness[0.005]}},
 SymbolShape → {MakeSymbol[{Line[{{2, 2}, {-2, -2}}], Line[{{-2, 2}, {2, -2}}]}], MakeSymbol[RegularPolygon[5, 3]],
  PlotSymbol[Star], PlotSymbol[Triangle], MakeSymbol[RegularPolygon[7, 2.6, {0, 0}, 0, 3]],
  MakeSymbol[{Line[{{0, -2}, {0, 2}}], Line[{{-2, 0}, {2, 0}}]}]},
SymbolStyle → {GrayLevel[0], GrayLevel[.5], GrayLevel[.3], GrayLevel[0], GrayLevel[0], GrayLevel[0]},
PlotLegend -> {"SR, 100 %/R", "SR, 10 %/R", "SR, 1%/R", "SR, 0.1%/R", "SR, 0.01%/R", "SR, 0.001%/R"},
LegendPosition → {-.9, -0.95}, LegendSize → {2, .3}, LegendTextSpace → .05,
LegendLabel → "Steinberg-Reissner Max Vert Deflection over Reissner Max Vert Deflection with α,
   β , γ & ϵ by #/100:", LegendLabelSpace → 1.5, LegendOrientation → Horizontal,
LegendBackground → GrayLevel[0.95], LegendShadow → {.03, -.03}, Background → GrayLevel[.9],
PlotJoined → True, AxesFront → True, Axes → True, AxesOrigin → {5, 0}, AxesLabel → {"X ", "Y [SR, #%/R]"},
Frame → True, FrameLabel → {"a/h", "Max Vert Deflection S-R / Max Vert Deflection R"},
PlotRange → {{5, 30.5}, {0, 1.1}}]
```

```
MultipleListPlot[
 Table[{a, MaxΩD1MicrorotationVectorEringenIITikhonovRegularization[
     {114.3449, 2186.847, -2.8986, 4.347967, 1029.104, 0.1, a, a, Pi, 1, 1, 1, 0}]}, {a, 5, 30, 0.75}],
 Table[{a, MaxΩD1MicrorotationVectorEringenIITikhonovRegularization[
     {114.3449, 2186.847, -2.8986, 4.347967, 1029.104, 0.1, a, a, Pi, 1, 1, 10, 0}]}, {a, 5, 30, 0.75}],
 Table[{a, MaxΩD1MicrorotationVectorEringenIITikhonovRegularization[
     {114.3449, 2186.847, -2.8986, 4.347967, 1029.104, 0.1, a, a, Pi, 1, 1, 100, 0}]}, {a, 5, 30, 0.75}],
 Table[{a, MaxΩD1MicrorotationVectorEringenIITikhonovRegularization[
     {114.3449, 2186.847, -2.8986, 4.347967, 1029.104, 0.1, a, a, Pi, 1, 1, 1000, 0}]}, {a, 5, 30, 0.75}],
 Table[{a, MaxΩD1MicrorotationVectorEringenIITikhonovRegularization[
     {114.3449, 2186.847, -2.8986, 4.347967, 1029.104478, 0.1, a, a, Pi, 1, 1, 10000, 0.103}]}, {a, 5, 30, 0.75}],
 Table[{a, MaxΩD1MicrorotationVectorEringenIITikhonovRegularization[
     {114.3449, 2186.847, -2.8986, 4.347966, 1029.104, 0.1, a, a, Pi, 1, 1, 100000, 0.121}]}, {a, 5, 30, 0.75}],
 PlotStyle → {{Dashing[{Dot}], Peru, Thickness[0.005]}, {Dashing[{Dot}], RGBColor[0, 0, 1], Thickness[0.005]},
   {Dashing[{Dot}], RGBColor[1, 0, 1], Thickness[0.005]}, {Dashing[{Dot}], Violet, Thickness[0.005]},
   {Dashing[{Dot}], Orange, Thickness[0.005]}, {Dashing[{Dot}], RGBColor[0., 1., 0.], Thickness[0.005]}},
 SymbolShape → {MakeSymbol[{Line[{{2, 2}, {-2, -2}}], Line[{{-2, 2}, {2, -2}}]}], MakeSymbol[RegularPolygon[5, 3]],
   PlotSymbol[Star], PlotSymbol[Triangle], MakeSymbol[RegularPolygon[7, 2.6, {0, 0}, 0, 3]],
   MakeSymbol[{Line[{{0, -2}, {0, 2}}], Line[{{-2, 0}, {2, 0}}]}]},
 SymbolStyle → {GrayLevel[0], GrayLevel[.5], GrayLevel[.3], GrayLevel[0], GrayLevel[0], GrayLevel[0]},
 PlotLegend -> {"EII,100%", "EII,10%", "EII,1%", "EII,0.1%", "EII,0.01%", "EII,0.001%"},
 LegendPosition → {-.9, -0.95}, LegendSize → {2, .3}, LegendTextSpace → .05,
 LegendLabel → "Eringen II Max ΩD1 Microrotation Vector with α, β ,γ & ϵ by #/100:", LegendLabelSpace → 1.5,
 LegendOrientation → Horizontal, LegendBackground → GrayLevel[0.95], LegendShadow → {.03, -.03},
 Background → GrayLevel[.9], PlotJoined → True, AxesFront → True, Axes → True, AxesOrigin → {5, 0},
 AxesLabel → {"X ", "Y [E-II,#%]"}, Frame → True, FrameLabel → {"a/h", "Max ΩD1 Microrotation Vector E-II"},
 PlotRange -> {{5, 30.5}, {0, 0.8}}]
```

Tikhonov Regularization Method

Assume X, Y are Hilbert spaces. To obtain regularized solution to the linear system A x = y, choose -x- to fit data -y- in least-squares sense, but penalize solutions of "large norm". In other words, solve the following minimization problem:

$$x_{\alpha} = \arg\min_{x \in X} \|Ax - y\|_{Y}^{2} + \alpha \|x\|_{X}^{2}$$

$$= (A^{*}A + \alpha\, I)^{-1} A^{*} y$$

, where α is called the *regularization parameter or factor of regularization of Tikhonov*.

Output Data

Below we show tables for Maximum of the Vertical Deflection Corresponding to Plate Theories.

Table 1 Maximum Vertical Deflection of the Middle Plane of the Plate Corresponding to six Plate Theories Considering the 100 % of the effect of the Microstructure

Model	$a/h = 5$	$a/h = 10$	$a/h = 15$	$a/h = 20$	$a/h = 25$	$a/h = 30$
Steinberg - Reissner ($\alpha = 114$, $\lambda = 2186.8$, $\beta = -2.898$, $\gamma = 4.3$, $\epsilon = 0.135$, $\mu = 1029.$, $h = 0.1$)	0.00029	0.00147	0.00349	0.00633	0.00998	0.01445
Reissner ($\lambda = 2186.8$, $\mu = 1029$, $h = 0.1$)	0.00074	0.01037	0.0511	0.15998	0.38884	0.80436
Eringen I ($e_{33} = 0$; $\alpha = 114$, $\lambda = 2186.8$, $\beta = -2.898$, $\gamma = 4.3$, $\epsilon = 0.135$, $\mu = 1029$, $h = 0.1$)	0.00034	0.00379	0.01754	0.05359	0.12874	0.2646
Eringen I Reduction to Elastic Classic Materials ($e_{33} = 0$; $\alpha = 0$, $\lambda = 2186.8$, $\beta = \gamma = \epsilon = 0$, $\mu = 1029$, $h = 0.1$)	0.00058	0.00775	0.03784	0.11805	0.28648	0.59209
Eringen II ($t_{33} = 0$; $\alpha = 114$, $\lambda = 2186.8$, $\beta = -2.898$, $\gamma = 4.3$, $\mu = 1029$, $h = 0.1$)	0.00033	0.00242	0.00712	0.01439	0.02407	0.03608
Eringen II Reduction to Elastic Classic Materials ($t_{33} = 0$; $\alpha = 0$, $\lambda = 2186.8$, $\beta = \gamma = 0$, $\mu = 1029$, $h = 0.1$)	0.00074	0.01037	0.0511	0.15998	0.38885	0.80438

Table 2 Maximum Vertical Deflection of the Middle Plane of the Plate Corresponding to six Plate Theories Considering the 10% of the effect of the Microstructure

Model	$a/h=5$	$a/h=10$	$a/h=15$	$a/h=20$	$a/h=25$	$a/h=30$
Steinberg - Reissner ($\alpha=114/10$, $\lambda=2186.8$, $\beta=-2.898/10$, $\gamma=4.3/10$, $\epsilon=0.135/10$, $\mu=1029.$, $h=0.1$)	0.00063	0.00606	0.01952	0.04113	0.07039	0.10692
Reissner ($\lambda=2186.8$, $\mu=1029$, $h=0.1$)	0.00074	0.01037	0.0511	0.15998	0.38884	0.80436
Eringen I ($e_{33}=0$; $\alpha=114/10$, $\lambda=2186.8$, $\beta=-2.898/10$, $\gamma=4.3/10$, $\epsilon=0.135/10$, $\mu=1029$, $h=0.1$)	0.00054	0.007	0.03388	0.10532	0.25513	0.52676
Eringen I Reduction to Elastic Classic Materials ($e_{33}=0$; $\alpha=0$, $\lambda=2186.8$, $\beta=\gamma=\epsilon=0$, $\mu=1029$, $h=0.1$)	0.00058	0.00775	0.03784	0.11805	0.28648	0.59209
Eringen II ($t_{33}=0$; $\alpha=114/10$, $\lambda=2186.8$, $\beta=-2.898/10$, $\gamma=4.3/10$, $\mu=1029$, $h=0.1$)	0.00065	0.00778	0.03151	0.07935	0.15431	0.25666
Eringen II Reduction to Elastic Classic Materials ($t_{33}=0$; $\alpha=0$, $\lambda=2186.8$, $\beta=\gamma=0$, $\mu=1029$, $h=0.1$)	0.00074	0.01037	0.0511	0.15998	0.38885	0.80438

Table 3 Maximum Vertical Deflection of the Middle Plane of the Plate Corresponding to six Plate Theories Considering the 1% of the effect of the Microstructure

Model	a/h = 5	a/h = 10	a/h = 15	a/h = 20	a/h = 25	a/h = 30
Steinberg – Reissner ($\alpha = 114/100$, $\lambda = 2186.8$, $\beta = -2.898/100$, $\gamma = 4.3/100$, $\epsilon = 0.135/100$, $\mu = 1029.$, $h = 0.1$)	0.00073	0.00967	0.04387	0.12358	0.26609	0.48303
Reissner ($\lambda = 2186.8$, $\mu = 1029$, $h = 0.1$)	0.00074	0.01037	0.0511	0.15998	0.38884	0.80436
Eringen I ($e_{33} = 0$; $\alpha = 114/100$, $\lambda = 2186.8$, $\beta = -2.898/100$, $\gamma = 4.3/100$, $\epsilon = 0.135/100$, $\mu = 1029$, $h = 0.1$)	0.00057	0.00767	0.0374	0.11664	0.283	0.58483
Eringen I Reduction to Elastic Classic Materials ($e_{33} = 0$; $\alpha = 0$, $\lambda = 2186.8$, $\beta = \gamma = \epsilon = 0$, $\mu = 1029$, $h = 0.1$)	0.00058	0.00775	0.03784	0.11805	0.28648	0.59209
Eringen II ($t_{33} = 0$; $\alpha = 114/100$, $\lambda = 2186.8$, $\beta = -2.898/100$, $\gamma = 4.3/100$, $\mu = 1029$, $h = 0.1$)	0.00073	0.01003	0.04811	0.14522	0.33754	0.66289
Eringen II Reduction to Elastic Classic Materials ($t_{33} = 0$; $\alpha = 0$, $\lambda = 2186.8$, $\beta = \gamma = 0$, $\mu = 1029$, $h = 0.1$)	0.00074	0.01037	0.0511	0.15998	0.38885	0.80438

Table 4 Maximum Vertical Deflection of the Middle Plane of the Plate Corresponding to six Plate Theories

Considering the 0.1% of the effect of the Microstructure

Model	a/h = 5	a/h = 10	a/h = 15	a/h = 20	a/h = 25	a/h = 30
Steinberg - Reissner ($\alpha = 114/1000$, $\lambda = 2186.8$, $\beta = -2.898/1000$, $\gamma = 4.3/1000$, $\epsilon = 0.135/1000$, $\mu = 1029.$, $h = 0.1$)	0.00074	0.01029	0.05027	0.15539	0.37167	0.75411
Reissner ($\lambda = 2186.8$, $\mu = 1029$, $h = 0.1$)	0.00074	0.01037	0.0511	0.15998	0.38884	0.80436
Eringen I ($e_{33} = 0$; $\alpha = 114/1000$, $\lambda = 2186.8$, $\beta = -2.898/1000$, $\gamma = 4.3/1000$, $\epsilon = 0.135/1000$, $\mu = 1029$, $h = 0.1$)	0.00058	0.00774	0.03779	0.11791	0.28612	0.59135
Eringen I Reduction to Elastic Classic Materials ($e_{33} = 0$; $\alpha = 0$, $\lambda = 2186.8$, $\beta = \gamma = \epsilon = 0$, $\mu = 1029$, $h = 0.1$)	0.00058	0.00775	0.03784	0.11805	0.28648	0.59209
Eringen II ($t_{33} = 0$; $\alpha = 114/1000$, $\lambda = 2186.8$, $\beta = -2.898/1000$, $\gamma = 4.3/1000$, $\mu = 1029$, $h = 0.1$)	0.00074	0.01033	0.05079	0.15837	0.38303	0.78757
Eringen II Reduction to Elastic Classic Materials ($t_{33} = 0$; $\alpha = 0$, $\lambda = 2186.8$, $\beta = \gamma = 0$, $\mu = 1029$, $h = 0.1$)	0.00074	0.01037	0.0511	0.15998	0.38885	0.80438

Table 5 Maximum Vertical Deflection of the Middle Plane of the Plate Corresponding to six Plate Theories Considering the 0.01% of the effect of the Microstructure

Model	a/h = 5	a/h = 10	a/h = 15	a/h = 20	a/h = 25	a/h = 30
Steinberg – Reissner ($\alpha = 114/10000$, $\lambda = 2186.8$, $\beta = -2.898/10000$, $\gamma = 4.3/10000$, $\epsilon = 0.135/10000$, $\mu = 1029.$, $h = 0.1$)	0.00074	0.01036	0.05101	0.15949	0.38696	0.79859
Reissner ($\lambda = 2186.8$, $\mu = 1029$, $h = 0.1$)	0.00074	0.01037	0.0511	0.15998	0.38884	0.80436
Eringen I ($e_{33} = 0$; $\alpha = 114/10000$, $\lambda = 2186.8$, $\beta = -2.898/10000$, $\gamma = 4.3/10000$, $\epsilon = 0.135/10000$, $\mu = 1029$, $h = 0.1$)	0.00058	0.00775	0.03783	0.11804	0.28644	0.59201
Eringen I Reduction to Elastic Classic Materials ($e_{33} = 0$; $\alpha = 0$, $\lambda = 2186.8$, $\beta = \gamma = \epsilon = 0$, $\mu = 1029$, $h = 0.1$)	0.00058	0.00775	0.03784	0.11805	0.28648	0.59209
Eringen II ($t_{33} = 0$; $\alpha = 114/10000$, $\lambda = 2186.8$, $\beta = -2.898/10000$, $\gamma = 4.3/10000$, $\mu = 1029$, $h = 0.1$)	0.00074	0.01036	0.05089	0.15811	0.37825	0.75911
Eringen II Reduction to Elastic Classic Materials ($t_{33} = 0$; $\alpha = 0$, $\lambda = 2186.8$, $\beta = \gamma = 0$, $\mu = 1029$, $h = 0.1$)	0.00074	0.01037	0.0511	0.15998	0.38885	0.80438

Table 6 Maximum Vertical Deflection of the Middle Plane of the Plate Corresponding to six Plate Theories

Considering the 0.001% of the effect of the Microstructure

Model	$a/h=5$	$a/h=10$	$a/h=15$	$a/h=20$	$a/h=25$	$a/h=30$
Steinberg-Reissner ($\alpha=114/100000$, $\lambda=2186.8$, $\beta=-2.898/100000$, $\gamma=4.3/100000$, $\epsilon=0.135/100000$, $\mu=1029.$, $h=0.1$)	0.00074	0.01037	0.05109	0.15991	0.38854	0.8032
Reissner ($\lambda=2186.8$, $\mu=1029$, $h=0.1$)	0.00074	0.01037	0.0511	0.15998	0.38884	0.80436
EringenI ($e_{33}=0$; $\alpha=114/100000$, $\lambda=2186.8$, $\beta=-2.898/100000$, $\gamma=4.3/100000$, $\epsilon=0.135/100000$, $\mu=1029$, $h=0.1$)	0.00058	0.00775	0.03784	0.11805	0.28647	0.59208
Eringen IReductiontoElasticClassicMaterials ($e_{33}=0$; $\alpha=0$, $\lambda=2186.8$, $\beta=\gamma=\epsilon=0$, $\mu=1029$, $h=0.1$)	0.00058	0.00775	0.03784	0.11805	0.28648	0.59209
EringenII ($t_{33}=0$; $\alpha=114/100000$, $\lambda=2186.8$, $\beta=-2.898/100000$, $\gamma=4.3/100000$, $\mu=1029$, $h=0.1$)	0.00074	0.01036	0.05094	0.15825	0.37786	0.75425
EringenII ReductiontoElasticClassicMaterials ($t_{33}=0$; $\alpha=0$, $\lambda=2186.8$, $\beta=\gamma=0$, $\mu=1029$, $h=0.1$)	0.00074	0.01037	0.0511	0.15998	0.38885	0.80438

APPENDIX C

SOME BASIC THEOREMS OF ELASTICITY THEORY

Cauchy-Poisson Theorem

The **Cauchy-Poisson Theorem [5]** is one of the major results of continuum mechanics.

We assume given on $\bar{B}$ a continuous strictly positive function ρ called the ***density***; the mass of any part P of B is then

$$\int_P \rho dv.$$

Let $(0,t_0)$ denote a fixed interval of time. A ***motion*** of the body is a class C^2 vector field u on $B\times(0,t_0)$. The vector $u(x,t)$ is the ***displacement*** of x at time t, while the fields $\dot{u},\ddot{u},\ E=\frac{1}{2}(\nabla u+\nabla u^T),$ and $\dot{E}$ are the *velocity, acceleration, strain*, and *strain-rate*. We say that a motion is ***admissible*** if $u,\dot{u},\ddot{u},E,$ and $\dot{E}$ are continuous on $\bar{B}\times(0,t_0)$. Given an admissible motion and a part P of B,

$$l(P)=\int_P \dot{u}\rho dv$$

is the *linear momentum* of P, and

$$h(P)=\int_P p\times\dot{u}\rho dv$$

is the *angular momentum* (about the origin 0) of P. Note that, for P fixed, $l(P)$ and $h(P)$ are smooth functions of time on $[0,t_0)$; in fact,

$$\dot{l}(P)=\int_P \ddot{u}\rho dv, \qquad \dot{h}(P)=\int_P p\times\ddot{u}\rho dv.$$

A ***system of forces*** $\mathscr{f}$ for the body is defined by assigning to each $(x,t)\in\bar{B}\times[0,t_0)$ a vector $b(x,t)$ and, for each unit vector ***n***, a vector $s_n(x,t)$ such that:

(i) s_n is continuous on $\bar{B}\times[0,t_0)$ and of class $C^{1,0}$ on $B\times(0,t_0)$;

(ii) ***b*** is continuous on $\bar{B}\times[0,t_0)$.

We call $s_n(x,t)$ the ***stress vector*** at (x,t). Let $\mathscr{L}$ be an oriented regular surface in B with unit normal ***n***. Then $s_{n(x)}(x,t)$ is the force per unit area x exerted by the portion of B on the other side; thus

$$\int_{\mathscr{L}} s_n da = \int_{\mathscr{L}} s_{n(x)}(x,t) da_x$$

and

$$\int_{\mathscr{L}} p \times s_n da = \int_{\mathscr{L}} p(x) \times s_{n(x)}(x,t) da_x$$

represent the total force and moment across $\mathscr{L}$. The same consideration also applies when x is located on the boundary of B and ***n*** is the outward unit normal to ∂B at x; in this case $s_n(x,t)$ is called the ***surface traction*** at (x,t). The vector $b(x,t)$ is the ***body force*** at (x,t); it represents the force per unit volume exerted on the point x by bodies exterior to B. The *total force* $f\ \ P$ on a part P is the total surface force exerted across ∂P plus the total body force exerted on P by the external world:

$$f(P) = \int_{\partial P} s_n da + \int_{P} b dv.$$

Analogously, the *total moment* $m\ \ P$ on P (about 0) is given by

$$m(P) = \int_{\partial P} p \times s_n da + \int_{P} p \times b dv.$$

An ordered array $[u, \mathscr{f}]$, where u is an admissible motion and $\mathscr{f}$ a system of forces, is called a ***dynamical process*** if it obeys the following postulate**[5]**: for every part P of B

$$f(P) = \dot{l}(P)$$

and

$$m(P) = \dot{h}(P).$$

These two relations constitute the laws of ***balance of linear and angular momentum***.

Cauchy-Poisson Theorem [5]. Let u be an admissible motion and $\mathscr{f}$ a system of forces. Then $[u, \mathscr{f}]$ is a dynamical process if and only if the following two conditions are satisfied:

(i) there exists a class $C^{1,0}$ symmetric tensor field on $B \times (0, t_0)$, called the ***stress field***, such that for each unit vector n,

$$s_n = Sn;$$

(ii) u, S, and b satisfy the ***equation of motion:***

$$divS + b = \rho \ddot{u}.$$

The proof of this theorem is based on two lemmas. The first is usually referred to as the law of action and reaction (***Cauchy's reciprocal theorem.*** Let $[u, \mathscr{f}]$ be a dynamical process. Then given any unit vector n, $s_n = -s_{-n}$.).

Nonexistence Theorem

The boundary value problems of elastostatics [5]. We assume given an elasticity **C** field on B, body forces ***b*** on B, surface displacements $\hat{u}$ on $\mathscr{S}_1$, and surface forces $\hat{s}$ on $\mathscr{S}_2$, where $\mathscr{S}_1$ and $\mathscr{S}_2$ are complementary regular sub-surfaces of ∂B. And then the ***mixed problem of elastostatics*** is to find an elastic state u, E, S that corresponds to ***b*** and satisfies the ***displacement condition***

$$u = \hat{u} \quad \text{on } \mathscr{S}_1$$

and the ***traction condition***

$$s = Sn = \hat{s} \text{ on } \mathscr{S}_2.$$

We will call such an elastic state a ***solution of the mixed problem.*** When $\mathscr{S}_2$ is empty, so that $\mathscr{S}_1 = \partial B,$ the above boundary conditions reduce to

$$u = \hat{u} \quad \text{on } \partial B,$$

and the associated problem is called the ***displacement problem***. If $\mathscr{S}_2 = \partial B,$ the boundary conditions become

$$s = \hat{s} \quad \text{on} \quad \partial B,$$

and we refer to the resulting problem as the ***traction problem.***

Regularity assumptions:

(i) **C** is smooth on $\bar{B}$;
(ii) ***b*** is continuous on $\bar{B}$;
(iii) $\hat{u}$ is continuous on $\mathscr{S}_1$;
(iv) $\hat{s}$ is piecewise regular on $\mathscr{S}_2$.

Assumptions (ii)-(iv) are necessary for the existence of a solution to the mixed problem. In the definition of a solution u, E, S, the requirement that S be admissible is redundant; indeed, the required properties of S follow from (i), (ii), the admissibility of u, and the field equations.

By a ***displacement field corresponding to a solution of the mixed problem*** we mean a vector field u with the property that there exist fields E, S such that u, E, S is a solution of the mixed problem. We define a ***stress field corresponding to a solution of the mixed problem*** analogously.

For the traction problem a *necessary* condition for the existence of a solution is that the external forces be in equilibrium, i.e. that

$$\int_{\partial B} \hat{s}\, da + \int_B b\, dv = 0,$$
$$\int_{\partial B} p \times \hat{s}\, da + \int_B p \times b\, dv = 0.$$

A deeper result was established by *Ericksen*, who proved that, in general, *lack of uniqueness implies lack of existence,* or equivalently, that *existence implies uniqueness.*

Suppose there were two solutions to a given mixed problem, and that these solutions were not equal modulo a rigid displacement. Then their difference u, E, S would have $E \neq 0$, would satisfy

$$u = 0 \quad \text{on} \quad \mathscr{S}_1, \qquad s = Sn = 0 \quad \text{on} \quad \mathscr{S}_2,$$

and would correspond to vanishing body forces. We call an elastic state u, E, S with the above properties a ***non-trivial solution of the mixed problem with null data.***

Nonexistence Theorem [5]. Let the elasticity field be symmetric. Assume that there exists a non-trivial solution of the mixed problem with null data. Then there exists a continuous body force field $\boldsymbol{b}$ on $\bar{B}$ of class C^2 on B with the following property: the mixed problem corresponding to this body force field and to the null boundary condition (N) has no solution. Further, if $\mathscr{L}_1$ is empty, then $\boldsymbol{b}$ can be chosen so as satisfy

$$\int_B b dv = 0, \qquad \int_B p \times b dv = 0.$$

APPENDIX D

DIFFERENTIATION OF FOURIER SERIES AND MAXIMUM PRINCIPLE FOR ELLIPTIC PDE

Term-by-term Differentiation of Fourier Cosine Series

Theorem 1 [19]

If $f^{'}(x)$ is piecewise smooth, then the *Fourier cosine series* of a continuous function f, given in the form $f(x)=\sum_{n\in N}\alpha_n \cos(\frac{n\pi x}{L})$, $0\le x\le L,$ can be differentiated term by term, i.e.

$$f^{'}(x)\cong -\sum_{n\in N}(\frac{n\pi}{L})\alpha_n \sin(\frac{n\pi x}{L}),\ 0\le x\le L. \qquad \textbf{(D.1)}$$

Differentiation of Fourier Sine Series

Theorem 2[19]

If $f^{'}(x)$ is piecewise smooth, then the *Fourier sine series* of a continuous function $f(x)$, given in the following form $f(x)=\sum_{n\in N}\beta_n \sin(\frac{n\pi x}{L}),\ 0\le x\le L,$ cannot, in general be differentiated term by term. However,

$$f^{'}(x)\cong \frac{1}{L}[f(L)-f(0)]+\sum_{n\in N}[\frac{n\pi}{L}\beta_n+\frac{2}{L}((-1)^n f(L)-f(0))]\cos(\frac{n\pi x}{L}),\ 0\le x\le L. \qquad \textbf{(D.2)}$$

APPENDIX E

ELEMENTS OF THEORY OF ELLIPTIC PDE SYSTEMS

Elliptic Equations

Consider a system of differential equations

$$\sum_{j=1}^{N} \sum_{|k| \le m} A_k^{ij}(x) D_x^k u_j = f_i(x) \qquad (i = 1, \ldots, N), \tag{E.1}$$

and form the matrix

$$P(x, \xi) = \left(\sum_{|k|=m} A_k^{ij}(x) \xi^k \right). \tag{E.2}$$

If for any real vector $\xi \neq 0$, $\det P(x,\xi) \neq 0$, then we say that the system **(E.1)** is of *elliptic system* **[20]**. m is the *order* of the system. If the coefficients of the system are real, then m must be an even number. Indeed, if m is odd, then from $\det P(x, -\xi) = -\det P(x, \xi)$ follows the existence of real vectors $\xi^0 \neq 0$ such that $\det P(x, \xi^0) = 0.$

Maximum Principle for Elliptic PDE

We here present versions **[20]** of the maximum principle for elliptic operators for a single differential equation, and we expect analogous results for elliptic PDE systems.
Consider the linear differential operator of *elliptic type* at a point x^0, with coefficients defined in an n – dimensional domain D,

$$\mathrm{L}u = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u \tag{E.3}$$

The *strong* maximum *principle* for *elliptic* operators is the following theorem.

Theorem 3 [20]

Let L be an *elliptic* operator with continuous coefficients in a domain D and assume that $c(x) \le 0$ and that $Lu \ge 0$ $(Lu \le 0)$ in D. If $u \ne const.$ then u cannot have a positive maximum $(negative\ \max imum)$ in D.

The following consequence of **Theorem 3** is know as the *weak maximum principle*

Theorem 4 [20]

Let L be as in **Theorem 3**, let D be a bounded domain, and let u be continuous in $\bar{D}$ with $Lu \ge 0$ $(Lu \le 0)$ in D. If u has a positive maximum $(negative\ \max imum)$ in $\bar{D}$, then

$$\operatorname*{l.u.b.}_{x \in D} u(x) \le \max_{x \in \partial D} u(x) \qquad \left(\operatorname*{g.l.b.}_{x \in D} u(x) \ge \min_{x \in \partial D} u(x) \right),$$

where ∂D is the boundary of D.

The problem of finding a solution u to the elliptic equation

$$Lu(x) = f(x) \quad \text{in } D, \tag{E.4}$$

satisfying the *boundary condition*

$$u(x) = \varphi(x) \qquad \text{on } \partial D, \tag{E.5}$$

is know as the *first boundary value problem*, or the *Dirichlet problem.*

Unless the contrary is explicitly stated, φ is always assumed to be a continuous function on ∂D. When φ is a the solution is always understood to be continuous in $\bar{D}$.

From the *weak* maximum *principle* we obtain the following uniqueness theorem.

Theorem 5 [20]

Let L be elliptic operator with continuous coefficients in a bounded domain D and assume that $c(x) \le 0$. Then there exists at most one solution to the *Dirichlet problem* **(E.4), (E.5).**

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