TOWARDS A SOLUTION OF THE TRANSIENT PROBLEM FOR BOOLEAN MONOMIAL DYNAMICAL SYSTEMS

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A problem of interest in finite dynamical systems is to determine when such a system reaches equilibrium, i.e., under what conditions is it a fixed point system. Moreover, given a fixed point system, how many time steps are required to reach a fixed point, i.e., what is its transient? Bollman and Colón have shown that a Boolean Monomial Dynamical System (BMDS) f is a fixed point system if and only if every strongly connected component of the dependency graph G_f of f is primitive and in fact, when G_f is strongly connected, the transient of f is equal to the exponent of G_f.

Furthermore, every directed graph gives rise to a unique BMDS and hence every example of a primitive graph with known exponent gives us an example of a fixed point BMDS with known transient. Unfortunately, the general problem of determining the exponent of a primitive graph is unsolved. In this work we give several families of primitive graphs for which we can determine the exponent and hence the transient of the corresponding BMDS.

Resumen de Disertación Presentado a Escuela Graduada
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Requerimientos para el grado de Maestría en Ciencias

HACIA UNA SOLUCIÓN DEL PROBLEMA DE TIEMPO DE TRANSICIÓN PARA UN SISTEMA DINÁMICO MONOMIAL BOOLEANO

Por

Xavier A. Terán Batista

2014

Consejero: Dorothy Bollman
Departamento: Ciencias Matemáticas

Un problema de interés en sistemas dinámicos finitos es determinar cuándo tales sistemas alcanzan equilibrio; es decir, bajo cuales condiciones es un sistema de punto fijo. Por otra parte, dado un sistema de punto fijo, cuánta cantidad de pasos son requeridos para alcanzar el punto fijo; es decir, ¿cuál es su tiempo de transición?. Bollman y Colón han mostrado que un Sistema Dinámico Monomial Booleano (SDMB) f es un sistema de punto fijo sí y solo sí cada componente fuertemente conecto del grafo de dependencia G_f de f es primitivo y en efecto, cuando G_f es fuertemente conecto, el tiempo de transición de f es igual a el exponente de G_f.

Además, cada grafo dirigido da lugar a un único SDMB y por tanto todo ejemplo de un grafo primitivo con exponente conocido provee un ejemplo de un SDMB de punto fijo con tiempo de transición conocido. Desafortunadamente, el problema general de determinar el exponente de un grafo primitivo es abierto. En este trabajo se muestran varias familias de grafos primitivos para las cuales se puede determinar el exponente y por tanto el tiempo de transición de los correspondientes SDMB.

To my mother and the memory of Jose Miguel Batista

To my daughter Isabel C. Terán and all my family.

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First, I would like to thank to God; and I want to thank the members of the graduate committee for their advice and guidance, especially Dorothy Bollman for her help and accepting to be my advisor.

Furthermore, thank you to my family for your strong support, to Walberto Batista for helping to me when I traveled to Puerto Rico, and to my friends Ricela Feliciano, Daiver Velez, and Einstein Morales who always were there when I needed them.

Thanks to all

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LIST OF ABBREVIATIONS

FDS	Finite Dynamical Systems.
FP	Fixed Point
FPS	Fixed Point System
BMDS	Boolean Monomial Dynamical Systems.
uwp	Unique Walk Property

LIST OF SYMBOLS

t	Transient.
$\gamma(G)$	Exponent of the primitive graph G.
$gcd(m_1, m_2, ..., m_n)$	Greatest Common Divisor of $m_1, m_2, ..., m_n$.
$g(m_1, m_2, ..., m_n)$	The Frobenius number of $m_1, m_2, ..., m_n$.

Chapter 1
INTRODUCTION

A *finite dynamical system* (FDS) is a function $f : X \to X$, where X is a finite set and these systems are time discrete. Known examples include cellular automata and Boolean networks, which have found broad applications in engineering, computer science, and, more recently, computational biology.
More general multi-state systems have been used in control theory, the design and analysis of computer simulations, and in computational biology.

In [3], Bollman-Colón, talk about the importance of these systems in genetic modeling and their ability to model the dynamics of gene expressions and relations among genes. This approach enables geneticists to determine the long term impact of a gene on the other genes, see [4]. Dynamical systems over the field with two elements can be used to model Boolean networks which have applications in both cellular automata and computational biology, see [4].
B. Elspas also mentions in [8] applications of *linear dynamical systems* (LDS) in computer control circuits and communications systems. Some of these applications reach a point in time where they do not experience a change in the state they are in.

Dynamical systems that model such phenomena are said to reach a steady state or *fixed point* (FP); in all of these applications, an important problem is to give sufficient conditions for a system to be a *fixed point system* (FPS) and in such a case, to determine the maximum number of time steps necessary to reach a FP, i.e. the *transient*. Although various authors have given conditions for the existence of or for the number of fixed points systems, little is known about transients for such systems.

Every linear system over a finite field can be represented as a matrix A and the state space structure of f can then be determined by finding the factorization of the characteristic polynomial of A. Bollman-Colón in [3], establish that if f is a LDS, $f : \mathbb{Z}_q^n \to \mathbb{Z}_q^n$ then f is a FPS if and only if the characteristic polynomial of f is of the form $x^i(x-1)$ or simply x^i.

In [5], Colón describes a nonlinear system called a *boolean monomial dynamical system* (BMDS). He defines a discrete dynamical system $f : F_2^n \to F_2^n$ where F_2^n is the $n-fold$ cartesian product of a finite field with two elements. It is well known that f can be written as $f = (f_1, f_2, ..., f_n)$ where each f_i is a polynomial of the form $x_{i_1}^{\epsilon_{i1}} x_{i_2}^{\epsilon_{i2}} \cdots x_{i_r}^{\epsilon_{ir}}$ where $\epsilon_{ij} \in \{0,1\}$ or a constant $c \in \{0,1\}$. The dynamics of a BMDS f is encoded in its *state space* $S(f)$, which is a directed graph defined as follows. The vertices of $S(f)$ are the 2^n elements of F_2^n. There is a directed edge (a,b) in $S(f)$ if $f(a) = b$. In particular, a directed edge from a vertex to itself is admissible. That is, $S(f)$ encodes all state transitions of f, and has the property that every vertex has *outdegree* exactly equal to 1.
Every BMDS has an associated *dependency graph* G, whose vertices $1, 2, ..., n$ correspond to $f_1, f_2, ..., f_n$. There is a directed edge from i to j if x_j divides f_i.

The main result in [5], shows that the structure of the cycles of $S(f)$ can be determined from the dependency graph. A principal role is played by *strongly connected* graphs, that is, directed graphs in which there is a walk between any two vertices.

For such a case, Colón defines the *loop number* to be the minimum positive difference of lengths of *circuits* through the same vertex. The dependency graph can be decomposed into strongly connected components. Colón in [5], proves that if the *loop number* of each strongly connected component of the dependency graph of a BMDS is 1, then f is a FPS.

Furthermore, the loop number of each strongly connected component is equal to the greatest common divisor of the cycle lengths.

Another very important result was proved by Bollman and Colón in [2] that says that a BMDS f with a strongly connected dependency graph G_f is a FPS and its only fixed points are $(0, 0, ..., 0)$ and $(1, 1, ..., 1)$ if and only if G_f is *primitive*; that is, if there exist a positive integer k such that for any pair of vertices (i, j) of G there is a walk of length k from i to j. The smallest positive integer k is called the *exponent* of G.

It turns out that the exponent of a primitive dependency graph is precisely the transient of the corresponding BMDS. This implies that the problem of determining the transient of a fixed point BMDS reduces to the problem of determining the exponent of a primitive graph. However, methods for finding the exponent of a primitive graph are known only in special cases.

In 1964, Dulmage and Mendelsohn [6], introduce a family of graphs with known exponent. They proved that given positive integers $m_1 < m_2 < \cdots < m_k$ such that $gcd(m_1, m_2, ..., m_k) = 1$, then

$$g(m_1, m_2, ..., m_k) + r + 1,$$

is an upper bound for the exponent of a primitive graph G, where the m_i are the lengths of the cycles of G, r is the length of the longest shortest walk between two vertices that touch at least one vertex of each cycle and $g(m_1, m_2, ..., m_k)$ is the *Frobenius number*, i.e., the largest positive integer that is not a non-negative integer linear combination of the m_i. They also proved that for a particular family of graphs the above upper bound is also a lower bound.

The same year, Heap and Lynn in [9] defined a family of graphs called Frobenius graphs, and they proved that the exponent of such a graph is given by

$$g(m_1, m_2, ..., m_k) + 2m_k - 1.$$

In [1], Bollman and Colón determined that an upper bound for the exponent of a family of graphs consisting of an increasing chain of cycles of coprime lengths is given by the same formula of Dulmage and Mendelsohn, but although it is suspected that this formula is a lower bound, there isn't an established proof.

The rest of the work is organized as follows. In Chapter 2 we present more detailed explanations of the basic concepts of graph theory, finite dynamical systems and their graphs and the Frobenius number. In Chapter 3, we give several families of primitive graphs for which the exponent is known and some interesting results about the role it plays in determining when a BMDS is a fixed point system. In Chapter 4 we study the general problem of finding the transient of a BMDS and give our main results on transients, i.e., on the exponent of a family of graphs. We end with a brief discussion of the conclusions and future work.

Remark 1. The State Spaces in this work were created with the DVD software developed by the Applied Discrete Mathematics Group at the Virginia Bioinformatic Institute (http://dvd.vbi.vt.edu). The Dependency Graphs were created using an algorithm created by Xavier Terán-Batista in Wolfram Mathematica software.

Chapter 2
PRELIMINARIES

In this chapter we give some basic results of graph theory, discrete dynamical systems and number theory which will help us to understand the mathematical theory used to solve our problem.

2.1 Graphs

Definition 2.1.1. *A directed graph G (or digraph) is an ordered pair (V, E) consisting of a set V of vertices or nodes and a set E of ordered pairs of vertices called edges.*

Example 2.1.2. *Graph $G = (V, E)$ where $V = \{1, 2, 3, 4\}$ and*
$E = \{(1, 2), (2, 2), (2, 3), (3, 4), (4, 3), (4, 1)\}$.

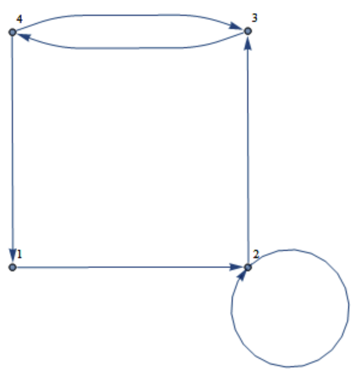


Figure 2–1: Digraph with 4 vertices

Definition 2.1.3. *For any vertex i in a directed graph, the number of edges of the form (i, j) is its outdegree.*

Definition 2.1.4. *A walk of length* n *in a digraph* $G = (V, E)$ *is a sequence of edges* $(v_1, v_2), (v_2, v_3), ..., (v_n, v_{n+1})$. *A walk in which* $v_1 = v_{n+1}$ *is a circuit. The number of edges in a walk* w *is called the length of* w *and is denoted by* $|w|$.

Definition 2.1.5. *A path is a walk* $(v_1, v_2), (v_2, v_3), ..., (v_n, v_{n+1})$ *where the* v_i *are distinct, except possibly* $v_1 = v_{n+1}$*; in such a case the path is a cycle of length* n. *A cycle of the form* (v_1, v_1) *is called a* <u>trivial</u> *cycle or* <u>loop</u>.

Definition 2.1.6. *A digraph is said to be strongly connected if there is a path between any two pair of vertices.*

An example of a strongly connected graph is given in the Example 2.1.2.

Definition 2.1.7. *The adjacency matrix of a graph* $G = (V, E)$ *with* n *vertices* $\{1, 2, ..., n\}$ *is an* $n \times n$ *matrix* $A_G = [a_{ij}]$ *such that*

$$a_{ij} = \begin{cases} 1, & \textit{if } (i, j) \in E, \\ 0, & \textit{otherwise.} \end{cases}$$

Example 2.1.8. *For the graph* G *in figure 2–1,*

$$A_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

The following theorem is well known and can be proved by induction.

Theorem 2.1.9. *Let* $G = (V, E)$ *be a digraph with adjacency matrix* A_G *and let* $A_G^k = [a_{ij}^{(k)}]$ *where* k *is a positive integer. Then* $a_{ij}^{(k)}$ *is equal to the number of walks of length* k *from* i *to* j.

Proof. *We proceed by induction on* k*. If* $k = 1$*,* $A^1 = A$ *and* $a_{ij}^{(1)} = a_{ij}$*; for vertices* i, j *of* G*. Therefore, the* a_{ij} *entry of the matrix* A *is the number of walks from* i *to* j *of length* 1 *in* G*.*

Assume, for a positive integer k*, that* $a_{ij}^{(k)}$ *is the number of different walks from* i *to* j *of length* k *in* G*. Observe that every walk from* i *to* j *of length* $k+1$ *in* G *is obtained from a walk from* i *to* v *of length* k *for some vertex* v *in* G *that is adjacent to* j*. Since* $A^{k+1} = A^k A$*, it follows that the* $a_{ij}^{(k+1)}$ *entry in* A^{k+1} *can be obtained by taking the inner product of row* i *of* A^k *and column* j *of* A*; i.e.,*

$$a_{ij}^{(k+1)} = a_{i1}^{(k)} a_{1j} + a_{i2}^{(k)} a_{2j} + \ldots + a_{in}^{(k)} a_{nj} = \sum_{v=1}^{n} a_{iv}^{(k)} a_{vj}. \quad (*)$$

By the induction hypothesis, for each integer v *with* $1 \leq v \leq n$*, the integer* $a_{iv}^{(k)}$ *is the number of different walks from* i *to* v *of length* k *in* G*.*

Now, if $a_{vj} = 1$*, then vertex* v *is adjacent to vertex* j *and so there are* $a_{iv}^{(k)}$ *different walks from* i *to* v *of length* $k+1$ *in* G *whose next to last vertex is* v*. Otherwise, if* $a_{vj} = 0$*, then vertex* v *is not adjacent to vertex* j *and there are no walks from* i *to* j *of length* $k+1$ *in* G *whose next to last vertex is* v*. In any case,* $a_{iv}^{(k)} a_{vj}$ *gives the number of different walks from* i *to* j *of length* $k+1$ *in* G *whose next to last vertex is* v*. Consequently, the total number of different walks from* i *to* j *of length* $k+1$ *in* G *is the sum in* $(*)$*, which is* $a_{ij}^{(k+1)}$*.*

Hence, by the Principle of Mathematical Induction, $a_{ij}^{(k)}$ *is the number of different walks from* i *to* j *of length* k *in* G *for every positive integer* k*.*

$\square$

Definition 2.1.10.

i. *A directed graph is said to be primitive if there exists a positive integer* t *such that for any two vertices* i *and* j *there is a walk from* i *to* j *of length* t*. The smallest such* t*, is called the exponent of the graph, and is denoted by* $\gamma(G)$*.*

ii. For any vertex i of G we also define the exponent of i, denoted $exp_i(G)$, to be the smallest positive integer w_i such that for every vertex j there is a walk from i to j of length w_i. Clearly, $\gamma(G) = \max\{exp_i(G)\}$.

Remark 2. Let w_i be the exponent of vertex i of the graph G, then if $p \geq w_i$, there exists a walk of length p from i to any vertex j.

Now, we mention a very important theorem cited by several authors and whose proof is given in [13] and [14].

Theorem 2.1.11 (Rosenblatt, [14])**.** *A digraph G is primitive if and only if G is strongly connected and $gcd(m_1, m_2, ..., m_n) = 1$, where $\{m_1, m_2, ..., m_n\}$ is the set of lengths of all cycles in G.*

2.2 Boolean Monomial Dynamical Systems

Definition 2.2.1. *A finite dynamical system (FDS) is a function $f : X \to X$, where X is a finite set.*

These systems are time discrete.

Definition 2.2.2. *The dynamics of an FDS $f : X \to X$ is represented by its state space, denoted $S(f)$, which consists of the directed graph whose vertex set is X and whose edges consist of all pairs (u, v) where $u, v \in X$ and $f(u) = v$.*

Example 2.2.3. *Let $f : X \to X$ be a function with $X = \{1, 2, 3, 4, 5\}$ such that $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 5$ and $f(5) = 3$. Then, the state space of f is*

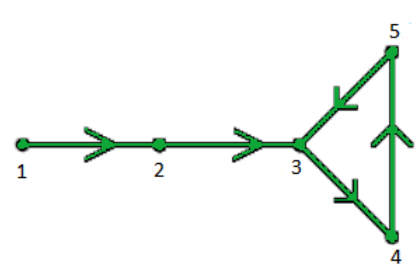


Figure 2–2: State Space of $f = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 3)\}$

Definition 2.2.4. *Given a dynamical system $f : X \to X$, and a state $a \in X$, the transient of a is the minimum value of t, for which $f^t(a)$ reaches an "attractor". That is, given a there exists $s = s(a)$ such that $f^{t+s}(a) = f^s(a)$, see figure 2–3. Moreover, the transient t_f of a system f is the maximum of the transients t, among all the states a.*

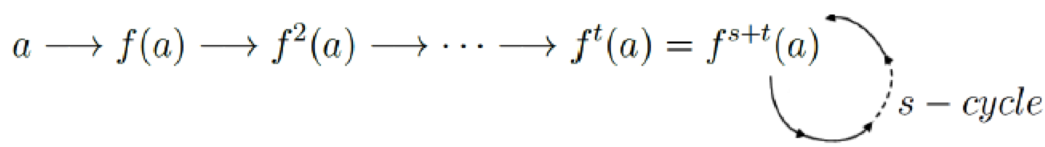


Figure 2–3: Transient of a dynamical system f

In Example 2.2.3 it can be seen that the transient is 2.

The cycle consisting of $f^t(a), f^{t+1}(a), ..., f^{s+t-1}(a)$ is called a limit cycle.
If $s = 1$, then $f^t(a) = f^{1+t}(a)$ is a fixed point (FP). f is a fixed point system (FPS) if for every $a \in X$, there exists a t for which $f^t(a) = f^{1+t}(a)$.

Definition 2.2.5. *A field F with a finite number of elements q is called a finite field and is denoted by F_q.*
It can be shown that the number q of elements in a finite field is a power of a prime.
$F_q[x_1, x_2, ..., x_n]$ denotes the polynomial ring with n variables over F_q.

Let $f : F_q^n \to F_q^n$ be a FDS where F_q^n is the $n-fold$ cartesian product of F_q. It is well known that every function defined on a finite field can be expressed as a polynomial. Therefore, f can be written as a tuple of polynomials, i.e., $f = (f_1, f_2, ..., f_n)$ where each $f_i \in F_q[x_1, x_2, ..., x_n]$.

Definition 2.2.6. *If all the f_i are linear polynomials without a constant term, then f is a linear system.*

When f is linear, then its dynamics can be completely determined from its matrix representation. Let A be a matrix representation of a linear system $f : F_q^n \to F_q^n$. Then the number of limit cycles and their length, as well as the structure of the transients, can be determined from the factorization of the characteristic polynomial of the matrix A.

For example, it is shown in [3] that a linear finite dynamical system is a FPS if and only if its minimal polynomial is of the form $x^i(x-1)$ or simply x^i and in this case, i is the transient.

Example 2.2.7. *Let $L : F_2^3 \to F_2^3$ be defined by the matrix*

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then the minimal polynomial is $\lambda^2(\lambda+1)$ and $L = (0, x_1, x_2 + x_3)$ has transient 2 as can be verified by the following graph.

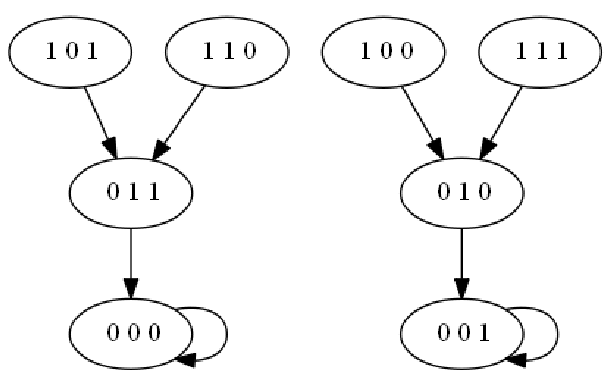


Figure 2–4: State Space of Linear Dynamical System L

Definition 2.2.8. *A boolean monomial dynamical system (BMDS) is a finite dynamical system* $f : F_2^n \to F_2^n$ *with* $f = (f_1, f_2, ..., f_n)$ *where each* f_i *is assumed to be a monomial, that is* f_i *is a polynomial of the form* $x_{i_1}^{a_{i1}} x_{i_2}^{a_{i2}} \cdots x_{i_r}^{a_{ir}}$*, where each* $a_{ij} \in \{0, 1\}$*, but not all* $a_{ij} = 0$*.*

Example 2.2.9. *A very simple example of a BMDS is given by*

$$f : F_2^7 \to F_2^7 \text{ such that } f = (x_2 x_4, x_3, x_1, x_5, x_6, x_7, x_1).$$

Another graph that plays an important role in determining the dynamics of a BMDS is the *"dependency graph"*.

Definition 2.2.10. *The dependency graph of a BMDS* $f : F_2^n \to F_2^n$ *is defined to be a directed graph whose vertices consist of* $1, 2, ..., n$ *and such that* (i, j) *is an edge if and only if* x_j *divides* f_i*.*

The dependency graph of the BMDS f of Example 2.2.9 is given in figure 2–5.

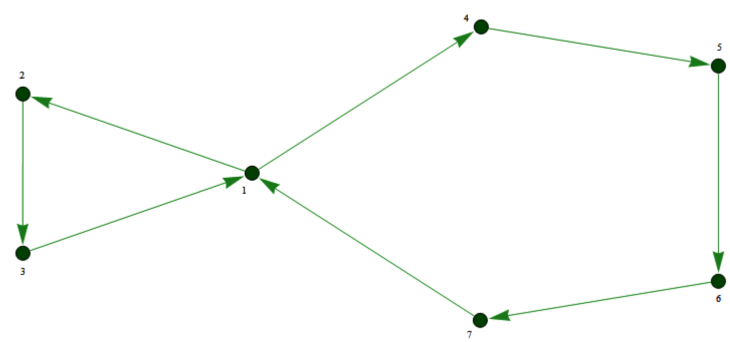


Figure 2–5: Dependency Graph of f

Dr. O. Colón in [5] defines one of the most important tools necessary to determine if a BMDS is a FPS called the *"loop number"*.

Definition 2.2.11. *Given a strongly connected graph* G*, let* v *be a vertex of* G*, the loop number of* v *is the minimum of all integers* $l \geq 1$ *with* $l = ||p| - |q||$*, for all walks* $p, q : v \to v$*.*

Example 2.2.12. *Consider the figure 2–6. The loop number of* 1 *is* 2*. This can be seen by considering walks of lengths* $p = 4$ *and* $q = 2$ *by walking around cycles of lengths* 4 *and* 2 *that contains 1.*

The loop number of every another vertex is also 2. It is not a coincidence that two different vertices have the same loop number. In fact, it is a result proved in [5].

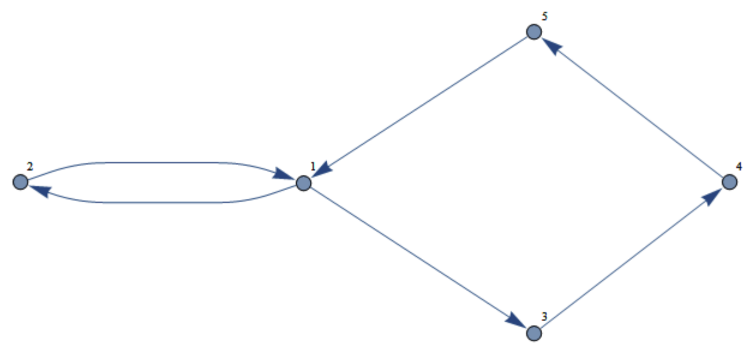


Figure 2–6: Graph with loop number 2

Some important results about loop numbers are given in the following Theorems.

Theorem 2.2.13 (O. Colón, [5])**.** *Let G be a strongly connected graph. The loop number of G is the greatest common divisor of the lengths of all cycles of G.*

Proof. *Observe that a circuit p can be decomposed into a number of cycles $m_1, m_2, ..., m_k$ sharing vertices.*

Let d be the greatest common divisor of the lengths of all cycles of G. Take a vertex i in G and suppose that the loop number of i is l. Let p and q be circuits through i for which $|p| - |q| = l$ of G. We want to show that $d = l$. Decompose p and q into a number of cycles of lengths $m_1, m_2, ..., m_k$ and $q_1, q_2, ..., q_r$. Then, we get $|p| = |m_1| + |m_2| + \cdots + |m_k|$ and $|q| = |q_1| + |q_2| + \cdots + |q_r|$. Hence

$$|p| - |q| = |m_1| + |m_2| + \cdots + |m_k| - |q_1| - |q_2| - \cdots - |q_r| = l,$$

so that d divides l.

Otherwise, assume that there is a cycle p' through of the vertex j of G whose length $s = |p'|$ is not divisible by l. Consider walks of lengths q_1 from i to j and q_2 from j to i. Let $m = |q_1| + |q_2|$ and let $l' = gcd(m, s, l)$. Then l' can be written as $l' = \alpha l - \beta m - \delta s$ with $\alpha, \delta \geq 0$ and $\beta > 0$. So

$$|p^\alpha| - |q^\alpha q_2 (q_1 q_2)^{\beta-1} (p')^\delta q_1| = l',$$

i.e., we have constructed two circuits whose lengths differ by l'. Hence $l' = l$ is the loop number of G. Then $l' = l$ divides $s = |p'|$, a contradiction. Thus all lengths of cycles in G are divisible by l, so that l divides d, hence $d = l$.

$\square$

Theorem 2.2.14 (O. Colón, [5])**.** *A BMDS f is a FPS if and only if every strongly connected component of its dependency graph has loop number* 1*.*

The dependency graph of the BMDS f given in Example 2.2.9 has loop number 1 and its state space is shown in figure 2–7. Furthermore, the system illustrated is a fixed point system, and reaches a steady state after 16 time steps, that is, the transient is 16.

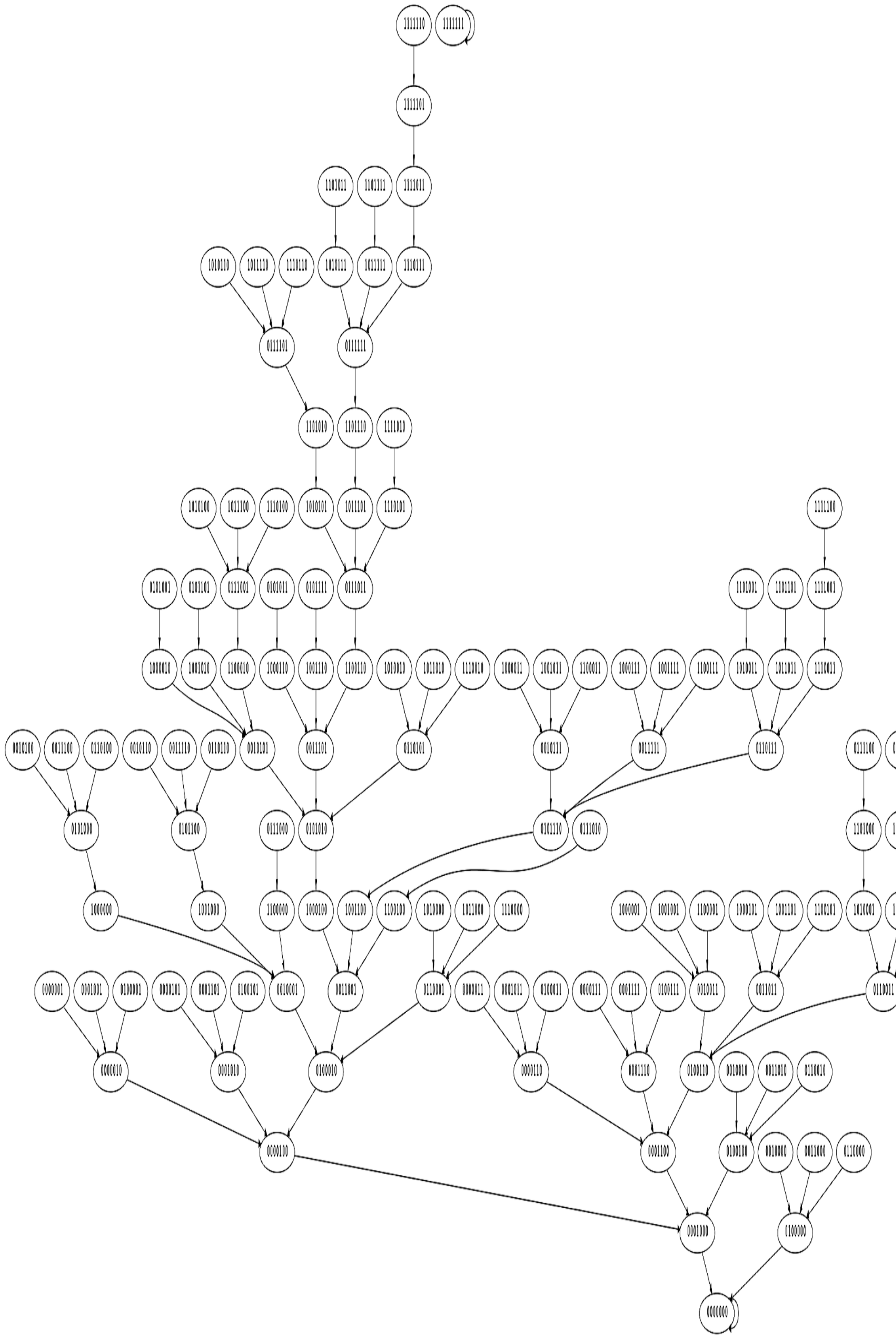


Figure 2–7: State Space of $f = (x_2x_4, x_3, x_1, x_5, x_6, x_7, x_1)$

2.3 Frobenius Number

Theorem 2.3.1 (Ramirez, [13]). *If $gcd(m_1, m_2, ..., m_n) = 1$, then there exists an integer N such that any integer $s \geq N$ is representable as a non-negative integer combination of $m_1, m_2, ..., m_n$.*

Proof. *There exists integers $a_1, a_2, ..., a_n$ s.t. $a_1m_1 + a_2m_2 + ... + a_nm_n = 1$. Denote by P and $-Q$ the sum of positive and negative terms respectively in this decomposition. Thus $P - Q = 1$.*

By the division algorithm, any integer $k \geq 0$ can be written as $hm_1 + k'$ with $h \geq 0$ and $0 \leq k' < m_1$. Then

$$\begin{aligned}(m_1 - 1)Q + k &= (m_1 - 1)Q + hm_1 + k' \\ &= (m_1 - 1)Q + hm_1 + k'(P - Q) \\ &= hm_1 + (m_1 - 1 - k')Q + k'P.\end{aligned}$$

P and Q belong to the semigroup W generated by $m_1, m_2, ..., m_n$. Hence,

$$(m_1 - 1)Q + k \in W \quad \forall k \geq 0$$

Let be $N = (m_1 - 1)Q$, then every integer $s \geq N$ is representable in terms of $m_1, m_2, ..., m_n$.
Therefore, there is a least integer $f(m_1, m_2, ..., m_n)$ such that each integer $s \geq f(m_1, m_2, ..., m_n)$ can be expressed as a non-negative linear combination of $m_1, m_2, ..., m_n$. Then,

$$g(m_1, m_2, ..., m_n) = f(m_1, m_2, ..., m_n) - 1.$$

$\square$

Definition 2.3.2. *Let $m_1, m_2, ..., m_n$ be distinct relatively prime positive integers and let $g(m_1, m_2, ..., m_n)$, the Frobenius Number, denote the largest positive integer which is not expressible in the form $a_1m_1 + a_2m_2 + ... + a_nm_n$ where each a_i is a non-negative integer.*

It is well known that, if p and q are relatively prime, then

$$g(p, q) = pq - p - q.$$

For example,

$$g(5, 6) = 5 * 6 - 5 - 6 = 29.$$

That is, 29 is the largest integer which is not a non-negative integer linear combination of 5 and 6.

Chapter 3
A FAMILY OF GRAPHS WITH KNOWN EXPONENT

In this chapter, we present several family of graphs for which the exponent is known and some interesting results and properties about particular families of BMDSs.

3.1 A Simple Example

It follows from Theorem 2.1.9 that the transient of a BMDS f is equal to the smallest power of the adjacency matrix of the dependency graph G_f for which all entries are positive. We illustrate this in the following example, but as we shall later see there is a much more efficient method for computing the transient of this system.

Example 3.1.1. *Consider the BMDS* $f = (x_2x_4, x_3, x_1, x_5, x_6, x_7, x_1)$*, whose dependency graph* G *is given in figure 2–5.*

Its adjacency matrix is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^{15} = \begin{bmatrix} 2 & 4 & 3 & 4 & 3 & 1 & 3 \\ 3 & 1 & 3 & 1 & 3 & 1 & 1 \\ 4 & 3 & 1 & 3 & 1 & 3 & 1 \\ 3 & 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 3 & 1 & 3 & 1 & 1 & 2 \\ 3 & 1 & 3 & 1 & 3 & 1 & 1 \\ 4 & 3 & 1 & 3 & 1 & 3 & 1 \end{bmatrix}, \quad A^{16} = \begin{bmatrix} 6 & 2 & 4 & 2 & 4 & 3 & 1 \\ 4 & 3 & 1 & 3 & 1 & 3 & 1 \\ 2 & 4 & 3 & 4 & 3 & 1 & 3 \\ 1 & 3 & 1 & 3 & 1 & 1 & 2 \\ 3 & 1 & 3 & 1 & 3 & 1 & 1 \\ 4 & 3 & 1 & 3 & 1 & 3 & 1 \\ 2 & 4 & 3 & 4 & 3 & 1 & 3 \end{bmatrix}.$$

Thus, the transient is $t = 16$*.*

Computing powers of matrices is computationally very expensive; In fact, just one multiplication of two $n \times n$ matrices has complexity $O(n^3)$. Our goal is to determine BMDSs whose transients can be computed by means of a formula. Some important results that advance this idea are given by the following Theorems.

Theorem 3.1.2. *A BMDS f whose dependency graph G_f is strongly connected is a FPS and its only fixed points are $(0, 0, ..., 0)$ and $(1, 1, ..., 1)$ if and only if G_f is primitive. Furthermore, the transient of f is equal to the exponent of G.*

Proof. *Let f be a BMDS and G_f its dependency graph. If G_f is primitive, then by Theorem 2.1.11, it follows that gcd$(m_1, m_2, ..., m_n) = 1$ where $m_1, m_2, ..., m_n$ are the cycle lengths. By Theorem 2.2.13, the loop number of a strongly connected graph G is equal to the greatest common divisor of the lengths of all cycles of G_f. Then by Theorem 2.2.14 it follows that f is a FPS.*

Let $1, 2, ..., n$ be the vertices of G and let $x_1, x_2, ..., x_n$ be the variables of f. By the definition of dependency graph, (i, j) is an edge if and only if x_j divides f_i. It can be shown by induction that there exists a walk of length m from i to j if and only if f_i^m contains the factor x_j. Now since G_f is primitive, there exists a smallest positive integer t such that $f_i^t = x_1 x_2 ... x_n$ for all i.

Hence $f^t(x_1, x_2, ..., x_n) = (x_1x_2...x_n, ..., x_1x_2...x_n)$ *for all* $(x_1, x_2, ..., x_n) \in \mathbb{F}_2^n$. *Also,* $f^k(0, 0, ..., 0) = (0, 0, ..., 0)$ *and* $f^k(1, 1, ..., 1) = (1, 1, ..., 1)$*, for all* $k \geq t$*, i.e.,* $(0, 0, ..., 0)$ *and* $(1, 1, ..., 1)$ *are fixed points and* t *is the transient.*

On the other hand, if f *is a FPS, then by Theorem 2.2.14 it follows that* G_f *has loop number* 1 *and by Theorem 2.2.13 the greatest common divisor of the cycle lengths in* G_f *is* 1*. Then by Theorem 2.1.11,* G_f *is primitive.*

□

Theorem 3.1.3. *Every BMDS* f *has a unique dependency graph and conversely, every dependency graph* G *all of whose vertices has outdegree at least one is the dependency graph of a BMDS whose state space is unique up to isomorphism.*

Proof. *Let* G *be a directed graph whose vertices all have outdegree* ≥ 1 *and let* $V = \{1, 2, 3, ..., n\}$ *be the set of vertices. Let* $f = (f_1, f_2, ..., f_n) : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ *where for each* f_i *is the product of all* x_j *for which* (i, j) *is an edge. Then,* f_i *has depedency graph* G*. Let* G' *be the graph obtained from* G *by relabeling the vertices, by a permutation* $\sigma : V \to V$ *and let*

$$\phi_\sigma(f) = (f_{\sigma^{-1}(1)}(x_{\sigma(1)}, ..., x_{\sigma(n)}), ..., f_{\sigma^{-1}(n)}(x_{\sigma(1)}, ..., x_{\sigma(n)})).$$

$\phi_\sigma(f)$ *is a well defined map. Now, consider the map* $\psi_\sigma : S(f) \to S(\phi_\sigma(f))$ *such that* $\psi_\sigma[(x_1, x_2, ..., x_n)] = (x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(n)})$*. Note that* ψ_σ *is well-defined too and is a one-to-one map. In fact, if* $\psi_\sigma[(x_1, x_2, ..., x_n)] = \psi_\sigma[(y_1, y_2, ..., y_n)]$*, then*

$$(x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(n)}) = (y_{\sigma^{-1}(1)}, ..., y_{\sigma^{-1}(n)})$$

$$\Rightarrow x_{\sigma^{-1}(i)} = y_{\sigma^{-1}(i)} \quad \forall i = 1, 2, ..., n.$$

Since σ^{-1} *is a bijective map, we have that for any* $j = 1, 2, ..., n$*, there exists* $i = 1, 2, ..., n$ *such that* $\sigma^{-1}(i) = j$*; hence* $x_j = y_j$ $\forall j \Rightarrow (x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n)$*. Therefore, there is a one-to-one correspondence between the vertices of* $S(f)$ *and*

$S(\phi_\sigma(f))$. Now we need to show that $S(f)$ and $S(\phi_\sigma(f))$ preserve edges, that is, suppose that $((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n))$ is an edge of $S(f)$, thus by definition $f((x_1, x_2, ..., x_n)) = (y_1, y_2, ..., y_n)$. Then

$$\begin{aligned}
\phi_\sigma(\psi_\sigma[(x_1, x_2, ..., x_n)]) &= \phi_\sigma((x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(n)})) \\
&= (f_{\sigma^{-1}(1)}(x_{\sigma(\sigma^{-1}(1))}, ..., x_{\sigma(\sigma^{-1}(n))}), ..., f_{\sigma^{-1}(n)}(x_{\sigma(\sigma^{-1}(1))}, ..., x_{\sigma(\sigma^{-1}(n))})) \\
&= (f_{\sigma^{-1}(1)}(x_1, ..., x_n), ..., f_{\sigma^{-1}(n)}(x_1, ..., x_n)) \\
&= (y_{\sigma^{-1}(1)}, y_{\sigma^{-1}(2)}, ..., y_{\sigma^{-1}(n)}) \\
&= \psi_\sigma[(y_1, y_2, ..., y_n)].
\end{aligned}$$

This shows that $(\psi_\sigma[(x_1, x_2, ..., x_n)], \psi_\sigma[(y_1, y_2, ..., y_n)])$ is an edge of $S(\phi_\sigma(f))$. Therefore, the state space is unique up to isomorphism.

□

Example 3.1.4. *Consider the BMDS $f = (x_2, x_2x_3, x_1x_2)$ whose dependency graph is given by*

Figure 3–1: G_1 dependency graph of $f = (x_2, x_2x_3, x_1x_2)$

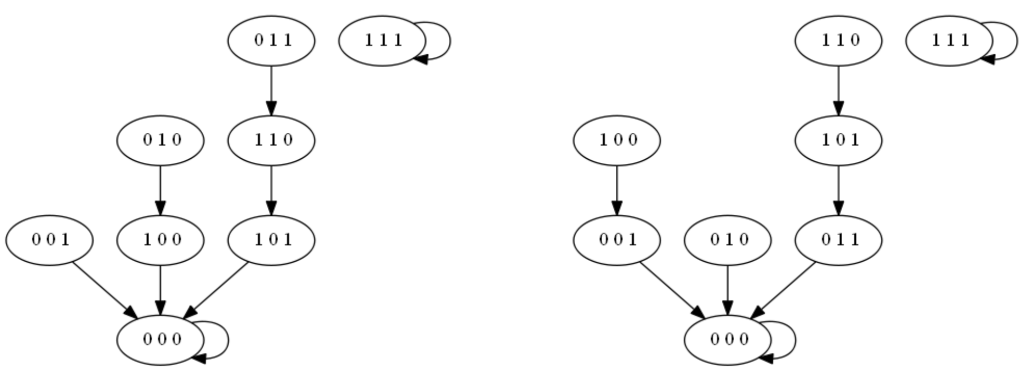
Now, consider the same graph obtained by relabeling the vertices, that is, applying the permutation $\sigma = (312)$, as it is shown in figure 3–2.

Figure 3–2: G_2 dependency graph of $\phi_\sigma(f)$

Then, by the map in Theorem 3.1.3, we have

$$\begin{aligned}\phi_\sigma(f) &= (f_{\sigma^{-1}(1)}(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}), f_{\sigma^{-1}(2)}(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}), f_{\sigma^{-1}(3)}(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}))\\ &= (f_2(x_3, x_1, x_2), f_3(x_3, x_1, x_2), f_1(x_3, x_1, x_2))\\ &= (x_1x_2, x_1x_3, x_1).\end{aligned}$$

That is, $\phi_\sigma(f) = (x_1x_2, x_1x_3, x_1)$ *is the BMDS corresponding to the graph* G_2 *in figure 3–2. Then, by Theorem 3.1.3 the state spaces are isomorphic, i.e.,* $S(f) \cong S(\phi_\sigma(f))$*; and for example in figure 3–3, we can observe that* $\psi_\sigma((0,1,0)) = (1,0,0)$*, that is, the vertex* $(0,1,0)$ *of* $S(f)$ *correspond to* $(1,0,0)$ *of* $S(\phi_\sigma(f))$*. Moreover, if we exchange the positions of the edges* $((0,0,1),(0,0,0))$ *and* $((1,0,0),(0,0,0))$ *of* $S(f)$*, the result is the same state space as* $S(\phi_\sigma(f))$*.*



(a) $S(f)$ state space of f **(b)** $S(\phi_\sigma(f))$ state space of $\phi_\sigma(f)$

Figure 3–3: State Space of $S(f)$ and $S(\phi_\sigma(f))$ of the BMDS $f = (x_2, x_2x_3, x_1x_2)$

Now, the problem of determining the transient of a fixed point BMDS reduces to the problem of determining the exponent of a primitive graph. However, methods for finding the exponent of a primitive graph are known only in special cases, as for example in the following theorem.

Theorem 3.1.5 (1.11.2 see [16] p.75-77)**.** *For each $n \geq 2$ define the adjacency matrix B_n of a graph G_n as follows:*

$$B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} \textit{ for } n \geq 4.$$

Then $\gamma(G_n) = (n-1)^2$.

Proof. *The graphs G_n associated with B_n are as follows:*

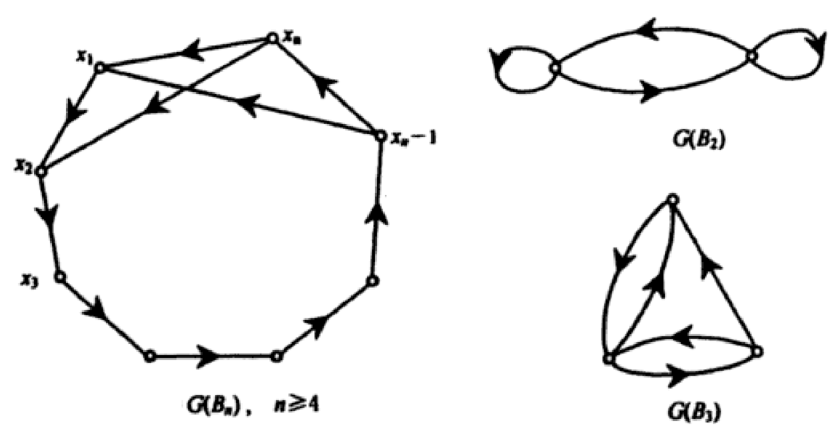


Figure 3–4: The digraph G_n associated with the matrix B_n

Clearly, G_n is strongly connected. Since G_n contains only three different directed cycles, one of length n, two of length $(n-1)$, and since $gcd(n, n-1) = 1$, we have, by Theorem 2.1.11, G_n is primitive.

Let $i, j \in G$ be two arbitrary vertices. Let P_{ij} be a shortest path from i to j in G_n, and denote its length by p. Then $p \leq n-1$.

Let

$$q = g(n, n-1) + n - p.$$

Since $n - p \geq 1$, we have $q \geq g(n, n-1) + 1$. Then, by definition of the Frobenius number, q must be a linear combination of n and $n-1$. Thus, there exists a walk from i to j of length $q + p$ in G_n.

Now, since i, j are arbitrary and $g(n, n-1) = n(n-1) - n - (n-1) = (n-1)^2 - n$. We have

$$\gamma(G_n) \leq q + p = g(n, n-1) + n - p + p = (n-1)^2 - n + n = (n-1)^2.$$

On the other hand, G_n contains a unique path from i to j of length n . Therefore, the length of every walk from i to j in G can be expressed as $n + r$, where r is a linear combination of n and $n-1$. Then G_n contains no directed walk of length $(n + g(n, n-1) - 1)$, because $g(n, n-1)$ is not a linear combination of n and $n-1$. Thus, we have

$$\gamma(G_n) \geq g(n, n-1) + n = (n-1)^2 - n + n = (n-1)^2.$$

$\square$

Corollary 3.1.6. *For every $n \geq 2$, let ϕ_n be the BMDS whose dependency graph is G_n. Then*

$$\phi_n = (x_2, x_3, ..., x_1x_n, x_1x_2)$$

and the transient of ϕ_n is $(n-1)^2$.

3.2 BMDSs whose Dependency Graphs have The Unique Walk Property

In this section, we describe a family of primitive graphs for which the exponent is known and we state some results introduced by Dulmage and Mendelsohn in [6].

Let G be a primitive graph in which the cycle of lengths are $m_1, m_2, ...,$ or m_k. For any ordered pair (i, j) of vertices, a non-negative integer r_{ij} is defined as follows. If $i = j$ and if for $s = 1, 2, ..., k$ there is a cycle through vertex i of length m_s then $r_{ij} = 0$; otherwise r_{ij} is the length of the shortest walk from i to j which has at least one vertex on some cycle of length m_s for $s = 1, 2, ..., k$.

$$\text{Let} \quad r = \max(r_{ij}) \quad \text{taken over all ordered pairs} \quad (i, j).$$

In other words, r is the length of the longest shortest walk between two vertices that includes at least one vertex of each cycle.

Theorem 3.2.1 ([6]). *If G is a primitive graph then*

$$\gamma(G) \leq g(m_1, m_2, ..., m_k) + r + 1.$$

Proof. *For any set of non-negative integers $a_1, a_2, ..., a_k$ and any ordered pair (i, j) of vertices of G, if $i \neq j$ there is a walk from vertex i to vertex j of length r_{ij} that contains a vertex on a cycle of length m_i for each i and hence a walk from i to j of length m_i.*

$$r_{ij} + a_1 m_1 + a_2 m_2 + ... + a_k m_k.$$

Thus there is a walk from vertex i to vertex j of length

$$g(m_1, m_2, ..., m_k) + r_{ij} + N$$

for every $N \geq 1$, by definition if $N \geq 1$, $g(m_1, m_2, ..., m_n) + N$ can be expressed as a linear combination of the m's and so there is a walk of length $g(m_1, m_2, ..., m_k) + r_{ij} + N$. Choosing $N = 1 + r - r_{ij}$, we have a walk from vertex i to vertex j of length

$$g(m_1, m_2, ..., m_k) + r + 1,$$

so that

$$w_i \leq g(m_1, m_2, ..., m_k) + r + 1 \quad \forall i,$$

where w_i is the exponent of the vertex i; thus by definition 2.1.10 (ii)

$$\gamma(G) = \max\{w_i\} \leq g(m_1, m_2, ..., m_k) + r + 1.$$

□

Definition 3.2.2. *An ordered pair (i, j) of vertices in a primitive graph G is said to have the unique walk property (uwp) if every walk from vertex i to vertex j which has length $\geq r_{ij}$ consists of some walk α of length r_{ij} augmented by a number of cycles each of which has a vertex in common with α. G has the uwp if it has a pair (i, j) of vertices with the uwp and $r = r_{ij}$.*

The figure 2–5 is an example of a graph with the *uwp*, and in particular, the pair $(4, 7)$ satisfies $r = r_{47}$.

Theorem 3.2.3 ([6])**.** *If G is a primitive graph in which the ordered pair of vertices (i, j) has the uwp, then*

$$\gamma(G) \geq g(m_1, m_2, ..., m_k) + r_{ij} + 1.$$

Proof. *Since G is a graph with the uwp, set the order pair of vertices (i, j) has the uwp; thus there is no walk from vertex i to vertex j of length*

$$\alpha = g(m_1, m_2, ..., m_k) + r_{ij}$$

for such a walk would imply the existence of non-negative $a_1, a_2, ..., a_k$ *with*

$$g(m_1, m_2, ..., m_k) = a_1 m_1 + a_2 m_2 + ... + a_n m_k.$$

Thus by Remark 2,

$$g(m_1, m_2, ..., m_k) + r_{ij} \leq w_i$$

where w_i *is equal to the exponent of* i*. But* $w_i \leq \max\{w_j\} = \gamma(G)$*. Then*

$$\gamma(G) \geq g(m_1, m_2, ..., m_k) + r_{ij} + 1.$$

□

Corollary 3.2.4. *If* G *is a graph with the uwp, then*

$$\gamma(G) = g(m_1, m_2, ..., m_k) + r + 1.$$

This is a very important result about graph exponents and even though it has been known for a long time, other researchers have not used it. For example, Theorem 3.1.5 has an alternate, much simpler proof.

In fact, the graph G_n in Figure 3–4 has the *uwp*, that is, $r = r_{ij}$ where $r_{ij} = n - 1$. Then by above corollary we have

$$\gamma(G) = g(n, n-1) + n - 1 + 1 = (n-1)^2.$$

3.3 BMDSs whose Dependency Graphs are The Frobenius Graphs

Now, we define another family of particular graphs given by Heap and Lynn in [9], which we will call the *Frobenius graphs*, associated with the relatively prime integers $\{m_i\}$. This is the graph G which is the finite directed graph whose adjacency matrix $A = (a_{ij})$ is of order $n = m_k + m_{k-1} - 1$, where,

$$a_{ij} = \begin{cases} 1, & \text{if } j = i+1 \ (i = 1, ..., n-1; i \neq m_{k-1}), \\ 1, & \text{if } j = 1, \ i = n \ \text{ or } \ m_s \ (s = 1, ..., k-1), \\ 1, & \text{if } i = 1, \ j = m_{k-1} + 1, \\ 0, & \text{otherwise.} \end{cases}$$

This Frobenius graph is illustrated in figure 3–5, the left hand cycle being of length m_k.

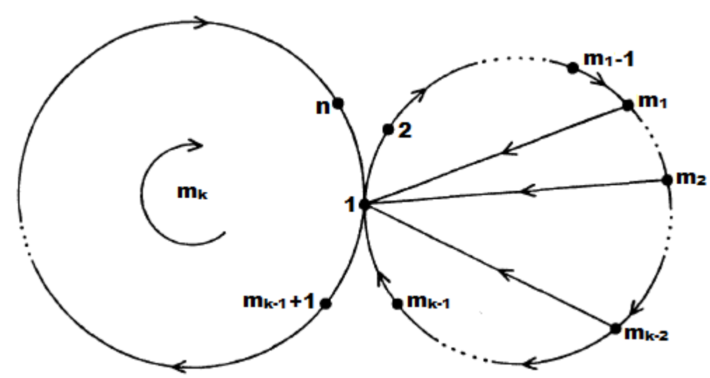


Figure 3–5: The Frobenius Graphs

The following Theorem was proved by Heap and Lynn in 1964, the same year which Dulmage and Mendelsohn presented their paper mentioned in this work, but although Heap and Lynn cited Dulmage and Mendelsohn in their paper, they did not take advantage of the fact that their Frobenius graph has the *uwp*.

Theorem 3.3.1 ([9])**.** *Let $0 < m_1 < m_2 < \cdots < m_k$ be relatively prime integers, and let G be the graph in figure 3–5. Then*

$$\gamma(G) = g(m_1, m_2, ..., m_k) + 2m_k - 1.$$

Proof. *Clearly, G is strongly connected and by hypothesis $gcd(m_1, m_2, ..., m_k) = 1$, where $\{m_s\}$ are the cycles lengths of G for $s = 1, 2, ..., k$. Thus G is primitive. Also, every pair of vertices of G has the uwp.*

Now, consider the pair $(m_{k-1}+1, n)$ *of vertices of* G*. We show that*

$$r_{m_{k-1}+1,n} = r$$

To prove this we consider 4 cases as follows: Let (i, j) *be an ordered pair of vertices.*

Case 1. If (i, j) *are on the cycle of length* m_k*, there is a positive integer* d *such that*

$$r_{ij} = m_k + d \leq m_k + m_k - 2 = 2m_k - 2$$

and the equality holds when $i = m_{k-1} + 1$ *and* $j = n$*.*

Case 2. If (i, j) *are on the cycle of length* m_{k-1}*, then*

$$r_{ij} = m_{k-1} + d \leq m_{k-1} + m_{k-1} - 2 = 2m_{k-1} - 2$$

and the equality holds when $i = 2$ *and* $j = m_{k-1}$*.*

Case 3. If the vertex i *is on the cycle of length* m_k *and vertex* j *is on the cycle of length* m_{k-1} *then*

$$r_{ij} = m_k - 1 + d \leq m_k - 1 + m_{k-1} - 1 = m_k + m_{k-1} - 2$$

and the equality holds when $i = m_{k-1} + 1$ *and* $j = m_{k-1}$*.*

Case 4. If the vertex i *is on the cycle of length* m_{k-1} *and vertex* j *is on the cycle of length* m_k *then*

$$r_{ij} = m_{k-1} + d \leq m_{k-1} - 1 + m_k - 1 = m_k + m_{k-1} - 2$$

and the equality holds when $i = 2$ *and* $j = n$*.*

Thus,

$$r = \max\{2m_k - 2, 2m_{k-1} - 2, m_k + m_{k-1} - 2\},$$

since $m_{k-1} < m_k$ *it is clear that*

$$2m_{k-1} - 2 < m_k + m_{k-1} - 2$$

and also

$$m_k > m_{k-1} \Rightarrow 2m_k > m_k + m_{k-1} \Rightarrow 2m_k - 2 > m_k + m_{k-1} - 2$$

Therefore,

$$r = 2m_k - 2.$$

Then, by Corollary 3.2.4 it follows that

$$\gamma(G) = g(m_1, m_2, ..., m_k) + r + 1 = g(m_1, m_2, ..., m_k) + 2m_k - 1.$$

$\square$

Example 3.3.2. *When $m_1 = 3$ and $m_2 = 5$, then $n = m_2 + m_1 - 1 = 7$ and the Frobenius graph is the same as the graph of figure 2–5.*

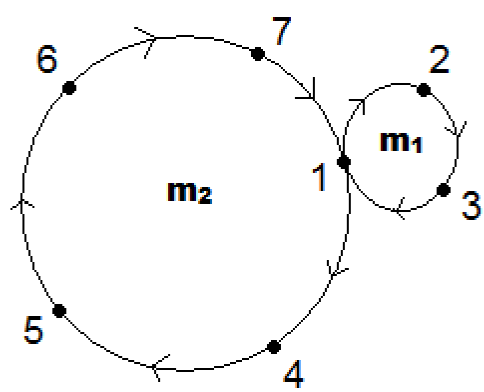


Figure 3–6: The Frobenius Graph G

Thus, the BMDS of the figure 3–6 is given by $f(x_2x_4, x_3, x_1, x_5, x_6, x_7, x_1)$ and its transient is

$$t = \gamma(G) = g(m_1, m_2) + 2m_k - 1 = g(3, 5) + 2(5) - 1 = 16.$$

In particular, when $k = 2$ the graph consists of an increasing chain of 2 cycles.

Chapter 4
THE GENERAL PROBLEM OF FINDING THE TRANSIENT OF A BMDS

In this chapter we consider BMDSs whose dependency graphs are *ordered wedges* of cycles.

4.1 The General Problem

For example, if the number of cycles is $k = 2$, then the graph G is a chain of two cycles with a common vertex (*adjacent cycles*), as is shown in figure 4–1. This is a special case of the graphs of Heap and Lynn, and also of Dulmage and Mendelsohn. In particular, this graph has the *uwp* and in fact,

$$r = r_{m_1+1,n} = m_2 + m_2 - 2 = 2m_2 - 2.$$

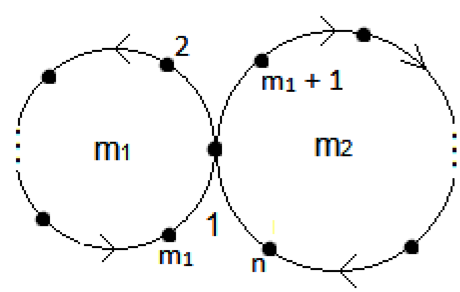


Figure 4–1: Chain with two cycles

Then, by Corollary 3.2.4, it follows that

$$\gamma(G) = g(m_1, m_2) + r_{m_1+1,n} + 1 = m_1m_2 - m_1 - m_2 + 2m_2 - 2 + 1 = m_1m_2 - m_1 + m_2 - 1$$

$$\therefore\ \gamma(G) = (m_2 - 1)(m_1 + 1).$$

We thus have the following theorem.

Theorem 4.1.1. *Let G be a digraph consisting of the union of two cycles of lengths m_1 and m_2 and having exactly one vertex in common. If $m_1 < m_2$ and $gcd(m_1, m_2) = 1$, then*

$$\gamma(G) = (m_2 - 1)(m_1 + 1).$$

It is natural to ask if this idea can be extended to cycles chains consisting of more than two cycles, that is for graphs of this form:

Let $C_{m_1}, C_{m_2}, ..., C_{m_k}$ be cycles of lengths $m_1, m_2, ..., m_k$ respectively, where $C_{m_i} = (V_i, E_i)$, be a finite sequence of cycles where

- For each $i = 1, 2, ..., k-1$ the edges of C_{m_i} and $C_{m_{i+1}}$ have opposite orientations.
- $V_i \cap V_j = \emptyset$ for $i < j$ except if $1 \leq i \leq k-1$ and $j = i+1$, then $V_i \cap V_j = \{u_i\}$.
- The u_i with $i = 1, 2, ..., k-1$ are distinct.

Here, one can see the general form of these graphs:

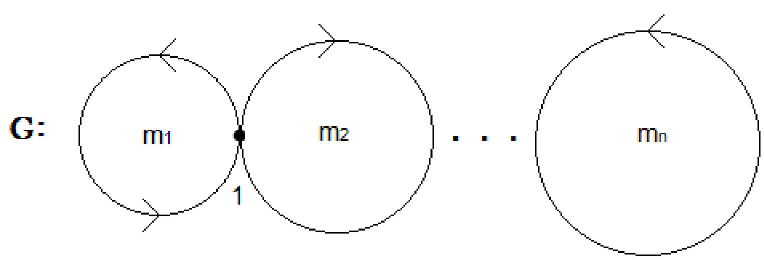


Figure 4–2: Cycles Chains of Coprime Lengths

In general, we denote G by $C_{m_1} \vee_{u_1} C_{m_2} \vee_{u_2} \cdots \vee_{u_{k-2}} C_{m_{k-1}} \vee_{u_{k-1}} C_{m_k}$. G is clearly strongly connected and if $gcd(m_1, m_2, ..., m_k) = 1$, G is primitive.
Thus, by Theorem 3.2.1 we have that $g(m_1, m_2, ..., m_k) + r + 1$ is an upper bound for the exponent of this family of graphs. However, we have not proved that it is a lower bound.

Note that this family of graphs does not have the *uwp* when $k \geq 3$. As an example, consider the graph $C_3 \vee C_4 \vee_5 C_5$ as shown in Figure 4–3.

Example 4.1.2. *Let* $f = (x_2x_4, x_3, x_1, x_5, x_6x_{10}, x_7, x_8, x_9, x_5, x_1)$ *be a BMDS whose associated dependency graph is given by*

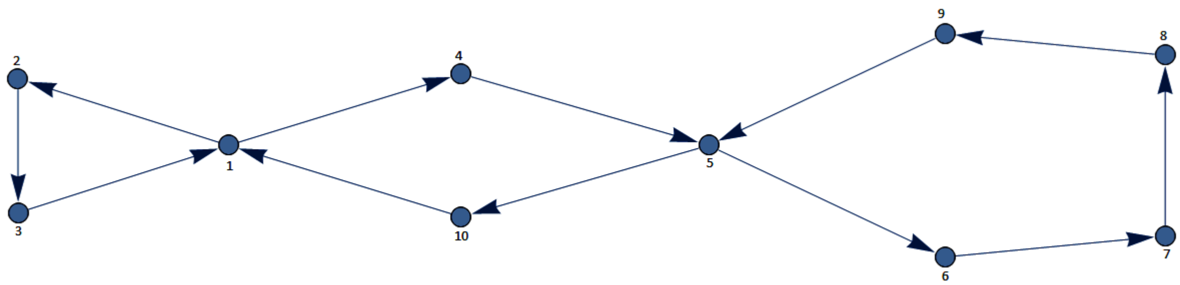


Figure 4–3: $G : C_3 \vee C_4 \vee_5 C_5$

Then $r = r_{6,9} = 12$. *Now let* α *be the walk* $6, 7, 8, 9, 5, 6, 7, 8, 9, 5, 6, 7, 8, 9$. *Then* $|\alpha| = 13 = r_{6,9} + 1$, *but* G *contains no "circuit" of length* 1. *Hence* G *does not have the uwp.*

But, in the next section we consider a similar family.

4.2 A Similar Family to the General Problem

We can construct a family of directed graphs with the *uwp* as follows. For each pair of positive integers u, v and s. Let $S_{u,v}(s)$ be the path graph $u \to ... \to v$ with s vertices.

For each positive integer s, let C_s be the directed cycle $(1, 2), (2, 3), ..., (n-1, n), (n, 1)$. Then, for relatively prime $m_1, m_2, ..., m_k$ with $2 \leq m_1 < m_2 < ... < m_k$, let

$$G(m_1, m_2, ..., m_k) = C_{m_k} \cup S_{1,m_k}(m_1) \cup S_{2,m_k-1}(m_2 - 2) \cup S_{3,m_k-2}(m_3 - 4) \cup$$

$$... \cup S_{i,m_k-i+1}(m_i - 2i + 2) \cup ... \cup S_{k-1,m_k-k+2}(m_{k-1} - 2k + 4).$$

This graph can be depicted as follows:

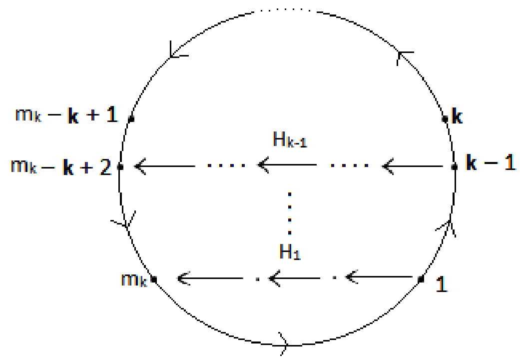


Figure 4–4: Family of Graphs with The *uwp*

where $H_i = S_{i,m_k-i+1}(m_i - 2i + 2)$ for $i = 1, 2, ..., k-1$.

Theorem 4.2.1. *For every sequence $m_1, m_2, ..., m_k$ of coprimes positive integers with $2 \leq m_1 < m_2 < ... < m_k$, there exists a BMDS with dependency graph $G(m_1, m_2, ..., m_k)$ and whose transient is*

$$g(m_1, m_2, ..., m_k) + 2m_k - 2k + 2.$$

Proof. *Let f be a BMDS whose dependency graph $G = G(m_1, m_2, ..., m_k)$ is given as in Figure 4–4; we need to prove that G is a graph with the uwp, that is, $r_{ij} = r$ for some vertices i, j of G with the uwp. If i and j are on the same cycle, say C_{m_p} then $r_{ij} \leq 2m_p - 2p + 1$. Suppose i and j are on distinct cycles C_{m_p} and $C_{m_{p'}}$ where $m_p > m_{p'}$. Then $r_{ij} \leq 2m_p - 2p + 1$.*

$$\therefore \quad r_{ij} \leq 2m_k - 2k + 1.$$

But,

$$r_{k,m_k-k+1} = m_k + m_k - 2k + 1 = 2m_k - 2k + 1 = r,$$

it follows by Corollary 3.2.4 that

$$\gamma(G) = g(m_1, m_2, ..., m_k) + 2m_k - 2k + 2.$$

□

Example 4.2.2 (see [11] p.826-829)**.** *The "lactose operon" can be regarded as a BMDS f whose dependency graph is*

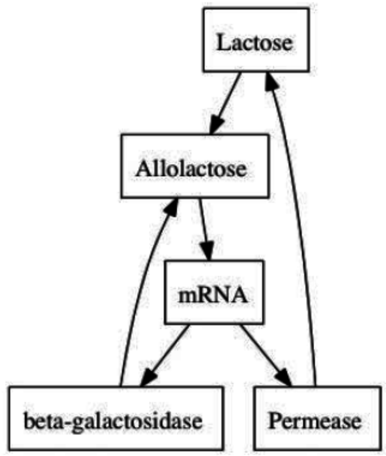


Figure 4–5: Dependency Graph of $f = (A, M, B \times P, L, A)$

where the $5-$*tuple* (L, A, M, P, B) *represents the states of Lactose, Allolactose, mRNA, Permease and* $\beta-$ *galactosidase.*

Renaming the nodes $1, 2, 3, 4, 5$ *and drawing the graph in the form of Figure 4–4 we have*

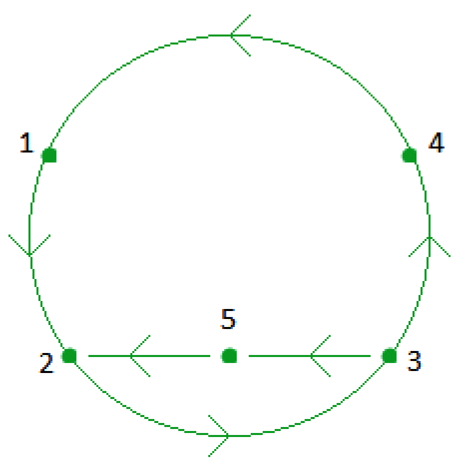


Figure 4–6: Dependency Graph of the BMDS $f = (x_2, x_3, x_4x_5, x_1, x_2)$

and the transient of f is

$$g(3, 4) + 2(4) - 2(2) + 2 = 5 + 8 - 4 + 2 = 11.$$

Chapter 5
DISCUSSION OF RESULTS

5.1 Conclusions

We have shown that a BMDS f with a strongly connected dependency graph G_f is a fixed point system if and only if G_f is primitive and the transient of f is equal to the exponent of G_f. Furthermore, every directed graph all of whose vertices have outdegree at least one is the dependency graph of a BMDS.

Thus, every example of a primitive graph with known exponent gives us an example of a BMDS with known transient. Consequently we have focused on primitive graphs with known exponent. We have presented an upper bound for the exponent of a primitive graph G given by

$$\gamma(G) \leq g(m_1, m_2, ..., m_k) + r + 1,$$

where $m_1, m_2, ..., m_k$ are the cycle lengths of G.

Moreover, we have described a particular family of graphs called "Graphs with the *uwp*" for which there is an explicit formula for the exponent. This family was defined by Dulmage and Mendelsohn in [6]. Also, we have described a type of graph called "The Frobenius graph" introduced by Heap and Lynn in [9] for which the exponent is given by

$$g(m_1, m_2, ..., m_k) + 2m_k - 1.$$

In chapter 4, we have presented a family of BMDS whose dependency graph is a chain of cycles of increasing coprime lengths $m_1, m_2, ..., m_k$ and we conjecture that its exponent is given by

$$g(m_1, m_2, ..., m_k) + 2m_k + m_{k-1} + \cdots + m_2 - 1. \qquad (1)$$

However, in general, this family does not have the *uwp*. Then, we constructed a similar family of graphs with the *uwp* and we proved that the exponent is given by

$$g(m_1, m_2, ..., m_k) + 2m_k - 2k + 2.$$

In general, we have shown that the Frobenius number can be expressed in terms of the exponent of a primitive graphs. Therefore, the problem of determining the exponent of a primitive graph is a class of problems that are, informally, "at least as hard as the hardest problems in NP", in fact, is NP-HARD.

5.2 Future Work

We present some important questions to advance the theory of FDS.

- Show that (1) is a lower bound for the problem of "Cycle Chains".
- Find an explicit formula for the transient of any BMDS.

We believe that answering these questions will contribute greatly to FDS. Since there are numerous applications such as reverse engineering problem and modeling of gene regulatory networks; exploring the above questions will provide interesting and challenging research directions as well.

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