THE NOTION OF SEPARATION FOR INTERIOR OPERATORS IN TOPOLOGY

By

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A notion of separation with respect to an interior operator in topology is introduced. After concrete examples, some properties of separation are presented. In particular, closure of separation with respect to subspaces and products is proved. This notion of separation with respect to an interior operator gives rise to a Galois connection between the collection of all Topological Spaces and the collection of all Interior Operators in Topology. Characterizations of the fixed points of this Galois connection are given. An equivalent definition of separation is introduced that makes possible a generalization to other categories as well. Examples are provided.

Resumen de Disertación Presentado a Escuela Graduada
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LA NOCIÓN DE SEPARACIÓN PARA OPERADORES INTERIORES EN TOPOLOGÍA

Por

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Se introduce una noción de separación con respecto a un operador de interior topológico. Después de proporcionar ejemplos concretos, se presentan algunas propiedades. En particular se demuestra que la separación es cerrada con respecto a subespacios y productos. Esta noción de separación con respecto a un operador de interior da origen a una conexión de Galois entre la colección de Espacios Topológicos y la colección de Operadores de Interior en Topología. Se presentan caracterizaciones de los puntos fijos de esta conexión de Galois. Se introduce una definición equivalente de separación que hace posible una generalización a otras categorías. Se presentan algunos ejemplos.

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CHAPTER 1
INTRODUCTION

In mathematics the first encounter with the Hausdorff separation axiom is in calculus, when it is used to show unique convergence of a sequence, although it is not explicitly mentioned. Then, in the branch of mathematics called Topology, Hausdorff separation is a very important property, that allows to prove many interesting results about compacteness. It is one of the most general separation axioms and at the same time one of the most deep. This work tries to extent this notion to a more abstract context.

It is well known that a topology on a set can be defined using the notion of closure operator, or equivalently, the notion of interior operator. In [1] the notion of closure operator was successfully extended from topology to other categories. But, what would happen if one tries to extend the notion of interior operator? Some results of a first attempt to answer this question can be found in [2]. This thesis is a continuation of the work in [2], and a further attempt to answer the question. Concretely, this thesis studies the Hausdorff separation axiom in the context of interior operators.

Some definitions and results of General Topology that are considered relevant for the thesis are given in Chapter 2. These results can be found in [3], [4], [5] and [6].

The aspects of the Interior Operator Theory that will be used in the main part of the thesis are summarized in Chapter 3. In particular, the notion of I-open set is fundamental because separation with respect to an interior operator is defined in terms of I-open sets.

Finally the notion of separation with respect to an interior operator on the category **Top** of topological spaces is defined in Chapter 4. Concrete examples of interior operators and their collections of separated spaces are given. Then, it is proved that the collection of separated spaces is closed under subspaces and products. After that, the collection of Topological Spaces and the collection of Interior Operators on **Top** are related via a Galois connection. Morever, the interior operator determined by a subcategory of **Top** is characterized and some categorical properties involved are stated. In the last part of the chapter, a definition and some results that try to generalise the notion of separation to other categories are given.

CHAPTER 2
TOPOLOGY REVIEW

In this chapter a number of definitions and results that will be used throughout this thesis are included.

The category of topological spaces will be denoted by **Top**. In this category, the *objects*, are the topological spaces, and the *morphisms* are the continuous functions between them.

Definition 1. Let X be a topological space.

1. X is an element of $\mathbf{Top_0}$ if for every pair of different elements $x, y \in X$ there is an open set U in X such that $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$.
2. X is an element of $\mathbf{Top_1}$ if every finite subset of X is a closed set in X.
3. X satisfies the *Hausdorff separation axiom* if for every pair of points x, y, with $x \neq y$, there are two open sets in X, U and V, such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. In this case, we write $X \in \mathbf{Top_2}$.
4. X is an element of $\mathbf{Top_{2\frac{1}{2}}}$, if for every pair of different elements $x, y \in X$ there are two open sets U, V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.

With the same idea, $\mathbf{Top_i}$ denotes the category of all the topological spaces which satisfy the separation axiom T_i.

The following results present other notions equivalent to the above definition of Hausdorff space. The symbol $\triangle_X$ denotes the *diagonal* of X, i.e., the subset of the cartesian product $X \times X$ that consists of all the ordered pairs having equal first and second coordinates.

Proposition 2. $X \in \mathbf{Top_2}$ *(in words, X is a Hausdorff space) if and only if $(X \times X) \setminus \triangle_X$ is open in $X \times X$ with the product topology, and consequently $\triangle_X$ is closed in $X \times X$.*

Proof. The complement of $\triangle_X$ with respect to $X \times X$ is denoted by $\complement\triangle_X := (X \times X) \setminus \triangle_X$, and a neighborhood of x by U_x. Given $x, y \in X$,

$$x \neq y \quad \text{if and only if} \quad (x, y) \in \complement\triangle_X.$$

Then

$$\begin{aligned} X \in \mathbf{Top_2} &\Leftrightarrow \Big(\forall\, x, y \in X\Big)\Big(x \neq y\Big)\Big(\exists\, U_x, U_y\Big), \quad U_x \cap U_y = \emptyset \\ &\Leftrightarrow \Big(\forall\, x, y \in X\Big)\Big(x \neq y\Big)\Big(\exists\, U_x, U_y\Big), \quad U_x \times U_y \subseteq \complement\triangle_X \\ &\Leftrightarrow \Big(\forall\, (x, y) \in \complement\triangle_X\Big)\Big(\exists\, U_{(x,y)} \text{ nbhd of } (x, y)\Big), \; U_{(x,y)} \subseteq \complement\triangle_X \\ &\Leftrightarrow \complement\triangle_X \text{ open set in } X \times X \\ &\Leftrightarrow \triangle_X \text{ closed set in } X \times X. \end{aligned}$$

□

Definition 3. If X, Y are sets, and $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are functions, the *equalizer* of f and g is the set

$$equ\,(f, g) := \{x \in X : f\,(x) = g\,(x)\}\,.$$

Proposition 4. $Y \in \mathbf{Top_2}$ *if and only if for every $X \in$ **Top** and for every pair of continuous functions $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, $M \subseteq X$ implies $\overline{M} \subseteq equ\,(f, g)$.*

Proof. For the "if" part, $Y \in$ **Top** and $Y \times Y \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} Y$ are considered, where π_1, π_2 are the projections on the first and second coordinates, respectively. If it can be shown that the diagonal $\triangle_Y$ is a subset of the equalizer of the functions, then the closure of $\triangle_Y$ is also a subset of the equalizer. But in this case $\triangle_Y = equ\,(\pi_1, \pi_2)$, so even more $\overline{\triangle_Y} \subseteq equ\,(\pi_1, \pi_2)$, or equivalently, $\overline{\triangle_Y} \subseteq \triangle_Y$. Therefore, $\triangle_Y$ is a closed set of $Y \times Y$, and $Y \in \mathbf{Top_2}$.

Now the proof of the "only if" part by contrapositive. It is assumed that $Y \in \mathbf{Top}$, and that there are $X \in \mathbf{Top}$, $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ continuous functions and a subset M of X such that $M \subseteq equ\,(f,g)$ but $\overline{M} \nsubseteq equ\,(f,g)$. Let $x \in \overline{M}$ such that $x \notin equ\,(f,g)$ so that $f\,(x) \neq g\,(x)$. Let $U_{f(x)}$, $U_{g(x)}$ be neighborhoods of $f\,(x)$, $g\,(x)$, respectively. Then $f^{-1}\left(U_{f(x)}\right) \cap g^{-1}\left(U_{g(x)}\right)$ is a neighborhood of x. Since $x \in \overline{M}$,

$$\left[f^{-1}\left(U_{f(x)}\right) \cap g^{-1}\left(U_{g(x)}\right)\right] \cap M \neq \emptyset.$$

Taking $m \in \left[f^{-1}\left(U_{f(x)}\right) \cap g^{-1}\left(U_{g(x)}\right)\right] \cap M$, then $f\,(m) = g\,(m) \in U_{f(x)} \cap U_{g(x)}$, so that $U_{f(x)} \cap U_{g(x)} \neq \emptyset$. Consequently, $Y \notin \mathbf{Top_2}$, since $U_{f(x)}$, $U_{g(x)}$ were arbitrary neighborhoods of $f\,(x)$, $g\,(x)$. $\square$

An immediate consequence of the above proposition is the following.

Corollary 5. *$equ\,(f,g)$ is closed in X, for $Y \in \mathbf{Top_2}$.*

Proposition 6. *$Y \in \mathbf{Top_2}$ if and only if for every pair of continuous functions $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ and for all x in X such that $f\,(x) \neq g\,(x)$, there is a neighborhood U_x of x in X such that for every $y \in U_x$, $f\,(y) \neq g\,(y)$.*

Proof. For the "if" part, it is assumed that $Y \in \mathbf{Top}$, $x, y \in Y$ with $x \neq y$, and $Y \times Y \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} Y$. Since $(x,y) \in \complement\triangle_Y \subset Y \times Y$, $\pi_1\,(x,y) = x \neq y = \pi_2\,(x,y)$. Then by hypothesis, there is a neighborhood $U_{(x,y)}$ of (x,y) in $Y \times Y$ such that for every $(w,z) \in U_{(x,y)}$, $\pi_1\,(w,z) \neq \pi_2\,(w,z)$. But $\pi_1\,(w,z) = w$ and $\pi_2\,(w,z) = z$, so that for every $(w,z) \in U_{(x,y)}$, $w \neq z$. Hence $U_{(x,y)} \subseteq \complement\triangle_Y$. Since there are U_x, U_y neighborhoods of x, y in Y such that $U_x \times U_y \subseteq U_{(x,y)}$, they satisfy $U_x \times U_y \subseteq \complement\triangle_Y$, and hence there are U_x, U_y neighborhoods of x, y in Y such that $U_x \cap U_y = \emptyset$. Thus $Y \in \mathbf{Top_2}$.

Now, for the "only if" part, $Y \in \mathbf{Top_2}$, $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ continuous functions, and $x \in X$ such that $f\,(x) \neq g\,(x)$ are considered. Therefore there are $U_{f(x)}$, $U_{g(x)}$

neighborhoods of $f(x)$, $g(x)$ in Y such that $U_{f(x)} \cap U_{g(x)} = \emptyset$. The set $f^{-1}\left(U_{f(x)}\right) \cap g^{-1}\left(U_{g(x)}\right)$ is a neighborhood of x in X, and for every element y in this neighborhood it is true that $y \in f^{-1}\left(U_{f(x)}\right)$, and $y \in g^{-1}\left(U_{g(x)}\right)$. Then $f(y) \in U_{f(x)}$ and $g(y) \in U_{g(x)}$. But since $U_{f(x)}$, $U_{g(x)}$ are disjoint neighborhoods, $f(y) \neq g(y)$. □

Definition 7. If $X, Y \in \mathbf{Top}$ and $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ continuous functions the *separator* set of f and g is defined by

$$sep(f, g) := \{x \in X : f(x) \neq g(x)\}.$$

Proposition 8. *If $Y \in \mathbf{Top_2}$ then $sep(f, g)$ is an open set in X.*

Proof. Two cases are considered. If $sep(f, g) = \emptyset$, it is obviously an open set in X. It is assumed then $sep(f, g) \neq \emptyset$. Taking $x \in sep(f, g)$ it follows that $f(x) \neq g(x)$, and using the previous proposition, there is U_x neighborhood of x such that for every $y \in U_x$, $f(y) \neq g(y)$. So that $U_x \subseteq sep(f, g)$, and therefore $sep(f, g)$ is open in X. □

With these results there is another way to prove

Corollary 9. *If $X \in \mathbf{Top_2}$, then $\triangle_X$ is closed in $X \times X$.*

Proof. Since $\complement \triangle_X = sep(\pi_1, \pi_2)$, the previous theorem lets say that $\complement \triangle_X$ is an open set in X, and therefore, $\triangle_X$ is closed in X. □

Definition 10. Let $X, Y \in \mathbf{Top}$ and let $q : X \longrightarrow Y$ be a surjective map. q is a *quotient map* if it satisfies that a subset V of Y is open in Y if and only if $q^{-1}(V)$ is open in X.

Definition 11. Let X be a topological space, N a set and $q : X \longrightarrow N$ a surjective map. Then, there is exactly one topology τ on N such that q is a quotient map relative to τ. τ is the *quotient topology* induced by q. τ is defined as follows: a subset $U \subseteq N$ is open with respect to τ if and only if $q^{-1}(U)$ is open in X.

Definition 12. Let X be a topological space and $\tilde{X}$ any partition of X. Let q be the map defined by

$$\begin{cases} q : X \longrightarrow \tilde{X} \\ q(x) := [x] , \end{cases}$$

where $[x]$ is the unique element of $\tilde{X}$ which contains x. Then $\tilde{X}$ with the quotient topology induced by q is a *quotient space* of X.

Definition 13. Let X_1, X_2 be arbitrary sets. The set

$$X_1 + X_2 := \bigcup_{i=1}^{2} \{(x, i) : x \in X_i\}$$

is the *disjoint union* of X_1 and X_2. If $X_1 = X_2 = X$, then $X_1 + X_2 = X \times \{1, 2\}$.

Definition 14. Let $X_1, X_2 \in \mathbf{Top}$. Let $X_1 + X_2$ be the disjoint union of X_1 and X_2, and let h, k be the functions defined by

$$\begin{cases} h : X_1 \longrightarrow X_1 + X_2 \\ h(x) := (x, 1) \end{cases} , \quad \text{and,} \quad \begin{cases} k : X_2 \longrightarrow X_1 + X_2 \\ k(x) := (x, 2) \end{cases} .$$

The *disjoint union topology* on $X_1 + X_2$ is defined as follows: U^* subset of $X_1 + X_2$ is open in $X_1 + X_2$ if and only if $h^{-1}(U^*)$ is open in X_1 and $k^{-1}(U^*)$ is open in X_2.

Remark. When it is used the notation $X_1 + X_2 \in \mathbf{Top}$, or when $X_1 + X_2$ is written as a member of a subcategory of $\mathbf{Top}$, it is assumed that the disjoint union $X_1 + X_2$ has the disjoint union topology.

Proposition 15. *Let X be a topological space with the assumption $X_1 = X_2 = X$. If $X \in \mathbf{Top_2}$, then $X_1 + X_2 \in \mathbf{Top_2}$.*

Proof. Let $x^*, y^* \in X_1 + X_2$, with $x^* \neq y^*$. Therefore there are $x, y \in X$ and $i, j \in \{1, 2\}$ such that $x^* = (x, i)$ and $y^* = (y, j)$. Two cases are considered.

Case 1 $x = y$.

Therefore $i \neq j$. Without loss of generality, it is assumed that $x^* = (x, 1)$ and $y^* = (x, 2)$. Let U be neighborhood of x in X. Then $U \times \{1\}$ is a neighborhood of x^* in $X_1 + X_2$ since $x^* = (x, 1) \in U \times \{1\}$, $h^{-1}(U \times \{1\}) = U$ is an open set in $X_1 = X$, and $k^{-1}(U \times \{1\}) = \emptyset$ is an open set in $X_2 = X$. Similarly, $U \times \{2\}$ is a neighborhood of y^* in $X_1 + X_2$. These neighborhoods satisfy

$$(U \times \{1\}) \cap (U \times \{2\}) = \emptyset.$$

Case 2 $x \neq y$.

Then there are U_x, U_y neighborhoods of x, y in X, respectively, such that $U_x \cap U_y = \emptyset$. As Case 1, $U_x \times \{i\}, U_y \times \{j\}$ are neighborhoods of x^*, y^*, respectively, and satisfy

$$(U_x \times \{i\}) \cap (U_y \times \{j\}) = \emptyset.$$

Hence, $X_1 + X_2 \in \mathbf{Top_2}$. □

Proposition 16. *Let X be a topological space, M a subset of X and $X_1 = X_2 = X$. Let x^* be a point in $X_1 + X_2$, so that there are $x \in X$ and $i \in \{1, 2\}$ such that $x^* = (x, i)$. With the definition*

$$[x^*] := \begin{cases} \{(x, i)\}, & x \notin M, \\ \{(x, 1), (x, 2)\}, & x \in M, \end{cases}$$

the set

$$\{[x^*] : x^* \in X_1 + X_2\}$$

is a partition of $X_1 + X_2$.

Notation. The partition in the previous definition is denoted as

$$(X_1 + X_2)/M := \bigcup \{[x^*] : x^* \in X_1 + X_2\}.$$

Remark. When it is used the notation $(X_1 + X_2)/M \in \mathbf{Top}$, or when $(X_1 + X_2)/M \in \mathbf{Top}$ is written as a member of a subcategory of $\mathbf{Top}$, it is assumed that $(X_1 + X_2)/M$

has the quotient topology induced by q,

$$\begin{cases} q : X_1 + X_2 \longrightarrow (X_1 + X_2)/M \\ \qquad q(x^*) := [x^*] . \end{cases}$$

Proposition 17. *Let X be a topological space, $M \subseteq X$, with M closed in X. Let $X_1 = X_2 = X$. If $x \in X \setminus M$, there is U_x neighborhood of x in X such that for $i \in \{1, 2\}$, $q(U_x \times \{i\})$ is a neighborhood of $[(x, i)]$ in $(X_1 + X_2)/M$.*

Proof. Since M is a closed set in X, $X \setminus M$ is open in X. Then, there is a neighborhood U_x of x in X such that $U_x \subseteq X \setminus M$. Clearly $[(x, i)] \in q(U_x \times \{i\})$, for $i = 1, 2$. It is claimed that

$$q^{-1}\big(q(U_x \times \{i\})\big) = U_x \times \{i\} .$$

Let (y, j) be a point in $q^{-1}\big(q(U_x \times \{i\})\big)$. Then $q(y, j)$ is a point in $q(U_x \times \{i\})$, so that there is $z \in U_x$ such that

$$q(y, j) = q(z, i) = [(z, i)] .$$

But this means that $[(y, j)] = \{(z, i)\}$, and hence $(y, j) = (z, i)$, so that $(y, j) \in U_x \times \{i\}$. Since $U_x \times \{i\} \subseteq q^{-1}\big(q(U_x \times \{i\})\big)$ is always true we obtain equality. □

Corollary 18. *Let X be a topological space, $M \subseteq X$, with M closed in X. Let $X_1 = X_2 = X$. If $x \in X \setminus M$, there is U_x neighborhood of x in X such that for $i \in \{1, 2\}$, $q([U_x \times \{1\}] \cup [U_x \times \{2\}])$ is a neighborhood of $[(x, i)]$ in $(X_1 + X_2)/M$.*

Proof. By the previous proposition there is U_x neighborhood of x in X such that $q(U_x \times \{1\})$ and $q(U_x \times \{2\})$ are open sets in $(X_1 + X_2)/M$. Since $[(x, i)]$ is a point in $q([U_x \times \{1\}] \cup [U_x \times \{2\}])$, and

$$\begin{aligned} q^{-1}\big(q([U_x \times \{1\}] \cup [U_x \times \{2\}])\big) &= q^{-1}\big(q(U_x \times \{1\}) \cup q(U_x \times \{2\})\big) \\ &= q^{-1}\big(q(U_x \times \{1\})\big) \cup q^{-1}\big(q(U_x \times \{2\})\big), \end{aligned}$$

the result is true. □

Proposition 19. *Let X be a topological space and $M \subseteq X$, with M closed in X. Let $X_1 = X_2 = X$. Let x be a point of X, and U_x a neighborhood of x in X. Then for $i \in \{1,2\}$, $q([U_x \times \{1\}] \cup [U_x \times \{2\}])$ is a neighborhood of $[(x,i)]$ in $(X_1 + X_2)/M$.*

Proof. If $U_x \subseteq X \backslash M$, is case of the previous corollary. It is assumed that $U_x \cap M \neq \emptyset$. If

$$U := (X \setminus M) \cap U_x,$$

$$C := M \cap U_x,$$

$U_x = U \cup C$. It is clear that U is an open set in X that satisfies $U \subseteq X \setminus M$, and that $[(x,i)]$ is a point in the set $q([U_x \times \{1\}] \cup [U_x \times \{2\}])$. It is claimed that

$$q^{-1}\big(q([U_x \times \{1\}] \cup [U_x \times \{2\}])\big) = [U_x \times \{1\}] \cup [U_x \times \{2\}]$$

and this proves the assertion.

Let (y,j) be a point in $q^{-1}\big(q([U_x \times \{1\}] \cup [U_x \times \{2\}])\big)$. Then $q(y,j)$ is a point in $q([U_x \times \{1\}] \cup [U_x \times \{2\}])$. But

$$\begin{aligned} q([U_x \times \{1\}] \cup [U_x \times \{2\}]) &= q([U \times \{1\}] \cup [C \times \{1\}] \cup [U \times \{2\}] \cup [C \times \{2\}]) \\ &= q([U \times \{1\}] \cup [U \times \{2\}]) \cup q([C \times \{1\}] \cup [C \times \{2\}]), \end{aligned}$$

so that $q(y,j)$ is a point in $q([U \times \{1\}] \cup [U \times \{2\}]) \cup q([C \times \{1\}] \cup [C \times \{2\}])$, and the two image sets are disjoint. Two cases are considered.

Case 1 $q(y,j) \in q([U \times \{1\}] \cup [U \times \{2\}])$.

Then, since is case of the previous corollary

$$\begin{aligned} (y,j) \in q^{-1}\big(q([U \times \{1\}] \cup [U \times \{2\}])\big) &= [U \times \{1\}] \cup [U \times \{2\}] \\ &\subseteq [U_x \times \{1\}] \cup [U_x \times \{2\}]. \end{aligned}$$

Case 2 $q(y,j) \in q([C \times \{1\}] \cup [C \times \{2\}])$.

Then there are $z \in C$ and $k \in \{1,2\}$ such that $q(y,j) = q(z,k) = [(z,k)]$. But then $[(y,j)] = \{(z,1),(z,2)\}$, so that $(y,j) = (z,1)$, or $(y,j) = (z,2)$. Hence

$$(y,j) \in [C \times \{1\}] \cup [C \times \{2\}] \subseteq [U_x \times \{1\}] \cup [U_x \times \{2\}].$$

Since the other inclusion is always true, equality is obtained. □

Proposition 20. *Let X be a topological space, $M \subseteq X$, with M closed in X. Let $X_1 = X_2 = X$. If $X \in \mathbf{Top_2}$, then $(X_1 + X_2)/M \in \mathbf{Top_2}$.*

Proof. Let $[x^*] \neq [y^*]$ be points in $(X_1 + X_2)/M$, with $x^* = (x,i)$, $y^* = (y,j)$, where $x, y \in X$ and $i, j \in \{1,2\}$. Two cases are considered.

Case 1 $x \neq y$.

Then there are U_x, U_y disjoint neighborhoods of x, y, respectively. Therefore,

$$(U_x \times \{1\}) \cap (U_y \times \{1\}) = \emptyset,$$

and

$$(U_x \times \{2\}) \cap (U_y \times \{2\}) = \emptyset.$$

Using these identities,

$$[(U_x \times \{1\}) \cup (U_x \times \{2\})] \cap [(U_y \times \{1\}) \cup (U_y \times \{2\})] = \emptyset.$$

But in the proof of Proposition 19 it was shown that

$$q^{-1}\big(q([U_x \times \{1\}] \cup [U_x \times \{2\}])\big) = [U_x \times \{1\}] \cup [U_x \times \{2\}],$$

and similarly it is possible to write

$$q^{-1}\big(q([U_y \times \{1\}] \cup [U_y \times \{2\}])\big) = [U_y \times \{1\}] \cup [U_y \times \{2\}].$$

Thus,

$$q^{-1}\big(q\left([(U_x \times \{1\}) \cup (U_x \times \{2\})]\right)\big) \cap q^{-1}\big(q\left([(U_y \times \{1\}) \cup (U_y \times \{2\})]\right)\big) = \emptyset,$$

so that

$$q^{-1}\big(q\left([(U_x \times \{1\}) \cup (U_x \times \{2\})]\right) \cap q\left([(U_y \times \{1\}) \cup (U_y \times \{2\})]\right)\big) = \emptyset.$$

Since q is surjective,

$$q\left([(U_x \times \{1\}) \cup (U_x \times \{2\})]\right) \cap q\left([(U_y \times \{1\}) \cup (U_y \times \{2\})]\right) = \emptyset,$$

and from Proposition 19 it has been found $q\left([(U_x \times \{1\}) \cup (U_x \times \{2\})]\right)$ neighborhood of $[x^*]$ that is disjoint from $q\left([(U_y \times \{1\}) \cup (U_y \times \{2\})]\right)$ neighborhood of $[y^*]$.

Case 2 $x = y$.

Then $x \notin M$. Without loss of generality, it is assumed that $x^* = (x, 1)$ and $y^* = (x, 2)$. There is U_x neighborhood of x in X such that $U_x \subseteq X \setminus M$. Then

$$\begin{aligned}
\emptyset &= (U_x \times \{1\}) \cap (U_x \times \{2\}) \\
&= q^{-1}\big(q\left((U_x \times \{1\})\right)\big) \cap q^{-1}\big(q\left((U_x \times \{2\})\right)\big) \\
&= q^{-1}\big(q\left((U_x \times \{1\})\right) \cap q\left((U_x \times \{2\})\right)\big),
\end{aligned}$$

and since q is surjective

$$[q\left((U_x \times \{1\})\right)] \cap [q\left((U_x \times \{2\})\right)] = \emptyset,$$

and it has been found $q\left((U_x \times \{1\})\right)$ neighborhood of $[x^*]$ that is disjoint of $q\left((U_x \times \{2\})\right)$ neighborhood of $[y^*]$.

□

Proposition 21. *Let $X \in \mathbf{Top_2}$, $M \subseteq X$, with M closed in X. Then there are $Y \in \mathbf{Top_2}$ and continuous functions $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ such that $M = equ\,(f, g)$.*

Proof. Let $X_1 = X_2 = X$. Since $X \in \mathbf{Top_2}$, by the previous proposition $(X_1 + X_2)\,/M \in \mathbf{Top_2}$. From the diagram

$$X \overset{h}{\underset{k}{\rightrightarrows}} X_1 + X_2 \xrightarrow{q} (X_1 + X_2)\,/M$$

$M = equ\,(q \circ h, q \circ k)$, where h and k are as in Definition 14. □

Definition 22. Let $X, Y \in \mathbf{Top}$. A function $f : X \longrightarrow Y$ is *sequentially continuous* if for every sequence $(x_n)_{n \in \mathbb{N}}$ in X, the convergence $x_n \longrightarrow x$ in X implies the convergence $f\,(x_n) \longrightarrow f\,(x)$ in Y.

Proposition 23. *Every continuous function is sequentially continuous.*

Proof. Let $f : X \longrightarrow Y$ be a continuous function. Let $x_n \longrightarrow x$, and let V be a neighborhood of $f\,(x)$. Since f is continuous at x, there is a neighborhood U of x such that $f\,(U) \subseteq V$. Since $x_n \longrightarrow x$, exists n_0 natural number such that for every natural number n with $n \geq n_0$, $x_n \in U$. Therefore, this n_0 satisfies that for every natural number $n \geq n_0$, $f(x_n) \in f\,(U) \subseteq V$. Hence $f\,(x_n) \longrightarrow f\,(x)$. □

The following are results related to the notion of connectedness.

Definition 24. Let $X \in \mathbf{Top}$. A *separation* of X consists of two disjoint open sets U, V such that $X = U \cup V$. If there is not a non-trivial separation of X then X is called a *connected space.*

Remark. Let U, V be a separation of X. Then U, V are open sets in X, so that $X \setminus U$, $X \setminus V$ are closed sets in X. But since $X = U \cup V$ and U, V are disjoint, $V = X \setminus U$ and $U = X \setminus V$. Therefore U, V are closed sets in X, and hence U, V are *clopen* sets in X.

Definition 25. Let $X \in \mathbf{Top}$ and $M \subseteq X$. M is a *connected set* in X if M is a connected space with the subspace topology.

Definition 26. Let $X \in \mathbf{Top}$ and $H, K \subseteq X$. H and K are *separated* if

$$\overline{H} \cap K = H \cap \overline{K} = \emptyset.$$

Theorem 27. *Let $X \in \mathbf{Top}$ and $H, K, M \subseteq X$, with $M = H \cup K$. Then, H, K are separated if and only if $H \cap K = \emptyset$, and H, K are open sets in M with the subspace topology.*

Proof. First, assuming that H, K are separated sets

$$H \cap K \subseteq \overline{H} \cap K = \emptyset,$$

and thus $H \cap K = \emptyset$. From $H \cap \overline{K} = \emptyset$, $H \subseteq X \setminus \overline{K}$, that is an open set in X such that

$$M \cap \left(X \setminus \overline{K}\right) = (H \cup K) \cap \left(X \setminus \overline{K}\right) = H \cap \left(X \setminus \overline{K}\right) = H,$$

therefore H is an open set in M with the subspace topology. Similarly, from $\overline{H} \cap K = \emptyset$, K is an open set in M with the subspace topology.

Now the assumptions are $H \cap K = \emptyset$, and H, K are open sets in M with the subspace topology. Let U be an open set in X such that $H = M \cap U$. Then $H \subseteq U$ and

$$K \cap U \subseteq M \cap U = H,$$

but since $H \cap K = \emptyset$, $K \cap U = \emptyset$. Therefore, U is a neighborhood of every point of H that satisfies

$$U \cap K = \emptyset,$$

and thus $H \cap \overline{K} = \emptyset$. Similarly, $\overline{H} \cap K = \emptyset$. But then H and K are separated. $\square$

A consequence of the previous theorem is the following.

Corollary 28. *Let X be a topological space, and $M \subseteq X$. M is connected if and only if M is not the union of two nonempty separated sets.*

Proof. M is connected if and only if M is not the union of two disjoint nonempty open sets in M with the subspace topology. By the previous theorem, M is connected if and only if M is not the union of two nonempty separated sets. □

Lemma 29. *If H, K are separated sets, and $H' \subseteq H$ and $K' \subseteq K$, then H', K' are separated sets.*

Proof. $\overline{H'} \cap K' \subseteq \overline{H} \cap K = \emptyset$, and, $H' \cap \overline{K'} \subseteq H \cap \overline{K} = \emptyset$. □

Theorem 30. *Let $X \in$ **Top**. Under the assumptions $H, K \subseteq X$ separated sets and $M \subseteq H \cup K$, if M is connected then M lies in either H or K.*

Proof. Let M be a connected set having nonempty intersection with H and with K. Therefore, $M = (M \cap H) \cup (M \cap K)$, where $M \cap H$, $M \cap K$ are nonempty separated sets, by the previous lemma. But this is a contradiction to Corollary 28, so that M lies in H or in K. □

Proposition 31. *Let $\{M_\alpha\}_{\alpha \in A}$ be a family of connected sets such that $\bigcap_{\alpha \in A} M_\alpha \neq \emptyset$. Then, $\bigcup_{\alpha \in A} M_\alpha$ is connected.*

Proof. Let $x \in \bigcap_{\alpha \in A} M_\alpha$. Assuming that $\bigcup_{\alpha \in A} M_\alpha = H \cup K$, where H, K are separated sets and that $x \in H$ then, since for every $\alpha \in A$ the set M_α is connected, by Theorem 30 $M_\alpha \subseteq H$, and thus $\bigcup_{\alpha \in A} M_\alpha \subseteq H$. Hence, $K = \emptyset$, and $\bigcup_{\alpha \in A} M_\alpha$ is a connected set. □

Theorem 32. *The direct image of a connected space under a continuous function is a connected space.*

Proof. Let $X, Y \in$ **Top** being X a connected set and let $f : X \longrightarrow Y$ be a continuous function. The function with restricted range

$$g : X \longrightarrow f(X)$$

is also continuous. Then, without loss of generality, $f : X \longrightarrow Y$ is assumed continuous and surjective. If Y is not connected, there is V, W a nontrivial separation of Y. But then, $f^{-1}(V)$ and $f^{-1}(W)$ are open sets in X, since f is continuous, also

$$X = f^{-1}(Y) = f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W),$$

and finally

$$f^{-1}(V) \neq \emptyset \neq f^{-1}(W),$$

since f is surjective. Hence, $f^{-1}(V)$, $f^{-1}(W)$ are a nontrivial separation of X, in contradiction with the fact that X is connected. So that Y is connected. $\square$

Definition 33. Let $X \in \mathbf{Top}$ and $x, y \in X$. x and y are *connected in* X $(x \sim y)$ if there is a connected set of X containing them both. This relation between points is an equivalence relation on X, and the equivalence classes are the *components* of X.

Remark.

1. The components are connected subsets of X:

   Let $x \in X$. Let C_x be the component of x in X. It is claimed that

   $$C_x = \bigcup_{y \sim x} C_{(x,y)},$$

   where $C_{(x,y)}$ is a connected set containing x and y. Let $y \in C_x$. Then $y \sim x$, and hence there is $C_{(x,y)}$ connected set in X such that $x, y \in C_{(x,y)}$. So that $y \in C_{(x,y)}$, and thus $y \in \bigcup_{y \sim x} C_{(x,y)}$.

   The set $\bigcup_{y \sim x} C_{(x,y)}$ is the union of connected sets that have x as a common element. Therefore, $\bigcup_{y \sim x} C_{(x,y)}$ is a connected set that contains x, so that by definition $\bigcup_{y \sim x} C_{(x,y)} \subseteq C_x$.

2. The components are maximal connected subsets of X:

   Let C be a connected set such that $x \in C$. Since $x \sim x$, $C \subseteq \bigcup_{y \sim x} C_{(x,y)} = C_x$. Therefore, for every connected set in X such that $x \in C$, $C \subseteq C_x$.

Proposition 34. *Every clopen is the union of the connected components of its elements.*

Proof. Let C be a clopen in X, and let x be a point in C. Let C_x be the connected component of x in X. Since C, $X \setminus C$ are a separation of X and C_x is a connected set in X, then C_x lies in C or C_x lies in $X \setminus C$. But $x \in C$, so that $C_x \subseteq C$, and since x was arbitrary, $C = \bigcup_{x \in C} C_x$. □

Proposition 35. *The finite product of connected sets is connected.*

Proof. Let X, Y be connected spaces. What is going to show is that $X \times Y$ has only one component. Let (x, y) be a fixed point in $X \times Y$, and let (u, v) be an arbitrary point in $X \times Y$. (x, y) lies in the connected set $\{x\} \times Y$ (this set is connected because is homeomorphic to Y), and (u, v) lies in the connected set $X \times \{v\}$ (this set is connected because is homeomorphic to X). But

$$(\{x\} \times Y) \cap (X \times \{v\}) = \{(x, v)\} \neq \emptyset,$$

then $(\{x\} \times Y) \cup (X \times \{v\})$ is a connected set that contains (x, y) and (u, v). Therefore, for every point (u, v) in $X \times Y$, $(u, v) \in C_{(x,y)}$, the connected component of (x, y) in $X \times Y$. Hence,

$$X \times Y = C_{(x,y)},$$

so that $X \times Y$ is connected. □

Definition 36. Let $X \in \mathbf{Top}$. X is *totally disconnected* if the components are singletons.

Definition 37. Let $X \in \mathbf{Top}$. X is *totally separated* if for every pair of different points $x, y \in X$, there is a separation U, V of X such that $x \in U$ and $y \in V$.

Remark. In reference [3] one can find examples of spaces that are totally separated but not discrete.

CHAPTER 3
INTERIOR OPERATORS

In this chapter the notion of interior operator is introduced, examples that illustrate this notion are provided and the notion of a set being open with respect to an interior operator, or I-open, is established. Then the closure of the notion of interior operator under arbitrary unions is studied, and finally, the infimum of a family of interior operators is defined.

Definition 38. The indexed family $I := (i_X)_{X \in \mathbf{Top}}$ is an *interior operator* on the category **Top**, if for every $X \in \mathbf{Top}$, i_X is a function

$$\begin{cases} i_X : S(X) \longrightarrow S(X) \\ \quad M \longmapsto i_X(M), \end{cases}$$

where $S(X)$ is the collection of all subclasses of X, ordered by inclusion, and i_X satisfies

- Contractibility: For every $M \in S(X)$, $i_X(M) \subseteq M$.
- Monotonicity: For every pair $M_1, M_2 \in S(X)$ such that $M_1 \subseteq M_2$, $i_X(M_1) \subseteq i_X(M_2)$.
- Continuity: For every $X, Y \in \mathbf{Top}$, $f : X \longrightarrow Y$ continuous function and $N \in S(Y)$, $f^{-1}(i_Y(N)) \subseteq i_X(f^{-1}(N))$.

The class of all interior operators on **Top** is denoted by $IN(\mathbf{Top})$.

Definition 39. Let $I := (i_X)_{X \in \mathbf{Top}}$, $J := (j_X)_{X \in \mathbf{Top}}$ interior operators on **Top**. $I \leq J$ if and only if for every $X \in \mathbf{Top}$, and for every $M \subseteq X$

$$i_X(M) \subseteq j_X(M).$$

Remark. From the definition, $\leq$ is a partial order on $IN(\mathbf{Top})$. A fortiori, $(IN(\mathbf{Top}), \leq)$ is a pre-order class (in other words, the relation $\leq$ is reflexive and transitive).

For each one of the following examples, let $X, Y \in \mathbf{Top}$, $M, M_1, M_2 \in S(X)$ with $M_1 \subseteq M_2$, $N \in S(Y)$ and let f be a continuous function $f : X \longrightarrow Y$. Example 42 is new; the others can be found in [2].

Example 40. It is defined

$$k_X(M) := \bigcup \{O \subseteq M : O \text{ is open in } X\}.$$

This function satisfies the contractibility property, since by definition is a union of subsets of M, so that $k_X(M) \subseteq M$. If O is a subset of X with $O \subseteq M_1$, then $O \subseteq M_2$, and hence

$$\{O \subseteq M_1 : O \text{ is open in } X\} \subseteq \{O \subseteq M_2 : O \text{ is open in } X\},$$

and thus

$$\bigcup \{O \subseteq M_1 : O \text{ is open in } X\} \subseteq \bigcup \{O \subseteq M_2 : O \text{ is open in } X\},$$

or equivalently, $k_X(M_1) \subseteq k_X(M_2)$.

For the continuity, by definition $k_Y(N)$ is an open set in Y, so that $f^{-1}(k_Y(N))$ is an open set in X and since $f^{-1}(k_Y(N)) \subseteq f^{-1}(N)$,

$$f^{-1}(k_Y(N)) \in \left\{O \subseteq f^{-1}(N) : O \text{ is open in } X\right\},$$

and then

$$f^{-1}(k_Y(N)) \subseteq \bigcup \left\{O \subseteq f^{-1}(N) : O \text{ is open in } X\right\},$$

or $f^{-1}(k_Y(N)) \subseteq k_X(f^{-1}(N))$. Hence, $K := (k_X)_{X \in \mathbf{Top}}$ is an interior operator on $\mathbf{Top}$. This is the usual interior operator induced by the topology.

Example 41. It is defined

$$h_X(M) := \bigcup \{C \subseteq M : C \text{ is closed in } X\}.$$

By definition, $h_X(M) \subseteq M$, since is the union of subsets of M. If C is a subset of X with $C \subseteq M_1$, then $C \subseteq M_2$, and thus

$$\{C \subseteq M_1 : C \text{ is closed in } X\} \subseteq \{C \subseteq M_2 : C \text{ is closed in } X\},$$

and then

$$\bigcup \{C \subseteq M_1 : C \text{ is closed in } X\} \subseteq \bigcup \{C \subseteq M_2 : C \text{ is closed in } X\},$$

or $h_X(M_1) \subseteq h_X(M_2)$, so that the monotonicity is proved.

For the continuity,

$$\begin{aligned} f^{-1}(h_Y(N)) &= f^{-1}\left(\bigcup \{C \subseteq N : C \text{ is closed in } Y\}\right) \\ &= \bigcup \left\{f^{-1}(C) : C \subseteq N, \text{ and, } C \text{ is closed in } Y\right\}. \end{aligned}$$

Since $C \subseteq N$, C is a closed set in Y and f is continuous, $f^{-1}(C) \subseteq f^{-1}(N)$ and $f^{-1}(C)$ is closed in X, and therefore

$$\begin{aligned} f^{-1}(h_Y(N)) &\subseteq \bigcup \left\{f^{-1}(C) \subseteq f^{-1}(N) : f^{-1}(C) \text{ is closed in } X\right\} \\ &\subseteq \bigcup \left\{C \subseteq f^{-1}(N) : C \text{ is closed in } X\right\} \\ &= h_X\left(f^{-1}(N)\right). \end{aligned}$$

Hence, $f^{-1}(h_Y(N)) \subseteq h_X(f^{-1}(N))$ and $H = (h_X)_{X \in \mathbf{Top}}$ is an interior operator on **Top**.

This interior operator can be also expressed as

$$\tilde{h}_X(M) := \left\{x \in M : \overline{\{x\}} \subseteq M\right\}.$$

Let $X \in \mathbf{Top}$ and $M \in S(X)$. That $\tilde{h}_X(M) \subseteq h_X(M)$ it is clear, because if $x \in \tilde{h}_X(M)$, $\overline{\{x\}}$ is a closed set such that $x \in \overline{\{x\}} \subseteq M$, so that

$$x \in \bigcup \{C \subseteq M : C \text{ is a closed set in } X\} = h_X(M).$$

Now for $h_X(M) \subseteq \tilde{h}_X(M)$, let $x \in h_X(M)$. Then there is a closed set C in X such that $x \in C \subseteq M$. But since C is closed and contains x, then $\overline{\{x\}} \subseteq C \subseteq M$, and thus $x \in \tilde{h}_X(M)$.

Example 42. It is defined

$$b_X(M) := \left\{x \in X : \exists U_x \text{ nbhd of } x \text{ such that } U_x \cap \overline{\{x\}} \subseteq M\right\}.$$

Clearly, $b_X(M) \subseteq M$.

For the monotonicity

$$\begin{aligned} b_X(M_1) &= \left\{x \in X : \exists U_x \text{ nbhd of } x \text{ such that } U_x \cap \overline{\{x\}} \subseteq M_1 \subseteq M_2\right\} \\ &\subseteq \left\{x \in X : \exists U_x \text{ nbhd of } x \text{ such that } U_x \cap \overline{\{x\}} \subseteq M_2\right\} \\ &= b_X(M_2). \end{aligned}$$

Finally, for the continuity, take $x \in f^{-1}(b_Y(N))$. Then, $f(x) \in b_Y(N)$, so that there is a neighborhood $V_{f(x)}$ of $f(x)$ such that $V_{f(x)} \cap \overline{\{f(x)\}} \subseteq N$. Therefore

$$\begin{gathered} f^{-1}\left(V_{f(x)} \cap \overline{\{f(x)\}}\right) \subseteq f^{-1}(N) \\ f^{-1}\left(V_{f(x)}\right) \cap f^{-1}\left(\overline{\{f(x)\}}\right) \subseteq f^{-1}(N), \end{gathered}$$

where the set $f^{-1}\left(V_{f(x)}\right)$ is a neighborhood of x and the set $f^{-1}\left(\overline{\{f(x)\}}\right)$ is closed in X and contains x. Hence,

$$f^{-1}\left(V_{f(x)}\right) \cap \overline{\{x\}} \subseteq f^{-1}\left(V_{f(x)}\right) \cap f^{-1}\left(\overline{\{f(x)\}}\right) \subseteq f^{-1}(N),$$

and consequently $f^{-1}\left(V_{f(x)}\right)$ is a neighborhood of x such that

$$f^{-1}\left(V_{f(x)}\right) \cap \overline{\{x\}} \subseteq f^{-1}(N).$$

But this means that $x \in b_X\left(f^{-1}(N)\right)$, and then $f^{-1}\left(b_Y(N)\right) \subseteq b_X\left(f^{-1}(N)\right)$, so that $B := (b_X)_{X \in \mathbf{Top}}$ is an interior operator on **Top**.

Example 43. It is defined

$$\theta_X(M) := \left\{x \in M : \exists U_x \text{ nbhd of } x \text{ s.t. } \overline{U}_x \subseteq M\right\}.$$

By definition $\theta_X(M)$ is a set that consists of elements of M, so that $\theta_X(M) \subseteq M$, and the contractibility is proved. For the monotonicity,

$$\begin{aligned}\theta_X(M_1) &= \left\{x \in M_1 : \exists U_x \text{ nbhd of } x \text{ s.t. } \overline{U}_x \subseteq M_1 \subseteq M_2\right\} \\ &\subseteq \left\{x \in M_2 : \exists U_x \text{ nbhd of } x \text{ s.t. } \overline{U}_x \subseteq M_2\right\} \\ &= \theta_X(M_2).\end{aligned}$$

For the continuity, let $x \in f^{-1}\left(\theta_Y(N)\right)$, so that $f(x) \in \theta_Y(N)$. Therefore, there is a neighborhood $V_{f(x)}$ of $f(x)$ such that $\overline{V_{f(x)}} \subseteq N$. But then $f^{-1}\left(V_{f(x)}\right)$ is a neighborhood of x such that

$$f^{-1}\left(V_{f(x)}\right) \subseteq f^{-1}\left(\overline{V_{f(x)}}\right) \subseteq f^{-1}(N),$$

and since $f^{-1}\left(\overline{V_{f(x)}}\right)$ is a closed set in X, $\overline{f^{-1}\left(V_{f(x)}\right)} \subseteq f^{-1}(N)$. So $f^{-1}\left(V_{f(x)}\right)$ is a neighborhood of x with

$$\overline{f^{-1}\left(V_{f(x)}\right)} \subseteq f^{-1}(N),$$

and this means that $x \in \theta_X\left(f^{-1}(N)\right)$. It was proved $f^{-1}\left(\theta_Y(N)\right) \subseteq \theta_X\left(f^{-1}(N)\right)$, and thus $\Theta = (\theta_X)_{X \in \mathbf{Top}}$ is an interior operator on **Top**.

Example 44. It is defined

$$l_X(M) := \{x \in X : C_x \subseteq M\},$$

where C_x is the connected component of x in X.

For the contractibility, if $x \in l_X(M)$, then $x \in C_x \subseteq M$, and $l_X(M)$ consists of elements of M, so that $l_X(M) \subseteq M$.

For the monotonicity,

$$\begin{aligned} l_X(M_1) &= \{x \in X : C_x \subseteq M_1 \subseteq M_2\} \\ &\subseteq \{x \in X : C_x \subseteq M_2\} \\ &= l_X(M_2). \end{aligned}$$

For the continuity, what is going to be proved is that

$$x \notin l_X\left(f^{-1}(N)\right) \Rightarrow x \notin f^{-1}(l_Y(N)).$$

Let x be an element in X such that $x \notin l_X(f^{-1}(N))$. Then $C_x \nsubseteq f^{-1}(N)$, where C_x is the connected component of x in X. In terms of the direct images across f

$$f(C_x) \nsubseteq f\left(f^{-1}(N)\right),$$

because if it is assumed that

$$f(C_x) \subseteq f\left(f^{-1}(N)\right)$$

this implies that

$$C_x \subseteq f^{-1}(f(C_x)) \subseteq f^{-1}\left(f\left(f^{-1}(N)\right)\right) = f^{-1}(N),$$

so that $C_x \subseteq f^{-1}(N)$, that is a contradicction. Therefore

$$f(C_x) \nsubseteq f\left(f^{-1}(N)\right) \subseteq N,$$

and hence $f(C_x) \nsubseteq N$. But $f(C_x)$ is the image of a connected set across a continuous function. Then $f(C_x)$ is a connected set that contains $f(x)$, so that $f(C_x) \subseteq C_{f(x)}$, the connected component of $f(x)$ in Y, and $C_{f(x)} \nsubseteq N$ is concluded. But the last relation means that $f(x) \notin l_Y(N)$, or equivalently, $x \notin f^{-1}(l_Y(N))$. Hence, $L := (l_X)_{X \in \mathbf{Top}}$ is an interior operator on **Top**.

Example 45. It is defined

$$t_X(M) := \{x \in M : x_n \longrightarrow x \text{ implies } x_n \in M \text{ eventually}\},$$

and $x_n \in M$ eventually means that there is a natural number $n_{(x_n)}$ such that for every natural number n with $n \geq n_{(x_n)}$, $x_n \in M$.

By definition, $t_X(M)$ consists of elements of M, so that $t_X(M) \subseteq M$ and the contractibility property is verified.

For the monotonicity,

$$\begin{aligned} t_X(M_1) &= \{x \in M_1 : x_n \longrightarrow x \text{ implies } x_n \in M_1 \subseteq M_2 \text{ eventually}\} \\ &\subseteq \{x \in M_2 : x_n \longrightarrow x \text{ implies } x_n \in M_2 \text{ eventually}\} \\ &= t_X(M_2), \end{aligned}$$

and then $t_X(M_1) \subseteq t_X(M_2)$.

For the continuity, let $x \in f^{-1}(t_Y(N))$. What is want to be proved is that

$$x \in t_X\left(f^{-1}(N)\right),$$

so let $x_n \longrightarrow x$. Since $f(x) \in t_Y(N)$ and f is sequentially continuous (to see a proof of the fact that a continuous function is sequentially continuous, see Proposition 23), $f(x_n) \longrightarrow f(x)$. Therefore, there exists a natural number $n_{(x_n)}$ such that for every natural number $n \geq n_{(x_n)}$, $f(x_n) \in N$. But then there is a natural number $n_{(x_n)}$ such that for every natural number $n \geq n_{(x_n)}$,

$$x_n \in f^{-1}(N).$$

But this means that $x \in t_X\left(f^{-1}(N)\right)$. Hence,

$$f^{-1}\left(t_Y(N)\right) \subseteq t_X\left(f^{-1}(N)\right),$$

and thus $T := \left(t_X\right)_{X \in \mathbf{Top}}$ is an interior operator on **Top**.

Example 46. It is defined

$$q_X(M) := \bigcup \{C \subseteq M : C \text{ is clopen in } X\}.$$

By definition, $q_X(M)$ is a union of subsets of M, and thus is a subset of M. So the contractibility is verified.

For the monotonicity, if C is a subset of X that is a subset of M_1, then it is also a subset of M_2. Therefore

$$\{C \subseteq M_1 : C \text{ is clopen in } X\} \subseteq \{C \subseteq M_2 : C \text{ is clopen in } X\},$$

and then

$$\bigcup \{C \subseteq M_1 : C \text{ is clopen in } X\} \subseteq \bigcup \{C \subseteq M_2 : C \text{ is clopen in } X\},$$

that is,

$$q_X(M_1) \subseteq q_X(M_2).$$

For the continuity,

$$\begin{aligned} f^{-1}\left(q_Y(N)\right) &= f^{-1}\left(\bigcup \{C \subseteq N : C \text{ is clopen in } Y\}\right) \\ &= \bigcup \left\{f^{-1}(C) : C \subseteq N, \text{ and, } C \text{ is clopen in } Y\right\}. \end{aligned}$$

Since $C \subseteq N$, C is a clopen in Y and f is continuous, $f^{-1}(C) \subseteq f^{-1}(N)$ and $f^{-1}(C)$ is clopen in X, and then

$$\begin{aligned} f^{-1}(q_Y(N)) &\subseteq \bigcup \left\{ f^{-1}(C) \subseteq f^{-1}(N) : f^{-1}(C) \text{ is clopen in } X \right\} \\ &\subseteq \bigcup \left\{ C \subseteq f^{-1}(N) : C \text{ is clopen in } X \right\} \\ &= q_X\left(f^{-1}(N)\right). \end{aligned}$$

Thus $f^{-1}(q_Y(N)) \subseteq q_X(f^{-1}(N))$, and $Q := (q_X)_{X \in \mathbf{Top}}$ is an interior operator on **Top**.

Definition 47. Let $I := (i_X)_{X \in \mathbf{Top}}$ be an interior operator on **Top**. Let $X \in \mathbf{Top}$ and $M \subseteq X$. M is *I-open* in X if and only if $i_X(M) = M$.

Remark. Since in the previous definition I is an interior operator, by the contractibility property, $i_X(M) \subseteq M$. Therefore, M is I-open in X if and only if $M \subseteq i_X(M)$.

Proposition 48. *Let $I := (i_X)_{X \in \mathbf{Top}}$ be an interior operator on* **Top**. *Let $X, Y \in$* **Top**, *$N \subseteq Y$ and let $f : X \longrightarrow Y$ be a continuous function. If N is I-open in Y, then $f^{-1}(N)$ is I-open in X.*

Proof. If N is I-open in Y, $N \subseteq i_Y(N)$, and hence

$$f^{-1}(N) \subseteq f^{-1}(i_Y(N)) \subseteq i_X\left(f^{-1}(N)\right),$$

where the last containment follows from the continuity property of I. Thus

$$f^{-1}(N) \subseteq i_X\left(f^{-1}(N)\right),$$

and $f^{-1}(N)$ is I-open in X. □

Proposition 49. *Let $I := (i_X)_{X \in \mathbf{Top}}$ be an interior operator on* **Top**. *Let $X \in$* **Top** *and consider the indexed family $\{M_\alpha\}_{\alpha \in A} \subseteq S(X)$. If M_α is I-open in X for each $\alpha \in A$, then $\bigcup_{\alpha \in A} M_\alpha$ is I-open in X.*

Proof. It is true that

$$\bigcup_{\alpha\in A} M_\alpha \subseteq \bigcup_{\alpha\in A} i_X\left(M_\alpha\right) \subseteq \bigcup_{\alpha\in A} i_X\left(\bigcup_{\beta\in A} M_\beta\right) = i_X\left(\bigcup_{\beta\in A} M_\beta\right),$$

where the first containment holds because for every $\alpha \in A$ M_α is I-open, and the second containment it is true because the monotonicity property of I. Therefore

$$\bigcup_{\alpha\in A} M_\alpha \subseteq i_X\left(\bigcup_{\alpha\in A} M_\alpha\right),$$

and thus $\bigcup_{\alpha\in A} M_\alpha$ is I-open in X. □

Proposition 50. *Let* $\{I_k\}_{k\in K}$ *be an indexed family of interior operators on* **Top**, *with* $I_k := ((i_k)_X)_{X\in\mathbf{Top}}$. *It is defined* $\bigwedge_{k\in K} I_k := \left((i_{\wedge I_k})_X\right)_{X\in\mathbf{Top}}$, *where if* $X \in$ **Top** *and* $M \subseteq X$,

$$(i_{\wedge I_k})_X(M) := \bigcap_{k\in K} (i_k)_X(M).$$

Then $\bigwedge_{k\in K} I_k$ *is an interior operator on* **Top**, *and is the infimum of the indexed family* $\{I_k\}_{k\in K}$.

Proof. Let X be a topological space, and $M \subseteq X$. Then

$$(i_{\wedge I_k})_X(M) \subseteq \bigcap_{k\in K} (i_k)_X(M) \subseteq \bigcap_{k\in K} M \subseteq M,$$

where the second containment it is true by the contractibility property of I_k, for every $k \in K$. Thus

$$(i_{\wedge I_k})_X(M) \subseteq M.$$

Now let $M_1, M_2 \subseteq X$, with $M_1 \subseteq M_2$. Then

$$(i_{\wedge I_k})_X(M_1) \subseteq \bigcap_{k\in K} (i_k)_X(M_1) \subseteq \bigcap_{k\in K} (i_k)_X(M_2) = (i_{\wedge I_k})_X(M_2),$$

where the second containment holds by the monotonicity of I_k, for every $k \in K$. Then

$$(i_{\wedge I_k})_X(M_1) \subseteq (i_{\wedge I_k})_X(M_2) \text{ if } M_1 \subseteq M_2.$$

For the continuity property, let $Y \in \textbf{Top}$, $N \subseteq Y$ and let $f : X \longrightarrow Y$ be a continuous function. Then

$$\begin{aligned} f^{-1}\left(\left(i_{\wedge I_k}\right)_Y(N)\right) &= f^{-1}\left(\bigcap_{k\in K}(i_k)_Y(N)\right) \\ &= \bigcap_{k\in K} f^{-1}\left((i_k)_Y(N)\right) \\ &\subseteq \bigcap_{k\in K}(i_k)_X\left(f^{-1}(N)\right) \\ &= \left(i_{\wedge I_k}\right)_X\left(f^{-1}(N)\right), \end{aligned}$$

where the containment is consequence of the continuity property of I_k, for every $k \in K$. Hence

$$f^{-1}\left(\left(i_{\wedge I_k}\right)_Y(N)\right) \subseteq \left(i_{\wedge I_k}\right)_X\left(f^{-1}(N)\right),$$

and consequently $\bigwedge_{k\in K} I_k := \left(\left(i_{\wedge I_k}\right)_X\right)_{X\in\textbf{Top}}$ is an interior operator on **Top**.

Now what is going to be showed is that $\bigwedge_{k\in K} I_k$ is the infimum of the indexed family $\{I_k\}_{k\in K}$. Let $X \in \textbf{Top}$ and $M \subseteq X$. In the first place, for every $k \in K$

$$\left(i_{\wedge I_k}\right)_X(M) = \bigcap_{k\in K}(i_k)_X(M) \subseteq (i_k)_X(M),$$

therefore, for every $k \in K$

$$\left(i_{\wedge I_k}\right)_X(M) \subseteq (i_k)_X(M),$$

so that for every $k \in K$,

$$\bigwedge_{k\in K} I_k \leq I_k,$$

and hence $\bigwedge_{k\in K} I_k$ is a lower bound for the indexed family $\{I_k\}_{k\in K}$.

Assuming that $J := (j_X)_{X\in\textbf{Top}}$ is an interior operator on **Top**, and that is a lower bound for $\{I_k\}_{k\in K}$, for every $k \in K$ it is true that

$$J \leq I_k.$$

If X is a topological space, and $M \subseteq X$, then for every $k \in K$

$$j_X(M) \subseteq (i_k)_X(M),$$

and hence

$$j_X(M) \subseteq \bigcap_{k \in K} (i_k)_X(M).$$

But then

$$j_X(M) \subseteq (i_{\wedge I_k})_X(M),$$

so that $J \leq \bigwedge_{k \in K} I_k$. It means that $\bigwedge_{k \in K} I_k$ is the greatest lower bound, the infimum, of the indexed family $\{I_k\}_{k \in K}$. □

CHAPTER 4
I-SEPARATION

For an interior operator I, the notion of I-separation in the category **Top** is introduced. As it will be see, it is a modification of a characterization of the notion of Hausdorff Topological Space. It is recalled from Chapter 2 that the separator of two functions $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is the set

$$sep\,(f, g) = \{x \in X : f\,(x) \neq g\,(x)\}\,,$$

and that a subset M of a topological space X is open with respect to the interior topological operator $I = (i_X)_{X \in \mathbf{Top}}$ (or I-open) if $i_X\,(M) = M$.

Definition 51. Let I be an interior operator on the category **Top**. $Y \in$ **Top** is I-*separated* if and only if for every $X \in$ **Top** and for every pair of continuous functions $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, $sep\,(f, g)$ is I-open.

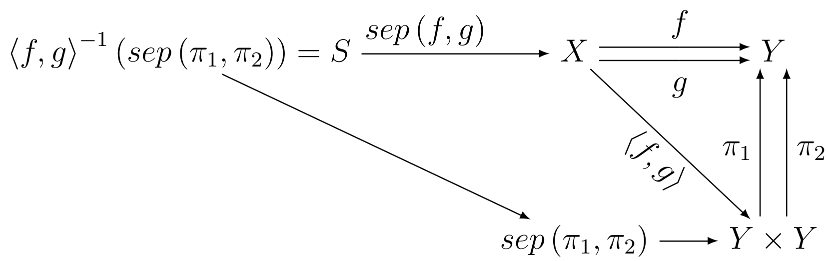
Corollary 52. *Y is I-separated if and only if $\complement \triangle_Y$ is I-open.*

Proof. The "only if" part follows from $\complement \triangle_Y = sep\,(\pi_1, \pi_2)$, where

$$\pi_1, \pi_2 : Y \times Y \longrightarrow Y$$

are the projection functions.

For the "if" part, let $X \in \mathbf{Top}$ and the continuous functions $X \overset{f}{\underset{g}{\rightrightarrows}} Y$. The following diagram is commutative,



where $\langle f,g\rangle$ is the continuous function defined by

$$\begin{cases} \langle f,g\rangle : X \longrightarrow Y \times Y \\ \langle f,g\rangle(x) := (f(x), g(x)). \end{cases}$$

It is enough to prove that $sep(f,g) = \langle f,g\rangle^{-1}\left(\complement\triangle_Y\right)$, since by hypothesis, $\complement\triangle_Y$ is I-open, and the function $\langle f,g\rangle$ is continuous, in which case $\langle f,g\rangle^{-1}\left(\complement\triangle_Y\right)$ is I-open. So let $s \in X$,

$$\begin{aligned} s \in sep(f,g) &\Leftrightarrow f(s) \neq g(s) \\ &\Leftrightarrow (f(s), g(s)) \in sep(\pi_1,\pi_2) = \complement\triangle_Y \\ &\Leftrightarrow \langle f,g\rangle(s) \in \complement\triangle_Y \\ &\Leftrightarrow s \in \langle f,g\rangle^{-1}\left(\complement\triangle_Y\right). \end{aligned}$$

Hence, $sep(f,g)$ is I-open. □

For each one of the following examples, let $X \in \mathbf{Top}$ and $M \subseteq X$. Since the examples are from the category **Top**, it is assumed that if $A \subseteq X$, $\complement A$ denote the complement of A in the set X, as is usual in Set Theory.

Example 53. It is considered the interior operator K of Example 40 defined by

$$k_X(M) := \bigcup \{O \subseteq M : O \text{ is open in } X\}.$$

Let $S(K)$ be the collection of objects of **Top** that are K-separated. Note that $X \in S(K)$ iff

$$\complement\triangle_X \subseteq k_{X^2}\left(\complement\triangle_X\right) = \bigcup\left\{O \subseteq \complement\triangle_X : O \text{ is open in } X \times X\right\}.$$

It is claimed that $S(K) = \mathbf{Top_2}$; the equality of these classes is a consequence of the equivalences

$$\begin{aligned} X \in S(K) &\Leftrightarrow \complement\triangle_X \subseteq k_{X^2}\left(\complement\triangle_X\right) \\ &\Leftrightarrow \complement\triangle_X = k_{X^2}\left(\complement\triangle_X\right) \\ &\Leftrightarrow \complement\triangle_X \text{ is open in } X \times X \\ &\Leftrightarrow \triangle_X \text{ is closed in } X \times X \\ &\Leftrightarrow X \in \mathbf{Top_2}. \end{aligned}$$

Here it had been used the fact that $k_{X^2}\left(\complement\triangle_X\right) \subseteq \complement\triangle_X$ by definition of interior operator.

Example 54. It is considered the interior operator H of Example 41 defined by

$$h_X(M) := \bigcup\{C \subseteq M : C \text{ is closed in } X\} = \left\{x \in M : \overline{\{x\}} \subseteq M\right\}.$$

Let $S(H)$ be the collection of objects of **Top** that are H-separated. $X \in S(H)$ iff

$$\begin{aligned} \complement\triangle_X \subseteq h_{X^2}\left(\complement\triangle_X\right) &= \bigcup\left\{C \subseteq \complement\triangle_X : C \text{ is closed in } X \times X\right\} \\ &= \left\{(x,y) \in \complement\triangle_X : \overline{\{(x,y)\}} \subseteq \complement\triangle_X\right\}. \end{aligned}$$

It is claimed that $S(H) = \mathbf{Top_1}$.

a. $\boldsymbol{Top_1 \subseteq S(H)}$ Let $X \in \mathbf{Top_1}$ and $(x,y) \in \complement\triangle_X$. Then

$$\overline{\{(x,y)\}} = \overline{\{x\}} \times \overline{\{y\}} = \{x\} \times \{y\} = \{(x,y)\} \subseteq \complement\triangle_X,$$

that means that $(x,y) \in h_{X^2}\left(\complement\triangle_X\right)$. Therefore $\complement\triangle_X \subseteq h_{X^2}\left(\complement\triangle_X\right)$, and thus $X \in S(H)$.

b. $\boldsymbol{S(H) \subseteq Top_1}$ This is going to be proved by the contrapositive. Let $X \notin \mathbf{Top_1}$. Let $x, y \in X$ such that $x \neq y$ and $y \in \overline{\{x\}}$. Thence

$$(y,y) \in \overline{\{x\}} \times \overline{\{y\}} = \overline{\{(x,y)\}}.$$

But then $\overline{\{(x,y)\}} \cap \triangle_X \neq \emptyset$, so there exists a pair $(x,y) \in \complement\triangle_X$ such that $\overline{\{(x,y)\}} \nsubseteq \complement\triangle_X$, or that is the same, $(x,y) \notin h_{X^2}\left(\complement\triangle_X\right)$. Therefore $\complement\triangle_X \nsubseteq h_{X^2}\left(\complement\triangle_X\right)$ and thus $X \notin S(H)$.

Example 55. Let B be the interior operator of Example 42 defined by

$$b_X(M) := \left\{x \in X : \exists\, U_x \text{ nbhd of } x \text{ s.t. } U_x \cap \overline{\{x\}} \subseteq M\right\}.$$

Let $S(B)$ be the collection of objects of **Top** that are B-separated. $X \in S(B)$ iff $\complement\triangle_X \subseteq b_{X^2}\left(\complement\triangle_X\right)$, where

$$b_{X^2}\left(\complement\triangle_X\right) = \left\{(x,y) \in X \times X : \exists\, U_{(x,y)} \text{ nbhd of } (x,y) \text{ s.t. } U_{(x,y)} \cap \overline{\{(x,y)\}} \subseteq \complement\triangle_X\right\}.$$

It is claimed that $S(B) = \mathbf{Top_0}$.

a. $\boldsymbol{Top_0 \subseteq S(B)}$ Let $X \in \mathbf{Top_0}$ and $(x,y) \in \complement\triangle_X$. Since $x \neq y$, there is an open neighborhood U_x of x such that $y \notin U_x$, or, there is an open neighborhood U_y of y such that $x \notin U_y$. If $y \notin U_x$, then $\overline{\{y\}} \subseteq \complement U_x$, because U_x is an open set in X. So another way to say that $X \in \mathbf{Top_0}$ is that for x, y

$$\exists\, U_x \text{ nbhd of } x \text{ s.t. } \overline{\{y\}} \cap U_x = \emptyset, \text{ or, } \exists\, U_y \text{ nbhd of } y \text{ s.t. } \overline{\{x\}} \cap U_y = \emptyset.$$

In any case, there are neighborhoods U_x, U_y of x, y, respectively, that satisfy

$$\left(\overline{\{x\}} \cap U_y\right) \cap \left(\overline{\{y\}} \cap U_x\right) = \emptyset,$$

so that

$$\left(U_x \cap \overline{\{x\}}\right) \cap \left(U_y \cap \overline{\{y\}}\right) = \emptyset$$
$$\left(U_x \cap \overline{\{x\}}\right) \times \left(U_y \cap \overline{\{y\}}\right) \subseteq \complement\triangle_X$$
$$(U_x \times U_y) \cap \left(\overline{\{x\}} \times \overline{\{y\}}\right) \subseteq \complement\triangle_X$$
$$(U_x \times U_y) \cap \overline{\{(x,y)\}} \subseteq \complement\triangle_X.$$

Hence there is a neighborhood $U_x \times U_y$ of (x,y) such that

$$(U_x \times U_y) \cap \overline{\{(x,y)\}} \subseteq \complement\triangle_X,$$

so that $(x,y) \in b_{X^2}\left(\complement\triangle_X\right)$. Consequently $\complement\triangle_X \subseteq b_{X^2}\left(\complement\triangle_X\right)$, and then $X \in S(B)$.

b. $\boldsymbol{S(B)} \subseteq \mathbf{Top_0}$ Let $X \in S(B)$ and $x, y \in X$ with $x \neq y$. Thus $(x,y) \in \complement\triangle_X = b_{X^2}\left(\complement\triangle_X\right)$, and hence there exists $U_{(x,y)}$ neighborhood of (x,y) such that

$$U_{(x,y)} \cap \overline{\{(x,y)\}} \subseteq \complement\triangle_X.$$

Let U_x, U_y neighborhoods of x, y, respectively, such that $U_x \times U_y \subseteq U_{(x,y)}$.
Since $\left[(U_x \times U_y) \cap \left(\overline{\{x\}} \times \overline{\{y\}}\right)\right] \subseteq U_{(x,y)} \cap \overline{\{(x,y)\}} \subseteq \complement\triangle_X$, then

$$(U_x \times U_y) \cap \left(\overline{\{x\}} \times \overline{\{y\}}\right) \subseteq \complement\triangle_X$$
$$\left(U_x \cap \overline{\{x\}}\right) \times \left(U_y \cap \overline{\{y\}}\right) \subseteq \complement\triangle_X$$
$$\left(U_x \cap \overline{\{x\}}\right) \cap \left(U_y \cap \overline{\{y\}}\right) = \emptyset,$$

and thus $y \notin U_x \cap \overline{\{x\}}$. There are two posibilities:

Case 1 $y \notin U_x$. Then there is a neighborhood U_x of x such that $y \notin U_x$, and thus $X \in \mathbf{Top_0}$.

Case 2 $y \in U_x$. Then, since $y \notin U_x \cap \overline{\{x\}}$, $y \notin \overline{\{x\}}$. But this implies that there is a neighborhood U_y of y such that $U_y \cap \{x\} = \emptyset$; in other words, there is a neighborhood U_y of y such that $x \notin U_y$. Hence $X \in \mathbf{Top_0}$.

Example 56. It is considered the interior operator Θ of Example 43 defined by

$$\theta_X(M) := \left\{x \in M : \exists\, U_x \text{ nbhd of } x \text{ s.t. } \overline{U}_x \subseteq M\right\}.$$

Let $S(\Theta)$ be the collection of objects of **Top** that are Θ-separated. $X \in S(\Theta)$ iff

$$\complement\triangle_X \subseteq \theta_{X^2}\left(\complement\triangle_X\right) = \left\{(x,y) \in \complement\triangle_X : \exists\, U_{(x,y)} \text{ nbhd of } (x,y) \text{ s.t. } \overline{U}_{(x,y)} \subseteq \complement\triangle_X\right\}.$$

It is claimed that $S(\Theta) = \mathbf{Top_{2\frac{1}{2}}} = \mathbf{Ury}$.

a. $\boldsymbol{Top_{2\frac{1}{2}} \subseteq S(\Theta)}$ Let $X \in \mathbf{Top_{2\frac{1}{2}}}$ and $(x,y) \in \complement\triangle_X$. Since $x \neq y$, there are neighborhoods U_x, U_y of x, y, respectively, such that $\overline{U}_x \cap \overline{U}_y = \emptyset$, or equivalently, $\overline{U}_x \times \overline{U}_y \subseteq \complement\triangle_X$. Therefore $U_x \times U_y$ is a neighborhood of (x,y) that satisfies

$$\overline{U_x \times U_y} = \overline{U}_x \times \overline{U}_y \subseteq \complement\triangle_X,$$

so that $(x,y) \in \theta_{X^2}\left(\complement\triangle_X\right)$ and hence $\complement\triangle_X \subseteq \theta_{X^2}\left(\complement\triangle_X\right)$. Consequently, $X \in S(\Theta)$.

b. $\boldsymbol{S(\Theta) \subseteq TOP_{2\frac{1}{2}}}$ Let $X \in S(\Theta)$ and $x, y \in X$ with $x \neq y$. Since $(x,y) \in \complement\triangle_X = \theta_{X^2}\left(\complement\triangle_X\right)$, there is $U_{(x,y)}$ such that $\overline{U}_{(x,y)} \subseteq \complement\triangle_X$. Let U_x, U_y neighborhoods of x, y, respectively, with $U_x \times U_y \subseteq U_{(x,y)}$. Therefore

$$\overline{U_x} \times \overline{U_y} = \overline{U_x \times U_y} \subseteq \overline{U}_{(x,y)} \subseteq \complement\triangle_X,$$

so that $\overline{U_x} \times \overline{U_y} \subseteq \complement\triangle_X$. Thus U_x, U_y are neighborhoods of x, y such that $\overline{U}_x \cap \overline{U}_y = \emptyset$, and hence $X \in \mathbf{Top_{2\frac{1}{2}}}$.

Example 57. Given $x \in X$, C_x denotes the connected component of x in X. It is recalled the interior operator of Example 44 defined by

$$l_X(M) := \{x \in X : C_x \subseteq M\}.$$

Let $S(L)$ be the collection of objects of **Top** that are L-separated. $X \in S(L)$ iff

$$\complement\triangle_X \subseteq l_{X^2}\left(\complement\triangle_X\right) = \left\{(x,y) \in X \times X : C_{(x,y)} \subseteq \complement\triangle_X\right\},$$

where $C_{(x,y)}$ is the connected component of (x,y) in $X \times X$. It is claimed that $S(L) =$ **Tot**.**Disc**., the collection of all the topological spaces that are totally disconnected, that is, the collection of all the topological spaces whose connected components are singletons.

a. ***Tot.Disc.*** $\subseteq$ $\boldsymbol{S(L)}$ Let $X \in$ **Tot**.**Disc**. and $(x,y) \in \complement\triangle_X$. Then, $X \times X \in$ **Tot**.**Disc**. and thus $C_{(x,y)} = \{(x,y)\} \subseteq \complement\triangle_X$. So that $(x,y) \in l_{X^2}\left(\complement\triangle_X\right)$, and hence $\complement\triangle_X \subseteq l_{X^2}\left(\complement\triangle_X\right)$; therefore $X \in S(L)$.

b. $\boldsymbol{S(L)} \subseteq$ ***Tot.Disc.*** Thi is going to be proved by the contrapositive. Let $X \notin$ **Tot**.**Disc**.; let $x, y \in X$ such that $x \neq y$ and $y \in C_x$. Since the cartesian product of two connected sets is connected,

$$(x,y) \in C_x \times C_x \subseteq C_{(x,y)}.$$

But then $(x,x) \in C_{(x,y)}$, so that $C_{(x,y)} \cap \triangle_X \neq \emptyset$. Therefore it has been found an ordered pair $(x,y) \in \complement\triangle_X$ such that $C_{(x,y)} \nsubseteq \complement\triangle_X$. Hence $(x,y) \notin l_{X^2}\left(\complement\triangle_X\right)$ and thus $\complement\triangle_X \nsubseteq l_{X^2}\left(\complement\triangle_X\right)$, so that $X \notin S(L)$.

Example 58. It is considered the interior operator T of Example 45 defined by

$$t_X(M) := \{x \in M : x_n \longrightarrow x \text{ implies } x_n \in M \text{ eventually}\},$$

where $x_n \in M$ *eventually* means that there is a natural number $n_{(x_n)}$ such that for every natural number n with $n \geq n_{(x_n)}$, $x_n \in M$.

Let $S(T)$ be the collection of objects of **Top** that are T-separated. $X \in S(T)$ iff $\complement\triangle_X \subseteq t_{X^2}\left(\complement\triangle_X\right)$, where

$$t_{X^2}\left(\complement\triangle_X\right) = \left\{(x,y) \in \complement\triangle_X : (x_n,y_n) \longrightarrow (x,y) \text{ implies } (x_n,y_n) \in \complement\triangle_X \text{ eventually}\right\}.$$

It is claimed that $S\left(T\right)=$ **UniqueConv.**, the subclass of objects in **Top** for which every sequence converges at most to one point.

a. ***UniqueConv.*** $\subseteq$ $\boldsymbol{S\left(T\right)}$ This is going to be proved by the contrapositive. Let $X\notin S\left(T\right)$. Then there is an ordered pair $(x,y)\in\complement\triangle_X$ such that $(x,y)\notin t_{X^2}\left(\complement\triangle_X\right)$. By definition this means that there is a sequence $\left((x_n,y_n)\right)_{n\in\mathbb{N}}$ such that $(x_n,y_n)\longrightarrow(x,y)$ but such that for every $k\in\mathbb{N}$ there is $n_k\in\mathbb{N}$ with $n_k\geq k$ that satisfies $(x_{n_k},y_{n_k})\notin\complement\triangle_X$, or what is the same, satisfies $(x_{n_k},y_{n_k})\in\triangle_X$. In few words this property says that the sequence $\left((x_n,y_n)\right)_{n\in\mathbb{N}}$ is *frequently* in $\triangle_X$, and then for $\left((x_n,y_n)\right)_{n\in\mathbb{N}}$ $x_n=y_n$ frequently. Since $(x_n,y_n)\longrightarrow(x,y)$, $x_n\longrightarrow x$ and $y_n\longrightarrow y$. But $x_n=y_n$ frequently, so that there is $\left(x_{n_k}\right)_{k\in\mathbb{N}}$ common subsequence of $\left(x_n\right)_{n\in\mathbb{N}}$ and $\left(y_n\right)_{n\in\mathbb{N}}$ that satisfies

$$x_{n_k}\longrightarrow x\quad\text{and}\quad x_{n_k}\longrightarrow y.$$

It had been showed a sequence $\left(x_{n_k}\right)_{k\in\mathbb{N}}$ in X such that

$$x_{n_k}\longrightarrow x,\quad x_{n_k}\longrightarrow y\quad\text{and}\quad x\neq y.$$

Hence $X\notin$ **UniqueConv**.

b. $\boldsymbol{S\left(T\right)}\subseteq$ ***UniqueConv.*** Again by contrapositive. Let $X\notin$ **UniqueConv**. So there is a sequence $\left(x_n\right)_{n\in\mathbb{N}}$ in X such that $x_n\longrightarrow x$ and $x_n\longrightarrow y$, where $x,y\in X$ and $x\neq y$. Therefore $(x_n,x_n)\longrightarrow(x,y)$. But for every $n\in\mathbb{N}$,

$$(x_n,x_n)\in\triangle_X,$$

or equivalently for every $n\in\mathbb{N}$,

$$(x_n,x_n)\notin\complement\triangle_X.$$

Hence there is $(x,y)\in\complement\triangle_X$ such that for this ordered pair

$$(x_n,x_n)\longrightarrow(x,y)\quad\text{and}\quad(x_n,x_n)\notin\complement\triangle_X,\text{ for every }n\in\mathbb{N}.$$

It has been found an ordered pair $(x,y) \in \complement\triangle_X$ such that $(x,y) \notin t_{X^2}\left(\complement\triangle_X\right)$, so that $X \notin S(T)$.

Example 59. Let Q be the interior operator of Example 46 defined by

$$q_X(M) := \bigcup \{C \subseteq M : C \text{ is clopen in } X\}.$$

Let $S(Q)$ be the collection of all objects of **Top** that are Q-separated. Note that $X \in S(Q)$ iff

$$\complement\triangle_X \subseteq q_{X^2}\left(\complement\triangle_X\right) = \bigcup \left\{C \subseteq \complement\triangle_X : C \text{ is clopen in } X \times X\right\}.$$

It is claimed that

$$\mathbf{Tot.Sep.} \subseteq S(Q) \subseteq \mathbf{Tot.Disc.} \cap \mathbf{Top}_{\mathbf{2\frac{1}{2}}},$$

where **Tot.Sep.** is the collection of all the topological spaces that are totally separated, and **Tot.Disc.** is the collection of all the topological spaces that are totally disconnected (Cf. Chapter 2).

To see that $\mathbf{Tot.Sep.} \subseteq S(Q)$, let $X \in \mathbf{Tot.Sep.}$ and let (x,y) be a point in $\complement\triangle_X$. Then, $x, y \in X$, with $x \neq y$. There is a separation C_x, C_y of X such that $x \in C_x$ and $y \in C_y$. But then

$$(x,y) \in C_x \times C_y \subseteq \complement\triangle_X,$$

where $C_x \times C_y$ is a clopen in $X \times X$, since C_x, C_y are clopen in X. Then there is a clopen $C_x \times C_y$ in $X \times X$ such that

$$(x,y) \in C_x \times C_y \subseteq \complement\triangle_X,$$

so that $(x,y) \in q_{X^2}\left(\complement\triangle_X\right)$, and thus $X \in S(Q)$.

Now that $S(Q) \subseteq \mathbf{Tot.Disc.}$ is going to be showed by contrapositive. Let $X \notin \mathbf{Tot.Disc.}$ Then, there exist $x, y \in X$, $x \neq y$, such that $\{x,y\} \subseteq C_x$, the connected component of x in X. Therefore, the connected set $C_x \times C_x$ satisfies

$$(x, y) \in C_x \times C_x \subseteq C_{(x,y)},$$

where $C_{(x,y)}$ is the connected component of (x, y) in $X \times X$. But this implies that for every C clopen in $X \times X$ such that $(x, y) \in C$,

$$C_x \times C_x \subseteq C_{(x,y)} \subseteq C,$$

or equivalently, for every C clopen in $X \times X$ such that $(x, y) \in C$, $C \nsubseteq \complement\triangle_X$. But this means that $(x, y) \notin q_{X^2}\left(\complement\triangle_X\right)$, hence $\complement\triangle_X \nsubseteq q_{X^2}\left(\complement\triangle_X\right)$, and thus $X \notin S(Q)$. To see that $S(Q) \subseteq \mathbf{Top_{2\frac{1}{2}}}$, let $X \in S(Q)$ and let x, y be points in X with $x \neq y$. Hence

$$(x, y) \in \complement\triangle_X \subseteq q_{X^2}\left(\complement\triangle_X\right).$$

Therefore, there is C clopen in $X \times X$ such that

$$(x, y) \in C \subseteq \complement\triangle_X.$$

Since C is open in $X \times X$, there are U_x, U_y neighborhoods of x, y, respectively, such that

$$(x, y) \in U_x \times U_y \subseteq C \subseteq \complement\triangle_X.$$

Since C is closed in $X \times X$,

$$(x, y) \in \overline{U_x} \times \overline{U_y} = \overline{U_x \times U_y} \subseteq C \subseteq \complement\triangle_X.$$

Hence, there are U_x, U_y neighborhoods of x, y, respectively, such that

$$(x, y) \in \overline{U_x} \times \overline{U_y} \subseteq \complement\triangle_X,$$

or equivalently, there are U_x, U_y neighborhoods of x, y, respectively, such that

$$\overline{U_x} \cap \overline{U_y} = \emptyset.$$

Thus, $X \in \mathbf{Top_{2\frac{1}{2}}}$.

A function $f : X \longrightarrow Y$ is injective, or a *monomorphism*, if and only if for $x, y \in X$, $f(x) = f(y)$ implies $x = y$.

Equivalently, f is a monomorphism if for any diagram of the form

$$Z \overset{h}{\underset{k}{\rightrightarrows}} X \xrightarrow{f} Y,$$

$f \circ h = f \circ k$ implies $h = k$.

The next step is to introduce a notion related to the one of monomorphism.

Definition 60. Let $\{Y_\lambda\}_{\lambda \in \Lambda}$ be an indexed family of sets and let X be a set. The indexed family of functions $\left(X \xrightarrow{f_\lambda} Y_\lambda\right)_{\lambda \in \Lambda}$ is called a *source*. $(f_\lambda)_{\lambda \in \Lambda}$ is a *monosource* if and only if given $x, y \in X$ the equalities $f_\lambda(x) = f_\lambda(y)$ for every $\lambda \in \Lambda$ imply $x = y$.

Equivalently, $(f_\lambda)_{\lambda \in \Lambda}$ is a monosource if for any diagram of the form

$$Z \overset{h}{\underset{k}{\rightrightarrows}} X \xrightarrow{f_\lambda} Y_\lambda,$$

it is true that $f_\lambda \circ h = f_\lambda \circ k$ for every $\lambda \in \Lambda$ implies $h = k$.

It is easy to see that every monomorphism is a monosource. The importance of the concept of monosource in this context is that topological products and subspaces can be seen as special case of monosources. Given X a set, $M \subseteq X$ and an indexed family of sets $\{Y_\lambda\}_{\lambda \in \Lambda}$, the projection functions

$$\prod_{\lambda \in \Lambda} Y_\lambda \xrightarrow{\pi_\lambda} Y_\lambda$$

and the inclusion

$$M \xrightarrow{\iota} X$$

are monosources.

The following proposition relates monosources and the objects that are separated with respect to an interior operator in **Top**. It is recalled from Chapter 2 that the objects of **Top** are the topological spaces and the morphisms, the continuous functions between topological spaces. Given an interior operator I on **Top**, $Sep\,(I)$ denotes the subcategory of the objects that are separated with respect to I.

Proposition 61. *$Sep\,(I)$ is closed under monosources.*

Proof. Let $\left(X \xrightarrow{f_\lambda} Y_\lambda\right)_{\lambda\in\Lambda}$ be a monosource, with $Y_\lambda \in Sep\,(I)$, for every $\lambda \in \Lambda$. What is going to be proved is that $X \in Sep\,(I)$. Let $Z \in$ **Top** and let f, g be continuous functions as in the following diagram

$$sep\,(f,g) \longrightarrow Z \underset{g}{\overset{f}{\rightrightarrows}} X \xrightarrow{f_\lambda} Y_\lambda.$$

It is claimed that $sep\,(f,g) = \bigcup_{\lambda\in\Lambda} sep\,(f_\lambda \circ f, f_\lambda \circ g)$. If $x \in sep\,(f,g)$, $f\,(x) \neq g\,(x)$. Since $(f_\lambda)_{\lambda\in\Lambda}$ is a monosource, there is an index $\lambda_0 \in \Lambda$ such that $f_{\lambda_0}\,(f\,(x)) \neq f_{\lambda_0}\,(g\,(x))$, so that

$$x \in sep\,(f_{\lambda_0} \circ f, f_{\lambda_0} \circ g) \subseteq \bigcup_{\lambda\in\Lambda} sep\,(f_\lambda \circ f, f_\lambda \circ g)\,.$$

Now let $x \in \bigcup_{\lambda\in\Lambda} sep\,(f_\lambda \circ f, f_\lambda \circ g)$. There is an index $\lambda_0 \in \Lambda$ such that $x \in sep\,(f_{\lambda_0} \circ f, f_{\lambda_0} \circ g)$, and therefore $(f_{\lambda_0} \circ f)\,(x) \neq (f_{\lambda_0} \circ g)\,(x)$. But then $f_{\lambda_0}\,(f\,(x)) \neq f_{\lambda_0}\,(g\,(x))$, and since f_{λ_0} is a function, $f\,(x) \neq g\,(x)$, so that $x \in sep\,(f,g)$. Hence, $sep\,(f,g)$ is I-open as a union of I-open subsets. Then $X \in Sep\,(I)$. □

The following corollary is fundamental in the theory of I-separation.

Corollary 62. *The product of I-separated spaces is I-separated. A subspace of an I-separated space is I-separated.*

Proof. The inclusion function and the projection functions are monosources. Then the previous proposition is applied. □

From now on $IN(\mathbf{Top})$ denotes the class of all interior operators on **Top** ordered as in Definition 39, and $S(\mathbf{Top})$ denotes the collection of all subclasses of **Top**, ordered by inclusion. A function is defined by

$$\begin{cases} S : IN(\mathbf{Top}) \longrightarrow S(\mathbf{Top}) \\ \qquad\qquad S(I) := \{X \in \mathbf{Top} : X \text{ is } I\text{-separated}\}. \end{cases}$$

Proposition 63. *S preserves infima.*

Proof. Let $\{I_k\}_{k\in K}$ be a family of interior operators in **Top**. The statement of the proposition is that

$$S\left(\bigwedge_{k\in K} I_k\right) = \bigcap_{k\in K} S(I_k).$$

By definition of infimum, for every $k \in K$, $\bigwedge_{k\in K} I_k \leq I_k$. If it is showed that S is order preserving half of the work is done because in that case, for each $k \in K$

$$S\left(\bigwedge_{k\in K} I_k\right) \subseteq S(I_k),$$

and therefore

$$S\left(\bigwedge_{k\in K} I_k\right) \subseteq \bigcap_{k\in K} S(I_k).$$

So it is proceeded to prove that S is order preserving. Let $I, J \in IN(\mathbf{Top})$, with

$$I := (i_X)_{X\in\mathbf{Top}}, \quad J := (j_X)_{X\in\mathbf{Top}}.$$

If $X \in S(I)$ and $I \leq J$,

$$\complement\triangle_X \subseteq i_{X^2}(\complement\triangle_X) \subseteq j_{X^2}(\complement\triangle_X),$$

and hence $\complement\triangle_X \subseteq j_{X^2}(\complement\triangle_X)$, so that $X \in S(J)$. Thus $S(I) \subseteq S(J)$ if $I \leq J$, and S is order preserving.

Now let $X \in \bigcap_{k\in K} S(I_k)$. Then for every $k \in K$, $X \in S(I_k)$. If $I_k := ((i_k)_X)_{X\in\mathbf{Top}}$, then the above can be written as $\complement\triangle_X \subseteq (i_k)_{X^2}(\complement\triangle_X)$, for each

$k \in K$, so that

$$\complement\triangle_X \subseteq \bigcap_{k\in K} (i_k)_{X^2} \left(\complement\triangle_X\right).$$

It had been defined

$$\bigwedge_{k\in K} I_k := \left((i_{\wedge I_k})_X\right)_{X\in\mathbf{Top}},$$

where

$$(i_{\wedge I_k})_X (M) := \bigcap_{k\in K} (i_k)_X (M),$$

for $M \subseteq X$ (this was proved in [2], Prop. 3.6). Consequently,

$$(i_{\wedge I_k})_{X^2} \left(\complement\triangle_X\right) = \bigcap_{k\in K} (i_k)_{X^2} \left(\complement\triangle_X\right),$$

and therefore

$$\complement\triangle_X \subseteq (i_{\wedge I_k})_{X^2} \left(\complement\triangle_X\right),$$

so that $X \in S\left(\bigwedge_{k\in K} I_k\right)$, and thus

$$\bigcap_{k\in K} S(I_k) \subseteq S\left(\bigwedge_{k\in K} I_k\right).$$

□

The concept of Galois connection is recalled from [1].

Definition 64. For pre-ordered classes $\mathcal{X} = (\mathbf{X}, \leq)$ and $\mathcal{Y} = (\mathbf{Y}, \leq)$ a *Galois connection* $\mathcal{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{Y}$ consists of order preserving functions f and g that satisfy $x \leq g(f(x))$ for every $x \in \mathbf{X}$ and $f(g(y)) \leq y$ for every $y \in \mathbf{Y}$.

Remark. If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are such that $f(x) = y$ and $g(y) = x$, then x and y are said to be corresponding fixed points of the Galois connection $(\mathcal{X}, f, g, \mathcal{Y})$.

The following result that will be very important in the context of the theory is recalled from [1].

Proposition 65. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two pre-ordered classes and assume that infima exist in $\mathcal{Y}$. Let $\mathcal{Y} \xrightarrow{g} \mathcal{X}$ be a function that preserves infima. Define $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ as* *follows: for every* $x \in \mathcal{X}$,

$$f(x) = \bigwedge \{y \in \mathcal{Y} : g(y) \geq x\}.$$

Then, $\mathcal{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{Y}$ *is a Galois connection.*

The following is recalled

1. $IN(\mathbf{Top})$ is the class of all interior operators in **Top**. Let $I, J \in IN(\mathbf{Top})$, with

$$I := (i_X)_{X \in \mathbf{Top}}, \quad J := (j_X)_{X \in \mathbf{Top}}.$$

With the relation $I \leq J$ if and only if for every $X \in \mathbf{Top}$ and for every $M \subseteq X$, $i_X(M) \subseteq j_X(M)$, $IN(\mathbf{Top})$ is a pre-ordered class, in which infima exist:

Let $\{I_k\}_{k \in K}$ an indexed family of interior operators in **Top**. Then the infimum of this family is $\bigwedge_{k \in K} I_k$, the interior operator that is known.

2. $S(\mathbf{Top})$ denotes the collection of all subclasses of **Top**, ordered by inclusion.
3. From Proposition 63 is known that the function

$$IN(\mathbf{Top}) \xrightarrow{S} S(\mathbf{Top})$$

preserves infima.

Therefore, using Proposition 65, the function defined by

$$\begin{cases} T : S(\mathbf{Top}) \longrightarrow IN(\mathbf{Top}) \\ \qquad T(\mathcal{A}) := \bigwedge \{I \in IN(\mathbf{Top}) : \mathcal{A} \subseteq S(I)\} \end{cases}$$

completes the Galois connection

$$S(\mathbf{Top}) \underset{S}{\overset{T}{\rightleftarrows}} IN(\mathbf{Top}).$$

From now on, unless otherwise stated, the notation

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

will mean that $X, Y \in \mathbf{Top}$ and that f, g are continuous functions from X to Y. It is remembered that the separator of f and g is the set

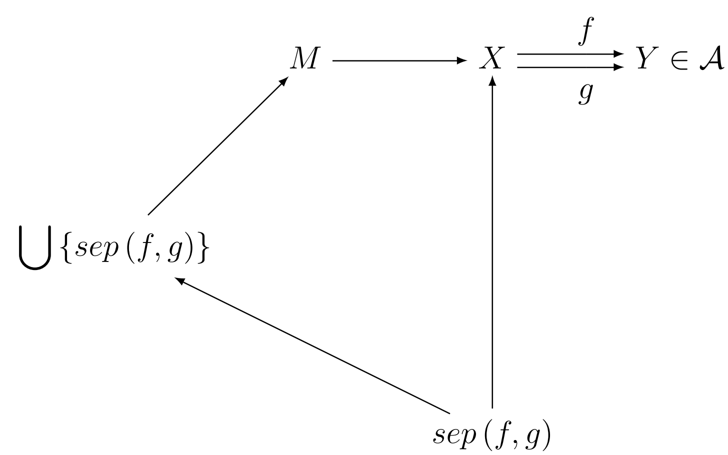
$$sep\,(f,g) := \{x \in X : f\,(x) \neq g\,(x)\}\,.$$

Definition 66 (Irregular interior operator)**.** Let $\mathcal{A} \in S(\mathbf{Top})$, $X \in \mathbf{Top}$ and $M \subseteq X$. It is defined

$$I_{\mathcal{A}} := \left((i_{\mathcal{A}})_X\right)_{X \in \mathbf{Top}}\,,$$

where

$$(i_{\mathcal{A}})_X\,(M) := \bigcup \left\{ sep\,(f,g) \subseteq M : X \underset{g}{\overset{f}{\rightrightarrows}} Y;\ Y \in \mathcal{A} \right\}.$$

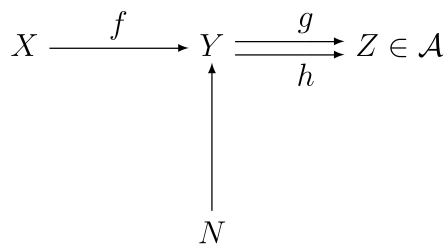


Proposition 67. *$I_{\mathcal{A}}$ is an interior operator on* **Top***.*

Proof. It is clear that $I_{\mathcal{A}}$ satisfies the contractibility property, since if $X \in \mathbf{Top}$ and $M \subseteq X$, by definition $(i_{\mathcal{A}})_X\,(M) \subseteq M$. To show the monotonicity property, let $M_1 \subseteq M_2$. Then

$$\left\{ sep\,(f,g) \subseteq M_1 : X \underset{g}{\overset{f}{\rightrightarrows}} Y;\ Y \in \mathcal{A} \right\} \subseteq \left\{ sep\,(f,g) \subseteq M_2 : X \underset{g}{\overset{f}{\rightrightarrows}} Y;\ Y \in \mathcal{A} \right\},$$

so that $(i_{\mathcal{A}})_X(M_1) \subseteq (i_{\mathcal{A}})_X(M_2)$. For the continuity property, let $X, Y \in \mathbf{Top}$, $N \subseteq Y$ and $f : X \longrightarrow Y$ a continuous function.



Then

$$\begin{aligned} f^{-1}\left((i_{\mathcal{A}})_Y(N)\right) &= f^{-1}\left(\bigcup\left\{sep\,(g,h) \subseteq N : Y \overset{g}{\underset{h}{\rightrightarrows}} Z;\ Z \in \mathcal{A}\right\}\right) \\ &= \bigcup\left\{f^{-1}\left(sep\,(g,h)\right) \subseteq f^{-1}(N) : Y \overset{g}{\underset{h}{\rightrightarrows}} Z;\ Z \in \mathcal{A}\right\}, \end{aligned}$$

but $f^{-1}\left(sep\,(g,h)\right) = sep\,(g \circ f, h \circ f)$, because

$$\begin{aligned} x \in f^{-1}\left(sep\,(g,h)\right) &\Leftrightarrow f(x) \in sep\,(g,h) \\ &\Leftrightarrow g\left(f(x)\right) \neq h\left(f(x)\right) \\ &\Leftrightarrow (g \circ f)(x) \neq (h \circ f)(x) \\ &\Leftrightarrow x \in sep\,(g \circ f, h \circ f), \end{aligned}$$

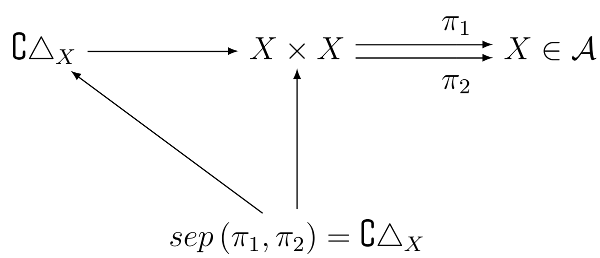
so that

$$\begin{aligned} f^{-1}\left((i_{\mathcal{A}})_Y(N)\right) &= \bigcup\left\{sep\,(g \circ f, h \circ f) \subseteq f^{-1}(N) : Y \overset{g}{\underset{h}{\rightrightarrows}} Z;\ Z \in \mathcal{A}\right\} \\ &\subseteq \bigcup\left\{sep\,(k,l) \subseteq f^{-1}(N) : X \overset{k}{\underset{l}{\rightrightarrows}} Z;\ Z \in \mathcal{A}\right\} \\ &= (i_{\mathcal{A}})_X\left(f^{-1}(N)\right). \end{aligned}$$

Hence, $I_{\mathcal{A}} \in IN(\mathbf{Top})$. □

Lemma 68. *If $X \in \mathcal{A}$, then X is $I_{\mathcal{A}}$-separated.*

Proof.



Since $\complement\triangle_X = sep\,(\pi_1, \pi_2) \subseteq (i_{\mathcal{A}})_{X^2}\left(\complement\triangle_X\right)$, then $\complement\triangle_X$ is $I_{\mathcal{A}}$-open. $\square$

It is remembered that $S(\mathbf{Top}) \underset{S}{\overset{T}{\rightleftarrows}} IN(\mathbf{Top})$ is a Galois connection, where

$$T\,(\mathcal{A}) = \bigwedge \{I \in IN(\mathbf{Top}) : \mathcal{A} \subseteq S\,(I)\}\,.$$

But by the previous lemma, $I_{\mathcal{A}} \in \{I \in IN(\mathbf{Top}) : \mathcal{A} \subseteq S\,(I)\}$, so that $T\,(\mathcal{A}) \leq I_{\mathcal{A}}$. Now, is it the case that $I_{\mathcal{A}} \leq T\,(\mathcal{A})$?

Proposition 69. *Denote*

$$T\,(\mathcal{A}) := \left(t\,(\mathcal{A})_X\right)_{X \in \mathbf{Top}}\,.$$

Then a characterization for $T\,(\mathcal{A})$ *is*

$$t\,(\mathcal{A})_X\,(M) = \bigcup \left\{ sep\,(f,g) \subseteq M : X \underset{g}{\overset{f}{\rightrightarrows}} Y;\ Y \in \mathcal{A} \right\}.$$

Proof. What is need to be proved is that $I_{\mathcal{A}} \leq T\,(\mathcal{A})$. Since $S(\mathbf{Top}) \underset{S}{\overset{T}{\rightleftarrows}} IN(\mathbf{Top})$ is a Galois connection, $\mathcal{A} \subseteq S\,(T\,(\mathcal{A}))$, so that for every $Y \in \mathcal{A}$, Y is $T\,(\mathcal{A})$-separated. Let $Y \in \mathcal{A}$, and let $X \in \mathbf{Top}$, $M \subseteq X$ and $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ with $sep\,(f,g) \subseteq M$. A new function is defined by

$$\begin{cases} \langle f, g\rangle : X \longrightarrow Y \times Y \\ \qquad\qquad \langle f, g\rangle\,(x) := (f\,(x)\,, g\,(x))\,. \end{cases}$$

Then $\langle f, g\rangle$ is a continuous function. Using the continuity of the interior operator $T(\mathcal{A})$,

$$\langle f, g\rangle^{-1}\left(t(\mathcal{A})_{Y^2}\left(\complement\triangle_Y\right)\right) \subseteq t(\mathcal{A})_X\left(\langle f, g\rangle^{-1}\left(\complement\triangle_Y\right)\right),$$

and since Y is $T(\mathcal{A})$-separated, this yields

$$\langle f, g\rangle^{-1}\left(\complement\triangle_Y\right) \subseteq t(\mathcal{A})_X\left(\langle f, g\rangle^{-1}\left(\complement\triangle_Y\right)\right).$$

But

$$\begin{aligned} x \in \langle f, g\rangle^{-1}\left(\complement\triangle_Y\right) &\Leftrightarrow \langle f, g\rangle(x) \in \complement\triangle_Y \\ &\Leftrightarrow (f(x), g(x)) \in \complement\triangle_Y \\ &\Leftrightarrow f(x) \neq g(x) \\ &\Leftrightarrow x \in sep(f, g); \end{aligned}$$

hence $\langle f, g\rangle^{-1}\left(\complement\triangle_Y\right) = sep(f, g)$. Therefore using this, the fact that $sep(f, g) \subseteq M$ and the monotonicity of $T(\mathcal{A})$

$$sep(f, g) \subseteq t(\mathcal{A})_X(sep(f, g)) \subseteq t(\mathcal{A})_X(M).$$

But Y, f and g were arbitrary, so that for every $Y \in \mathcal{A}$ and for every pair of continuous functions $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ such that $sep(f, g) \subseteq M$, $sep(f, g) \subseteq t(\mathcal{A})_X(M)$, so that

$$\bigcup\left\{sep(f, g) \subseteq M : X \underset{g}{\overset{f}{\rightrightarrows}} Y;\ Y \in \mathcal{A}\right\} \subseteq t(\mathcal{A})_X(M),$$

therefore $(i_{\mathcal{A}})_X(M) \subseteq t(\mathcal{A})_X(M)$, and thus $I_{\mathcal{A}} \leq T(\mathcal{A})$. □

Example 70. Let $\mathcal{A} = \mathbf{Ind}$, where $\mathbf{Ind}$ is the subcategory with objects all indiscrete topological spaces. The interior operator $T_{\mathbf{Ind}} = (t(\mathbf{Ind})_X)_{X\in\mathbf{Top}}$ is going to be studied, where if $X \in \mathbf{Top}$ and $M \subseteq X$,

$$t(\mathbf{Ind})_X(M) = \bigcup\left\{sep(f, g) \subseteq M : X \underset{g}{\overset{f}{\rightrightarrows}} Y;\ Y \in \mathbf{Ind}\right\}.$$

Let $Y = \{y_1, y_2\}$ be a set with the indiscrete topology, and let f, g be functions defined by

$$\begin{cases} f : X \longrightarrow Y \\ \qquad f(x) := y_1, \end{cases}$$

and by

$$\begin{cases} g : X \longrightarrow Y \\ \qquad g(x) := \begin{cases} y_2, & \text{if } x \in M \\ y_1, & \text{if } x \notin M \end{cases}. \end{cases}$$

Since $Y \in \mathbf{Ind}$, f and g are continuous functions, such that $sep(f, g) = M$. Thus $t(\mathbf{Ind})_X(M) = M$. Consequently for every $M \subseteq X$, M is $T_{\mathbf{Ind}}$-open, and therefore, X is $T_{\mathbf{Ind}}$-discrete.

Example 71. Let $\mathcal{A} = \mathbf{Top_1}$. It is wanted an explicit form of the operator $T_{\mathbf{Top_1}} = (t(\mathbf{Top_1})_X)_{X \in \mathbf{Top}}$, where if $X \in \mathbf{Top}$ and $M \subseteq X$,

$$t(\mathbf{Top_1})_X(M) = \bigcup \left\{ sep(f, g) \subseteq M : X \underset{g}{\overset{f}{\rightrightarrows}} Y;\ Y \in \mathbf{Top_1} \right\}.$$

A partition of $X \in \mathbf{Top}$ is called a *closed partition* if its members are closed sets in X. Consider the collection

$$\mathscr{C}(X) := \{\mathcal{C} : \mathcal{C} \text{ is a closed partition of } X\}.$$

$\mathscr{C}(X) \neq \emptyset$, since $\{X\} \in \mathscr{C}(X)$. Let $\mathcal{C}_1, \mathcal{C}_2 \in \mathscr{C}(X)$. A relation on $\mathscr{C}(X)$ is defined by

$$\mathcal{C}_1 \leq \mathcal{C}_2 \text{ iff } \left(\forall\, C_1 \in \mathcal{C}_1\right)\left(\forall\, C_2 \in \mathcal{C}_2\right), C_2 \subseteq C_1, \text{ or}, C_1 \cap C_2 = \emptyset.$$

It is claimed that $\leq$ is an order relation, and therefore, $(\mathscr{C}(X), \leq)$ is a poset.

To prove this, let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in \mathscr{C}(X)$. The reflexivity is clear, because every pair of different elements in a closed partition are disjoint. For the antisymmetry, it is assumed that $\mathcal{C}_1 \leq \mathcal{C}_2$ and $\mathcal{C}_2 \leq \mathcal{C}_1$. Let $C_1 \in \mathcal{C}_1$. Since C_1 is a nonempty subset of X and $\mathcal{C}_2$ is a partition of X, there is $C_2 \in \mathcal{C}_2$ such that $C_1 \cap C_2 \neq \emptyset$. Then by hypothesis $C_1 \subseteq C_2$ and $C_2 \subseteq C_1$, and thus $C_2 = C_1$. So every element in $\mathcal{C}_1$ is an element in $\mathcal{C}_2$. Hence, $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Using the same argument the second relation, $\mathcal{C}_2 \subseteq \mathcal{C}_1$, it is true, so that $\mathcal{C}_1 = \mathcal{C}_2$. For the transitivity, it is assumed that $\mathcal{C}_1 \leq \mathcal{C}_2$ and $\mathcal{C}_2 \leq \mathcal{C}_3$. Let $C_1 \in \mathcal{C}_1$ and $C_3 \in \mathcal{C}_3$ such that $C_1 \cap C_3 \neq \emptyset$. Since $\mathcal{C}_2$ is a partition, let $C_2 \in \mathcal{C}_2$ such that $C_2 \cap C_3 \neq \emptyset$. Then by hypothesis $C_3 \subseteq C_2$, and this implies $C_2 \cap C_1 \neq \emptyset$, and using the other part of the hypothesis, $C_2 \subseteq C_1$. Hence, $C_3 \subseteq C_1$, and thus $\mathcal{C}_1 \leq \mathcal{C}_3$.

Let $x \in X$ an arbitrary element. For every $\mathcal{C} \in \mathscr{C}(X)$ $C_{\mathcal{C}}(x)$ denotes the element of $\mathcal{C}$ such that $x \in C_{\mathcal{C}}(x)$. Therefore, $x \in \cap_{\mathcal{C} \in \mathscr{C}(X)} C_{\mathcal{C}}(x)$, and since x is arbitrary,

$$X \subseteq \bigcup \left\{ \cap_{\mathcal{C} \in \mathscr{C}(X)} C_{\mathcal{C}} : (\forall \mathcal{C} \in \mathscr{C}(X)), C_{\mathcal{C}} \in \mathcal{C} \right\}.$$

Taking two nonempty different elements of the collection

$$\left\{ \cap_{\mathcal{C} \in \mathscr{C}(X)} C_{\mathcal{C}} : (\forall \mathcal{C} \in \mathscr{C}(X)), C_{\mathcal{C}} \in \mathcal{C} \right\},$$

there are two indexed families $\{C_{\mathcal{C}}\}_{\mathcal{C} \in \mathscr{C}(X)}$, $\{D_{\mathcal{C}}\}_{\mathcal{C} \in \mathscr{C}(X)}$ such that these elements can be wrtten as $\cap_{\mathcal{C} \in \mathscr{C}(X)} C_{\mathcal{C}}$ and $\cap_{\mathcal{C} \in \mathscr{C}(X)} D_{\mathcal{C}}$, and

a. For every $\mathcal{C} \in \mathscr{C}(X)$, $C_{\mathcal{C}}, D_{\mathcal{C}} \in \mathcal{C}$.

b. $\cap_{\mathcal{C} \in \mathscr{C}(X)} C_{\mathcal{C}} \neq \emptyset \neq \cap_{\mathcal{C} \in \mathscr{C}(X)} D_{\mathcal{C}}$.

With the assumption $\cap_{\mathcal{C} \in \mathscr{C}(X)} C_{\mathcal{C}} \nsubseteq \cap_{\mathcal{C} \in \mathscr{C}(X)} D_{\mathcal{C}}$ let $x \in \cap_{\mathcal{C} \in \mathscr{C}(X)} C_{\mathcal{C}}$ with $x \notin \cap_{\mathcal{C} \in \mathscr{C}(X)} D_{\mathcal{C}}$. Then there is $\mathcal{C}_0 \in \mathscr{C}(X)$ such that $x \notin D_{\mathcal{C}_0}$. Obviously, $x \in C_{\mathcal{C}_0}$. Therefore,

$$\left(\cap_{\mathcal{C} \in \mathscr{C}(X)} C_{\mathcal{C}} \right) \cap \left(\cap_{\mathcal{C} \in \mathscr{C}(X)} D_{\mathcal{C}} \right) = \cap_{\mathcal{C} \in \mathscr{C}(X)} \left(C_{\mathcal{C}} \cap D_{\mathcal{C}} \right) \subseteq C_{\mathcal{C}_0} \cap D_{\mathcal{C}_0} = \emptyset,$$

so that different elements of the collection are disjoint. Thus the collection

$$\left\{ \cap_{\mathcal{C} \in \mathscr{C}(X)} C_{\mathcal{C}} \neq \emptyset : (\forall \mathcal{C} \in \mathscr{C}(X)), C_{\mathcal{C}} \in \mathcal{C} \right\}$$

is a partition of X. Even more, is a closed partition, since every collection $\mathcal{C}$ is a closed partition, and the intersections of elements of these collections are in fact arbitrary intersections of closed sets in X. It means that

$$\left\{\cap_{\mathcal{C}\in\mathscr{C}(X)}C_{\mathcal{C}} \neq \emptyset : (\forall \mathcal{C} \in \mathscr{C}(X)), C_{\mathcal{C}} \in \mathcal{C}\right\} \in \mathscr{C}(X),$$

and for every closed partition $\mathcal{D}$,

$$\mathcal{D} \leq \left\{\cap_{\mathcal{C}\in\mathscr{C}(X)}C_{\mathcal{C}} \neq \emptyset : (\forall \mathcal{C} \in \mathscr{C}(X)), C_{\mathcal{C}} \in \mathcal{C}\right\},$$

so that $\left\{\cap_{\mathcal{C}\in\mathscr{C}(X)}C_{\mathcal{C}} \neq \emptyset : (\forall \mathcal{C} \in \mathscr{C}(X)), C_{\mathcal{C}} \in \mathcal{C}\right\}$ is the maximal element of $\mathscr{C}(X)$.

Let $\mathcal{C}$ be the maximal closed partition of X. It is claimed that

$$t(\mathbf{Top_1})_X(M) = \bigcup \{C \subseteq M : C \in \mathcal{C}\}.$$

Two cases are considered.

Case 1 For every $C \in \mathcal{C}$, $C \nsubseteq M$.

In this case,

$$\{C \subseteq M : C \in \mathcal{C}\} = \emptyset.$$

Let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ such that $Y \in \mathbf{Top_1}$ and $sep(f,g) \neq \emptyset$; let $x \in sep(f,g)$. Then $f(x) \neq g(x)$. Since $Y \in \mathbf{Top_1}$, the singletons $\{f(x)\}$ and $\{g(x)\}$ are closed sets. Therefore, the collections

$$\left\{f^{-1}(\{y\}) : y \in f(X)\right\} \quad \text{and} \quad \left\{g^{-1}(\{y\}) : y \in g(X)\right\}$$

are closed partitions of X, that give origin to the closed partition

$$\left\{f^{-1}(\{y_1\}) \cap g^{-1}(\{y_2\}) : (y_1, y_2) \in f(X) \times g(X)\right\}.$$

The set $f^{-1}(\{f(x)\}) \cap g^{-1}(\{g(x)\})$ is a member of the previous partition, so that exists $C \in \mathcal{C}$ such that

$$C \subseteq f^{-1}(\{f(x)\}) \cap g^{-1}(\{g(x)\}) \subseteq sep(f,g).$$

Since $C \nsubseteq M$, $sep(f,g) \nsubseteq M$. Thus

$$\left\{ sep(f,g) \subseteq M : X \overset{f}{\underset{g}{\rightrightarrows}} Y;\ Y \in \mathbf{Top_1} \right\} = \emptyset,$$

and in this case,

$$t(\mathbf{Top_1})_X(M) = \emptyset = \bigcup \{C \subseteq M : C \in \mathcal{C}\}.$$

Case 2 There exists $C \in \mathcal{C}$ such that $C \subseteq M$.

Let $x \in t(\mathbf{Top_1})_X(M)$. Then there exist $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, with $Y \in \mathbf{Top_1}$ and $x \in sep(f,g) \subseteq M$. Hence, $f(x) \neq g(x)$. Following the same reasoning that in case 1, the set $f^{-1}(\{f(x)\}) \cap g^{-1}(\{g(x)\})$ is an element of a closed partition. Hence, there is $C_x \in \mathcal{C}$ such that

$$C_x \subseteq f^{-1}(\{f(x)\}) \cap g^{-1}(\{g(x)\}) \subseteq sep(f,g) \subseteq M,$$

so that there is $C_x \in \mathcal{C}$ such that $x \in C_x \subseteq M$. Thus $x \in \bigcup\{C \subseteq M : C \in \mathcal{C}\}$, and then

$$t(\mathbf{Top_1})_X(M) \subseteq \bigcup \{C \subseteq M : C \in \mathcal{C}\}.$$

To prove the other part, for every $C \in \mathcal{C}$ it is chosen a unique element of C denoted by $x(C)$. Let $C_0, C_1 \in \mathcal{C}$ such that $C_0 \subseteq M$ and $C_1 \neq C_0$. X_{cf} denotes the set X with the *cofinite* topology. The cofinite topology on X is the collection that consists of $\emptyset$ and every subset of X whose complement is a finite set. Two new functions are defined,

$$\begin{cases} f : X \longrightarrow X_{cf} \\ \qquad f(x) := x(C), \quad \text{if } x \in C, \end{cases}$$

and

$$\begin{cases} g : X \longrightarrow X_{cf} \\ \qquad g(x) := \begin{cases} f(x), & \text{if } x \notin C_0 \\ x(C_1), & \text{if } x \in C_0 \end{cases}. \end{cases}$$

Then f and g are continuous functions, such that $sep(f,g) = C_0$, and hence

$$\{C \subseteq M : C \in \mathcal{C}\} \subseteq \left\{ sep(f,g) \subseteq M : X \overset{f}{\underset{g}{\rightrightarrows}} Y;\ Y \in \mathbf{Top_1} \right\},$$

so that

$$\bigcup \{C \subseteq M : C \in \mathcal{C}\} \subseteq t(\mathbf{Top_1})_X(M).$$

In particular, for $X \in \mathbf{Top_1}$ and $M \subseteq X$, $t(\mathbf{Top_1})_X(M) = M$.

Example 72. Let $\mathcal{A} = \mathbf{Top_2}$. In this case, the operator is

$T_{\mathbf{Top_2}} = (t(\mathbf{Top_2})_X)_{X \in \mathbf{Top}}$, where if $X \in \mathbf{Top}$ and $M \subseteq X$,

$$t(\mathbf{Top_2})_X(M) = \bigcup \left\{ sep(f,g) \subseteq M : X \overset{f}{\underset{g}{\rightrightarrows}} Y;\ Y \in \mathbf{Top_2} \right\}.$$

An explicit description of this operator when $X \in \mathbf{Top_2}$ is going to be showed. First it is assumed that M is open in X. Therefore, $\complement M$ is a closed set in X, and by Proposition 21 there are $Y \in \mathbf{Top_2}$ and continuous functions $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ such that $\complement M = equ(f,g)$. But then $M = sep(f,g)$, so that $t(\mathbf{Top_2})_X(M) = M$, and M is I-open. Now let $M \subseteq X$ arbitrary but nonempty. Let f, g be continuous functions $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ such that $sep(f,g) \subseteq M$. By Proposition 8, $sep(f,g)$ is open in X, thus $sep(f,g) \subseteq M^{\circ}$, and then $t(\mathbf{Top_2})_X(M) \subseteq M^{\circ}$. On the other hand, if $U \neq \emptyset$ is an open subset of X with $U \subseteq M$, then $U = sep(f,g) \subseteq M$, $U \subseteq t(\mathbf{Top_2})_X(M)$ and hence $M^{\circ} \subseteq t(\mathbf{Top_2})_X(M)$. It is concluded that if M is an arbitrary nonempty subset of X, $t\left(\mathbf{Top_2}\right)_X(M) = M^{\circ}$, that is, $T_{\mathbf{Top_2}}$ coincides with the classical interior operator K on $\mathbf{Top_2}$.

Now a result that will be useful for the next part is stated.

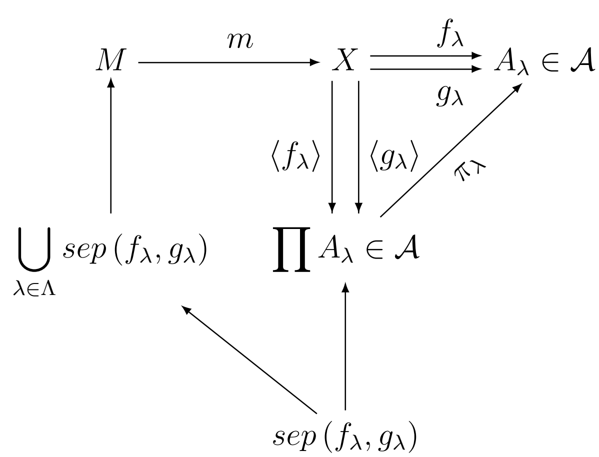
Proposition 73. *Let $\mathcal{A}$ be a subcategory of* **Top** *that is closed under products. Let $X \in$* **Top**, *$M \subseteq X$, and let Λ be a set of indexes. For every $\lambda \in \Lambda$ consider $X \underset{g_\lambda}{\overset{f_\lambda}{\rightrightarrows}} A_\lambda$, where $\{A_\lambda\}_{\lambda\in\Lambda}$ is an indexed family of topological spaces in $\mathcal{A}$, and $\{f_\lambda\}_{\lambda\in\Lambda}$, $\{g_\lambda\}_{\lambda\in\Lambda}$ are indexed families of continuous functions. Then*

$$\bigcup_{\lambda\in\Lambda} sep\left(f_\lambda, g_\lambda\right) = sep\left(\langle f_\lambda\rangle, \langle g_\lambda\rangle\right),$$

where

$$\begin{cases} X \underset{\langle g_\lambda\rangle}{\overset{\langle f_\lambda\rangle}{\rightrightarrows}} \prod_{\lambda\in\Lambda} A_\lambda \\ \langle f_\lambda\rangle(x) := \left(f_\lambda(x)\right)_{\lambda\in\Lambda} \\ \langle g_\lambda\rangle(x) := \left(g_\lambda(x)\right)_{\lambda\in\Lambda}. \end{cases}$$

Proof. The following is a commutative diagram



where π_λ is the λ-projection. Consequently,

$$
\begin{aligned}
x \in \bigcup_{\lambda\in\Lambda} sep\left(f_\lambda, g_\lambda\right) &\Leftrightarrow \left(\exists\,\lambda_0\right),\ x \in sep\left(f_{\lambda_0}, g_{\lambda_0}\right) \\
&\Leftrightarrow \left(\exists\,\lambda_0\right),\ f_{\lambda_0}\left(x\right) \neq g_{\lambda_0}\left(x\right) \\
&\Leftrightarrow \left(f_\lambda\left(x\right)\right)_{\lambda\in\Lambda} \neq \left(g_\lambda\left(x\right)\right)_{\lambda\in\Lambda} \\
&\Leftrightarrow \left\langle f_\lambda\right\rangle\left(x\right) \neq \left\langle g_\lambda\right\rangle\left(x\right) \\
&\Leftrightarrow x \in sep\left(\left\langle f_\lambda\right\rangle, \left\langle g_\lambda\right\rangle\right).
\end{aligned}
$$

□

Some classical notions of Category Theory that will be useful in this context. All of them can be found in [1]. The work in this part uses the assumptions made in [1] for a category $\mathcal{X}$. In particular, it is assumed that every subcategory of $\mathcal{X}$ is full and isomorphism-closed.

Definition 74. A subcategory of $\mathcal{X}$ is *full* if the morphisms in the category are exactly those morphims in $\mathcal{X}$ with both domain and codomain in the subcategory.

Definition 75. A subcategory of $\mathcal{X}$ is *isomorphism-closed* if given that an object belongs to the subcategory, then so does any other object isomorphic to it.

Definition 76. A morphism $X \xrightarrow{f} Y$ is called an *isomorphism* if there exists a morphism $Y \xrightarrow{g} X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$.

Definition 77. A morphism $X \xrightarrow{e} E$ is called an *epimorphism* if whenever $f, g : E \longrightarrow Y$ are morphisms in $\mathcal{X}$ such that $f \circ e = g \circ e$, then $f = g$.

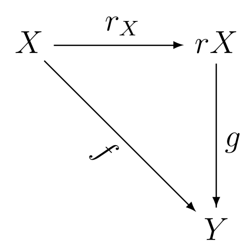
Definition 78.

a. A family of morphisms with common codomain $\left(X_\lambda \xrightarrow{f_\lambda} Y\right)_{\lambda\in\Lambda}$, indexed by a class Λ, is called a *sink*.

b. A sink $\left(X_\lambda \xrightarrow{f_\lambda} Y\right)_{\lambda\in\Lambda}$ is called an *episink* if for every pair of morphisms $h, k : Y \longrightarrow Z$, $h \circ f_\lambda = k \circ f_\lambda$, for every $\lambda \in \Lambda$ implies $h = k$.

Definition 79. An episink $\left(X_\lambda \xrightarrow{e_\lambda} E\right)_{\lambda\in\Lambda}$ is called *extremal* if whenever it factors through a sink $\left(X_\lambda \xrightarrow{f_\lambda} X\right)_{\lambda\in\Lambda}$ and a monomorphism $X \xrightarrow{m} E$, that is $e_\lambda = f_\lambda \circ m$, for each $\lambda \in \Lambda$, then m must be an isomorphism. If $|\Lambda| = 1$, then we speak of an *extremal epimorphism.*

Definition 80. A subcategory $\mathcal{A}$ of $\mathcal{X}$ is called a *reflective subcategory* of $\mathcal{X}$ if for every $X \in \mathcal{X}$ there is a morphism $X \xrightarrow{r_X} rX$ with $rX \in \mathcal{A}$ such that, for every morphism $X \xrightarrow{f} Y$ with $Y \in \mathcal{A}$, there is a unique morphism $rX \xrightarrow{g} Y$ that makes the following diagram commute:



The morphism $X \xrightarrow{r_X} rX$ is called the $\mathcal{A}$-*reflection* of X.

Remark. If for every $X \in \mathcal{X}$, the reflection is required to belong to a given class of morphisms $\mathcal{E}$, then it is called an $\mathcal{E}$-reflective subcategory.

In [1] it is made explicit that "under certain assumptions on $\mathcal{X}$ and $\mathcal{E}$, for any full subcategory $\mathcal{A}$ of $\mathcal{X}$, there exists a smallest $\mathcal{E}$-reflective subcategory of $\mathcal{X}$ containing $\mathcal{A}$. This subcategory is the intersection of all $\mathcal{E}$-reflective subcategories of $\mathcal{X}$ containing $\mathcal{A}$ and it is called the $\mathcal{E}$-*reflective hull* of $\mathcal{A}$ in $\mathcal{X}$."

The following proposition is also from [1].

Proposition 81. *Let $\mathcal{X}$ be an (extremal epi, monosource)-category and let $\mathcal{A} \subseteq \mathcal{X}$. Then the following are equivalent:*

(a) $\mathcal{A}$ is extremal epi-reflective in $\mathcal{X}$;

(b) $\mathcal{A}$ is closed under the formation of monosources.

From Category Theory is known that **Top** is an (extremal epi, monosource)-category, and subspaces and products are monosources in **Top**. Take a subcategory

$\mathcal{A}$ of **Top**. From the Galois connection

$$S(\mathbf{Top}) \underset{S}{\overset{T}{\rightleftarrows}} IN(\mathbf{Top}),$$

starting with $\mathcal{A}$, applying T and then S, the subcategory $S\left(T\left(\mathcal{A}\right)\right)$ of $T\left(\mathcal{A}\right)$-separated spaces is obtained. This subcategory is closed under products and subspaces as proved in Corollary 62, and by the Galois connection

$$\mathcal{A} \subseteq S\left(T\left(\mathcal{A}\right)\right).$$

By Propositions 61 and 81 $S\left(T\left(\mathcal{A}\right)\right)$ is extremal epi-reflective in **Top**.

Now a characterization of $Sep\left(T\left(\mathcal{A}\right)\right)$ is given. Since $Sep\left(T\left(\mathcal{A}\right)\right)$ is reflective, without loss of generality it is assumed that $\mathcal{A}$ is reflective.

Proposition 82. *Let $\mathcal{A}$ be a reflective subcategory of $\mathcal{X}$. Take $X, Y \in \mathcal{X}$. Let $X \xrightarrow{r_X} rX$, $Y \xrightarrow{r_Y} rY$ be the $\mathcal{A}$-reflections of X, Y, respectively. Then for every morphism $X \xrightarrow{f} Y$ there is a unique morphism $rX \xrightarrow{r(f)} rY$ that makes the following diagram commute:*

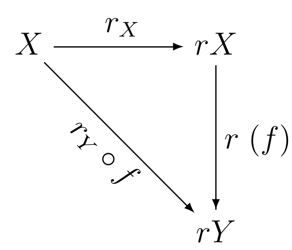
$$\begin{array}{ccc} X & \xrightarrow{r_X} & rX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle r\left(f\right)} \\ Y & \xrightarrow[r_Y]{} & rY \end{array}$$

Furthermore, if f is an isomorphism then $r\left(f\right)$ is an isomorphism.

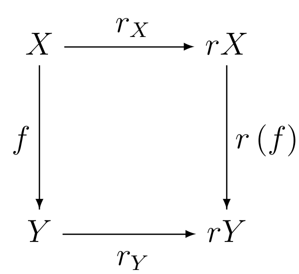
Proof. From the following diagram

$$\begin{array}{ccc} X & \xrightarrow{r_X} & rX \\ {\scriptstyle f}\downarrow & \searrow^{r_Y \circ f} & \\ Y & \xrightarrow[r_Y]{} & rY \end{array}$$

$X \xrightarrow{r_Y \circ f} rY$ is a morphism with $rY \in \mathcal{A}$. Since the morphism $X \xrightarrow{r_X} rX$ is the $\mathcal{A}$-reflection of X, there is a unique morphism $rX \xrightarrow{r(f)} rY$ such that the following diagram commutes:

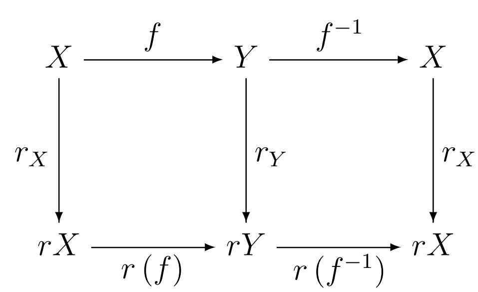


But this means that the diagram

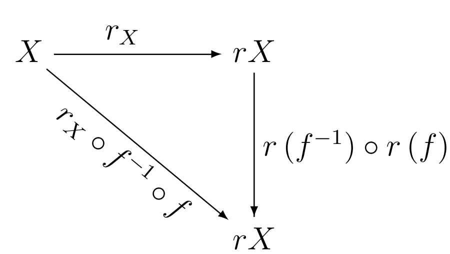


commutes.

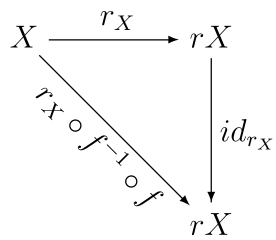
Now it is assumed that f is an isomorphism. From the diagram



follows the commutative diagram



but the following diagram is also commutative



Since $X \xrightarrow{r_X} rX$ is the $\mathcal{A}$-reflection of X, uniqueness of the morphism that completes the diagram yields

$$r\left(f^{-1}\right) \circ r\left(f\right) = id_{rX}.$$

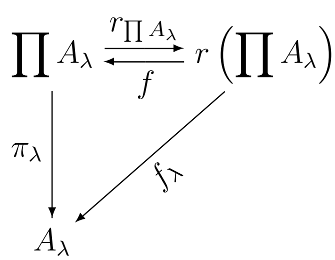
Similarly, it can be showed that

$$r\left(f\right) \circ r\left(f^{-1}\right) = id_{rY}.$$

Therefore, $r\left(f\right)$ is an isomorphism. □

Proposition 83. *Reflective categories are closed under products.*

Proof. Let $\mathcal{A}$ be a reflective subcategory of $\mathcal{X}$, and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an indexed family of objects in $\mathcal{A}$. Then the following is a commutative diagram



where for every $\lambda \in \Lambda$, f_λ is the unique morphism determinated by the reflection

$$\prod A_\lambda \xrightarrow{r_{\prod A_\lambda}} r\left(\prod A_\lambda\right),$$

and f is induced by the family of morphisms

$$\left(r\left(\prod A_\lambda\right) \xrightarrow{f_\lambda} A_\lambda\right)_{\lambda\in\Lambda}.$$

From the diagram, for every $\lambda \in \Lambda$

$$\pi_\lambda \circ f \circ r_{\prod A_\lambda} = f_\lambda \circ r_{\prod A_\lambda} = \pi_\lambda = \pi_\lambda \circ id_{\prod A_\lambda},$$

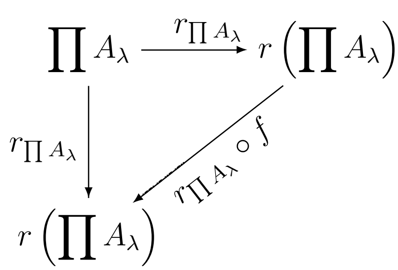
and since $\left(\prod A_\lambda \xrightarrow{\pi_\lambda} A_\lambda\right)_{\lambda \in \Lambda}$ is a monosource,

$$f_\lambda \circ r_{\prod A_\lambda} = id_{\prod A_\lambda}.$$

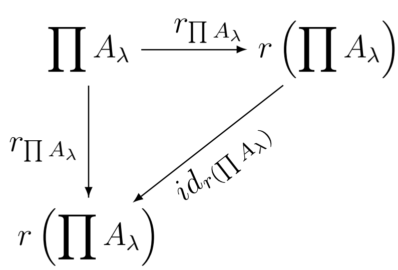
Now,

$$\begin{aligned}\left(r_{\prod A_\lambda} \circ f\right) \circ r_{\prod A_\lambda} &= r_{\prod A_\lambda} \circ \left(f \circ r_{\prod A_\lambda}\right) \\ &= r_{\prod A_\lambda} \circ id_{\prod A_\lambda} \\ &= id_{\prod A_\lambda}.\end{aligned}$$

A consequence is the commutative diagram



but also the following is a commutative diagram



Hence from uniqueness

$$r_{\prod A_\lambda} \circ f = id_{r(\prod A_\lambda)}.$$

Therefore $\prod A_\lambda$ and $r\left(\prod A_\lambda\right)$ are isomorphic. Since $\mathcal{A}$ is isomorphism-closed and $r\left(\prod A_\lambda\right) \in \mathcal{A}$, $\prod A_\lambda \in \mathcal{A}$. $\square$

Definition 84. Let $\mathcal{A}$ be reflective in **Top**. The class $Mono\,(\mathcal{A})$ is defined as

$$Mono\,(\mathcal{A}) := \{X \in \mathbf{Top} : \text{the } \mathcal{A}\text{-reflection } r_X : X \longrightarrow rX \text{ is a monomorphism}\}\,.$$

Remark. For $\mathcal{A}$ reflective subcategory, if $A \in \mathcal{A}$ then $A \xrightarrow{r_A} rA$ is an isomorphism and consequently a monomorphism. This implies that $\mathcal{A} \subseteq Mono\,(\mathcal{A})$.

Proposition 85. *Let $\mathcal{A}$ be reflective in* **Top**. *Then $X \in$* **Top** *belongs to $Mono\,(\mathcal{A})$ if and only if $\complement\triangle_X$ is $I_{\mathcal{A}}$-open (equivalently X is $I_{\mathcal{A}}$-separated).*

Proof. First the "only if" part. Let $X \in Mono\,(\mathcal{A})$, i.e. the $\mathcal{A}$-reflection $X \xrightarrow{r_X} rX$ is a monomorphism. The following diagram is considered

$$X \times X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow{r_X} rX.$$

If $(x, y) \in sep\,(r_X \circ \pi_1, r_X \circ \pi_2)$, then

$$(r_X \circ \pi_1)\,(x, y) \neq (r_X \circ \pi_2)\,(x, y)$$
$$r_X\,(\pi_1\,(x, y)) \neq r_X\,(\pi_2\,(x, y))$$
$$r_X\,(x) \neq r_X\,(y)\,,$$

and since r_X is a function, $x \neq y$ and thus $(x, y) \in \complement\triangle_X$. Now if $(x, y) \in \complement\triangle_X$, $x \neq y$, from the fact that r_X is a monomorphism,

$$r_X\,(x) \neq r_X\,(y)$$
$$r_X\,(\pi_1\,(x, y)) \neq r_X\,(\pi_2\,(x, y))$$
$$(r_X \circ \pi_1)\,(x, y) \neq (r_X \circ \pi_2)\,(x, y)\,,$$

and thus $(x, y) \in sep\,(r_X \circ \pi_1, r_X \circ \pi_2)$. Therefore, from the diagram, $\complement\triangle_X = sep\,(r_X \circ \pi_1, r_X \circ \pi_2)$. Since

$$(i_{\mathcal{A}})_X\,(\complement\triangle_X) = \bigcup \left\{ sep\,(f, g) \subseteq \complement\triangle_X : X \underset{g}{\overset{f}{\rightrightarrows}} Y; Y \in \mathcal{A} \right\},$$

then

$$(i_{\mathcal{A}})_X \left(\complement\triangle_X\right) = sep\,(r_X \circ \pi_1, r_X \circ \pi_2) = \complement\triangle_X,$$

so that $\complement\triangle_X$ is $I_{\mathcal{A}}$-open (equivalently X is $I_{\mathcal{A}}$-separated).

For the "if" part, it is assumed that X is $I_{\mathcal{A}}$-separated, that is, $\complement\triangle_X$ is $I_{\mathcal{A}}$-open. What it is going to be proved is that $X \xrightarrow{r_X} rX$ is a monomorphism (that is injective). Let $x, y \in X$ such that $r_X(x) = r_X(y)$. For convenience $T = \{t\}$ is considered and functions $T \overset{h}{\underset{k}{\rightrightarrows}} X$ such that $h(t) = x$ and $k(t) = y$. The following diagram is considered

$$\begin{array}{ccc} T \times X & \xrightarrow{r_{T\times X}} & r\,(T \times X) \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle r(\pi_X)} \\ X & \xrightarrow[r_X]{} & rX \end{array}$$

By Proposition 82 this is a commutative diagram. Since π_X is an isomorphism, also by Proposition 82 $r(\pi_X)$ is an isomorphism. Then

$$\begin{aligned} r(\pi_X)\left(r_{T\times X}(t, h(t))\right) &= r_X\left(\pi_X(t, h(t))\right) \\ &= r_X(h(t)) \\ &= r_X(x) \\ &= r_X(y) \\ &= r_X(k(t)) \\ &= r_X\left(\pi_X(t, k(t))\right) \\ &= r(\pi_X)\left(r_{T\times X}(t, k(t))\right). \end{aligned}$$

Since $r(\pi_X)$ is an isomorphism then

$$r_{T\times X}(t, h(t)) = r_{T\times X}(t, k(t)).$$

Since reflective subcategories are closed under products (Proposition 83), from Proposition 73, $\complement\triangle_X = sep(f,g)$, where $X \times X \overset{f}{\underset{g}{\rightrightarrows}} A$, with $A \in \mathcal{A}$ and f,g continuous functions. Also $\triangle_X = equ(f,g)$. Now the commutative diagram induced by the $\mathcal{A}$-reflection is

$$\begin{array}{ccc} T \times X & \xrightarrow{h \times id_X} & X \times X \\ \Big\downarrow r_{T\times X} & & f \Big\downarrow \Big\downarrow g \\ r(T \times X) & \overset{f_h}{\underset{g_h}{\rightrightarrows}} & A \end{array}$$

Then

$$\begin{aligned} f(h(t),k(t)) &= f(h \times id_X(t,k(t))) \\ &= f_h(r_{T\times X}(t,k(t))) \\ &= f_h(r_{T\times X}(t,h(t))) \\ &= f(h \times id_X(t,h(t))) \\ &= f(h(t),h(t)) \\ &= g(h(t),h(t)), \end{aligned}$$

where the last equation holds because $\triangle_X = equ(f,g)$. Then

$$\begin{aligned} f(h(t),k(t)) &= g(h(t),h(t)) \\ &= g(h \times id_X(t,h(t))) \\ &= g_h(r_{T\times X}(t,h(t))) \\ &= g_h(r_{T\times X}(t,k(t))) \\ &= g(h \times id_X(t,k(t))) \\ &= g(h(t),k(t)), \end{aligned}$$

so that

$$f(h(t), k(t)) = g(h(t), k(t)).$$

This implies that $(h(t), k(t)) \in equ(f, g) = \triangle_X$. Therefore

$$x = h(t) = k(t) = y,$$

that is r_X is injective. □

It is observed that in the category **Top** of topological spaces for a subcategory $\mathcal{A} \subseteq$ **Top**, $Mono(\mathcal{A})$ is the extremal epireflective hull of $\mathcal{A}$ (cf. [1]). The subcategories $\mathbf{Top_0}$, $\mathbf{Top_1}$, $\mathbf{Top_2}$ and $\mathbf{Top_{2\frac{1}{2}}}$ are all extremal epireflective and as a consequence of Proposition 85, they are fixed points of the Galois connection

$$S(\mathbf{Top}) \underset{S}{\overset{T}{\rightleftarrows}} IN(\mathbf{Top}).$$

Now an alternative definition of I-separation is given. The reason for this new definition is the following. The notion of I-separation given in Definition 51 has proved itself very successful in the category of Topological Spaces but it is not suitable for a generalization to an arbitrary category. For instance in the category **Grp** of Groups and homomorphisms, given two homomorphisms f, g, $sep(f, g)$ fails to be a subgroup. As a consequence a new definition of I-separation is going to be introduced that in the category **Top** is equivalent to the previous definition and that has potential for a generalization to an arbitrary category.

Definition 86. Let $X \in$ **Top** and let $I := (i_X)_{X \in \mathbf{Top}}$ be an interior operator on **Top**. X is I-separated if the set

$$\bigcup \{M \subset X \times X : M \cap \triangle_X = \emptyset\}$$

is I-open.

In the context of the category **Top** it is true that

$$\complement_X \triangle_X = \bigcup \{M \subset X \times X : M \cap \triangle_X = \emptyset\}$$

and consequently from Corollary 52 this definition is equivalent to Definition 51.

Notation. The suggestive notation is going to be used

$$\complement \triangle_X := \bigcup \{M \subset X \times X : M \cap \triangle_X = \emptyset\}$$

for the subset of $X \times X$ of the previous definition. Using this symbol, X is I-separated if $\complement \triangle_X$ is I-open, or equivalently if

$$\complement \triangle_X \subseteq i_{X^2} \left(\complement \triangle_X \right).$$

It is said "suggestive" because in the context of Set Theory, this is not just notation, but an identity, if $\complement \triangle_X$ denotes the complement of $\triangle_X$ in the set $X \times X$; in symbols $\complement_{X^2} \triangle_X$. Now, this is true in the context of the category **Top**, but we want to generalize the notion to others categories.

It has been seen that I-separated objects are closed under the formation of products and subspaces (Cf. Corollary 62). However the proof used there cannot be generalised to an arbitrary category. Therefore some proofs of those results are presented here that it is believed are susceptible of generalization to an arbitrary category.

Theorem 87. *The notion of I-separated is closed under the formation of subspaces. Let $X \in$ **Top**, $Y \subseteq X$ and $I := (i_X)_{X \in \mathbf{Top}}$ an interior operator on **Top**. If X is I-separated, then Y is I-separated.*

Proof. It is remembered that

$$\complement \triangle_X := \bigcup \{M \subset X \times X : M \cap \triangle_X = \emptyset\},$$

and

$$\complement\triangle_Y := \bigcup\{N \subset Y \times Y : N \cap \triangle_Y = \emptyset\}.$$

First consider the collections of sets

$$\mathcal{A} := \{M \cap (Y \times Y) : M \subset X \times X \text{ and } M \cap \triangle_X = \emptyset\},$$

and

$$\mathcal{B} := \{N \subset Y \times Y : N \cap \triangle_Y = \emptyset\}.$$

It is claimed that they are the same.

a. $\mathcal{A} \subseteq \mathcal{B}$ An element of $\mathcal{A}$ is of the form $M \cap (Y \times Y)$, where $M \subseteq X \times X$ and $M \cap \triangle_X = \emptyset$. Then $M \cap (Y \times Y) \subset Y \times Y$, and

$$\begin{aligned}(M \cap (Y \times Y)) \cap \triangle_Y &= M \cap ((Y \times Y) \cap \triangle_Y)\\ &= M \cap \triangle_Y\\ &\subseteq M \cap \triangle_X\\ &= \emptyset,\end{aligned}$$

so that $(M \cap (Y \times Y)) \cap \triangle_Y = \emptyset$ and hence $M \cap (Y \times Y) \in \mathcal{B}$.

b. $\mathcal{B} \subseteq \mathcal{A}$ The useful fact to prove this is

$$\triangle_Y = (Y \times Y) \cap \triangle_X.$$

Consider $N \in \mathcal{B}$. Then $N \subset Y \times Y \subseteq X \times X$. Furthermore,

$$\begin{aligned}N \cap \triangle_X &= (N \cap (Y \times Y)) \cap \triangle_X\\ &= N \cap ((Y \times Y) \cap \triangle_X)\\ &= N \cap \triangle_Y\\ &= \emptyset.\end{aligned}$$

Hence, $N = N \cap (Y \times Y) \in \mathcal{A}$.

Since these two collections are the same,

$$\bigcup\{N \subset Y \times Y : N \cap \triangle_Y = \emptyset\} = \bigcup\{M \cap Y \times Y : M \subset X \times X \text{ and } M \cap \triangle_X = \emptyset\}$$
$$= \left(\bigcup\{M \subset X \times X : M \cap \triangle_X = \emptyset\}\right) \cap (Y \times Y),$$

or by using the equivalent notation,

$$\complement\triangle_Y = \complement\triangle_X \cap (Y \times Y).$$

On the other hand, the inclusion map $\iota : Y \times Y \hookrightarrow X \times X$ is a continuous function, so that using the facts that I is an interior operator on **Top**

$$\iota^{-1}\left(i_{X^2}\left(\complement\triangle_X\right)\right) \subseteq i_{Y^2}\left(\iota^{-1}\left(\complement\triangle_X\right)\right),$$

and that X is I-separated,

$$\iota^{-1}\left(\complement\triangle_X\right) \subseteq i_{Y^2}\left(\iota^{-1}\left(\complement\triangle_X\right)\right).$$

But $\iota^{-1}\left(\complement\triangle_X\right) = \complement\triangle_X \cap (Y \times Y)$, so that

$$\complement\triangle_X \cap (Y \times Y) \subseteq i_{Y^2}\left(\complement\triangle_X \cap (Y \times Y)\right),$$

or equivalently,

$$\complement\triangle_Y \subseteq i_{Y^2}\left(\complement\triangle_Y\right).$$

Hence Y is I-separated. □

Remark. In the previous theorem the identity

$$\complement\triangle_Y = \complement\triangle_X \cap (Y \times Y).$$

was obtained. If it is translated to the language of Set Theory, with the symbol $\complement\triangle_Y$ meaning the complement with respect to the set $Y \times Y$ of the subset $\triangle_Y$, the right identity

$$\complement_{Y^2}\triangle_Y = \complement_{X^2}\triangle_X \cap (Y \times Y).$$

is obtained.

Theorem 88. *The notion of I-separated is closed under finite products. Let $X, Y \in$ **Top** and let $I := (i_X)_{X \in \mathbf{Top}}$ be an interior operator on* **Top**. *If X, Y are I-separated, then $X \times Y$ is I-separated.*

Proof. First let X, Y be sets. A map is defined by

$$\begin{cases} \phi : (X \times Y) \times (X \times Y) \longrightarrow (X \times X) \times (Y \times Y) \\ \quad \phi\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) := \left(\left(x_1, x_2\right), \left(y_1, y_2\right)\right). \end{cases}$$

It is easy verify that ϕ is a bijective function. But since $X, Y \in$ **Top**, ϕ is a homeomorphism. A basis element of $(X \times X) \times (Y \times Y)$ is of the form $(U_1 \times U_2) \times (V_1 \times V_2)$, where U_1, U_2 are open sets in X and V_1, V_2 are open sets in Y. Now

$$\phi^{-1}\left(\left(U_1 \times U_2\right) \times \left(V_1 \times V_2\right)\right) = \left(U_1 \times V_1\right) \times \left(U_2 \times V_2\right)$$

is a basis element in $(X \times Y) \times (X \times Y)$, and thus an open set in $(X \times Y) \times (X \times Y)$. Hence ϕ is continuous. Similarly, using the fact that every basis element in $(X \times Y) \times (X \times Y)$ is of the form

$$\left(U_1 \times V_1\right) \times \left(U_2 \times V_2\right),$$

with U_1, U_2, V_1, V_2 as before, it can be proved that ϕ^{-1} is continuous.

Another important remark that it is easy to verify is

$$\phi\left(\triangle_{X \times Y}\right) = \triangle_X \times \triangle_Y,$$

and this lets to claim that

$$\phi\left(\complement \triangle_{X \times Y}\right) = \left[\complement \triangle_X \times (Y \times Y)\right] \cup \left[(X \times X) \times \complement \triangle_Y\right],$$

where

$$\complement \triangle_{X \times Y} = \bigcup \left\{M \subset (X \times Y) \times (X \times Y) : M \cap \triangle_{X \times Y} = \emptyset\right\}.$$

In fact,

$$
\begin{aligned}
((x_1,x_2),(y_1,y_2)) \in \phi\left(\complement\triangle_{X\times Y}\right) &\Leftrightarrow ((x_1,y_1),(x_2,y_2)) \in \complement\triangle_{X\times Y} \\
&\Leftrightarrow ((x_1,y_1),(x_2,y_2)) \notin \triangle_{X\times Y} \\
&\Leftrightarrow ((x_1,x_2),(y_1,y_2)) \notin \phi\left(\triangle_{X\times Y}\right) \\
&\Leftrightarrow ((x_1,x_2),(y_1,y_2)) \notin \triangle_X \times \triangle_Y \\
&\Leftrightarrow (x_1,x_2) \notin \triangle_X, \text{ or, } (y_1,y_2) \notin \triangle_Y \\
&\Leftrightarrow (x_1,x_2) \in \complement\triangle_X, \text{ or, } (y_1,y_2) \in \complement\triangle_Y \\
&\Leftrightarrow ((x_1,x_2),(y_1,y_2)) \in \complement\triangle_X \times (Y\times Y), \text{ or,} \\
&\quad\ ((x_1,x_2),(y_1,y_2)) \in (X\times X) \times \complement\triangle_Y \\
&\Leftrightarrow ((x_1,x_2),(y_1,y_2)) \in \\
&\quad\ \left[\complement\triangle_X \times (Y\times Y)\right] \cup \left[(X\times X)\times \complement\triangle_Y\right].
\end{aligned}
$$

Since the projections

$$\pi_1 : (X\times X)\times(Y\times Y) \longrightarrow X\times X$$

and

$$\pi_2 : (X\times X)\times(Y\times Y) \longrightarrow Y\times Y$$

are continuous, and ϕ is a homeomorphism,

$$\pi_1\circ\phi : (X\times Y)\times(X\times Y) \longrightarrow X\times X$$

and

$$\pi_2\circ\phi : (X\times Y)\times(X\times Y) \longrightarrow Y\times Y$$

are continuous functions. By the continuity property of I,

$$(\pi_1\circ\phi)^{-1}\left(i_{X^2}\left(\complement\triangle_X\right)\right) \subseteq i_{(X\times Y)^2}\left((\pi_1\circ\phi)^{-1}\left(\complement\triangle_X\right)\right),$$

and given that X is I-separated,

$$\phi^{-1}\left(\pi_1^{-1}\left(\complement\triangle_X\right)\right) \subseteq i_{(X\times Y)^2}\left(\phi^{-1}\left(\pi_1^{-1}\left(\complement\triangle_X\right)\right)\right).$$

But $\pi_1^{-1}\left(\complement\triangle_X\right) = \complement\triangle_X \times (Y\times Y)$, so that

$$\phi^{-1}\left(\complement\triangle_X \times (Y\times Y)\right) \subseteq i_{(X\times Y)^2}\left(\phi^{-1}\left(\complement\triangle_X \times (Y\times Y)\right)\right),$$

and using the monotonicity of I,

$$\phi^{-1}\left(\complement\triangle_X \times (Y\times Y)\right) \subseteq i_{(X\times Y)^2}\left(\phi^{-1}\left(\complement\triangle_X \times (Y\times Y)\right) \cup \phi^{-1}\left((X\times X)\times \complement\triangle_Y\right)\right). \quad (\star)$$

Similarly, from

$$(\pi_2\circ\phi)^{-1}\left(i_{Y^2}\left(\complement\triangle_Y\right)\right) \subseteq i_{(X\times Y)^2}\left((\pi_2\circ\phi)^{-1}\left(\complement\triangle_Y\right)\right),$$

follows that

$$\phi^{-1}\left((X\times X)\times \complement\triangle_Y\right) \subseteq i_{(X\times Y)^2}\left(\phi^{-1}\left(\complement\triangle_X \times (Y\times Y)\right) \cup \phi^{-1}\left((X\times X)\times \complement\triangle_Y\right)\right). \quad (\star\star)$$

Joining $(\star)$ and $(\star\star)$ together,

$$\phi^{-1}\left(\complement\triangle_X \times (Y\times Y)\right) \cup \phi^{-1}\left((X\times X)\times \complement\triangle_Y\right) \subseteq$$
$$i_{(X\times Y)^2}\left(\phi^{-1}\left(\complement\triangle_X \times (Y\times Y)\right) \cup \phi^{-1}\left((X\times X)\times \complement\triangle_Y\right)\right). \quad (\star\star\star)$$

The identity

$$\phi\left(\complement\triangle_{X\times Y}\right) = \left(\complement\triangle_X \times (Y\times Y)\right) \cup \left((X\times X)\times \complement\triangle_Y\right)$$

is equivalent to

$$\complement\triangle_{X\times Y} = \phi^{-1}\left(\complement\triangle_X \times (Y\times Y)\right) \cup \phi^{-1}\left((X\times X)\times \complement\triangle_Y\right),$$

so $(\star\star\star)$ says that

$$\complement\triangle_{X\times Y} \subseteq i_{(X\times Y)^2}\left(\complement\triangle_{X\times Y}\right).$$

Hence, $X \times Y$ is I-separated. □

Remark.

1. $M \subset (X \times X) \times (Y \times Y)$ is assumed. Then it is false that

$$M \cap (\triangle_X \times \triangle_Y) = \emptyset \quad \Rightarrow \quad \pi_1(M) \cap \triangle_X = \emptyset \quad \text{or} \quad \pi_2(M) \cap \triangle_Y = \emptyset,$$

where

$$\pi_1 : (X \times X) \times (Y \times Y) \longrightarrow X \times X$$

and

$$\pi_2 : (X \times X) \times (Y \times Y) \longrightarrow Y \times Y.$$

The following set is considered

$$M := \{((x, x), (y_1, y_2)), ((x_1, x_2), (y, y))\},$$

where $x, x_1, x_2 \in X$ with $x_1 \neq x_2$ and $y, y_1, y_2 \in Y$ with $y_1 \neq y_2$. Therefore $\pi_1(M) = \{(x, x), (x_1, x_2)\}$, so that $\pi_1(M) \cap \triangle_X = \{(x, x)\}$, and $\pi_2(M) = \{(y_1, y_2), (y, y)\}$, so that $\pi_2(M) \cap \triangle_Y = \{(y, y)\}$. Consequently a set M has been found such that $M \cap (\triangle_X \times \triangle_Y) = \emptyset$ but $\pi_1(M) \cap \triangle_X \neq \emptyset$ and $\pi_2(M) \cap \triangle_Y \neq \emptyset$.

2. Let X, Y be sets, $A \subseteq X$ and $B \subseteq Y$. In Set Theory, the identity

$$\complement_{X \times Y} A \times B = \left(\complement_X A \times (Y \times Y)\right) \cup \left((X \times X) \times \complement_Y A\right).$$

is true. In the previous theorem, this identity could have been used to obtain

$$\complement_{X^2 \times Y^2} \triangle_X \times \triangle_Y = \left(\complement_{X^2} \triangle_X \times (Y \times Y)\right) \cup \left((X \times X) \times \complement_{Y^2} \triangle_Y\right).$$

Also, in the same theorem the identity

$$\phi\left(\complement \triangle_{X \times Y}\right) = \left[\complement \triangle_X \times (Y \times Y)\right] \cup \left[(X \times X) \times \complement \triangle_Y\right].$$

was found. Now the question is, how can the set $\phi\left(\complement\triangle_{X\times Y}\right)$ can be interpreted in terms of the set $\complement_{X^2\times Y^2}\triangle_X\times\triangle_Y$? It is true that

$$\begin{aligned}\phi\left(\complement\triangle_{X\times Y}\right)&=\phi\left(\bigcup\{M\subset(X\times Y)\times(X\times Y):M\cap\triangle_{X\times Y}=\emptyset\}\right)\\&=\bigcup\{\phi(M):M\subset(X\times Y)\times(X\times Y)\text{ and }M\cap\triangle_{X\times Y}=\emptyset\}.\end{aligned}$$

But

$$\begin{aligned}&\{\phi(M):M\subset(X\times Y)\times(X\times Y)\text{ and }M\cap\triangle_{X\times Y}=\emptyset\}=\\&\qquad\{N\subseteq(X\times X)\times(Y\times Y):N\cap\triangle_X\times\triangle_Y=\emptyset\},\end{aligned}$$

since ϕ is bijective, $\phi((X\times Y)\times(X\times Y))=(X\times X)\times(Y\times Y)$ and $\phi(\triangle_{X\times Y})=\triangle_X\times\triangle_Y$. Therefore

$$\phi\left(\complement\triangle_{X\times Y}\right)=\bigcup\{N\subseteq(X\times X)\times(Y\times Y):N\cap\triangle_X\times\triangle_Y=\emptyset\},$$

so that if the notation is used.

$$\complement\triangle_X\times\triangle_Y:=\bigcup\{N\subseteq(X\times X)\times(Y\times Y):N\cap\triangle_X\times\triangle_Y=\emptyset\},$$

the "identity" follows

$$\phi\left(\complement\triangle_{X\times Y}\right)=\complement\triangle_X\times\triangle_Y.$$

3. The previous theorem can be ended using a lightly different argument. The relations

$$\phi^{-1}\left(\complement\triangle_X\times(Y\times Y)\right)\subseteq i_{(X\times Y)^2}\left(\phi^{-1}\left(\complement\triangle_X\times(Y\times Y)\right)\right)$$

and

$$\phi^{-1}\left((X\times X)\times\complement\triangle_Y\right)\subseteq i_{(X\times Y)^2}\left(\phi^{-1}\left((X\times X)\times\complement\triangle_Y\right)\right),$$

have been found, from which can be said that $\phi^{-1}\left(\complement\triangle_X\times(Y\times Y)\right)$ and $\phi^{-1}\left((X\times X)\times\complement\triangle_Y\right)$ are I-open sets in $(X\times Y)\times(X\times Y)$. But the union of

I-open sets is an I-open set, so that

$$\complement\triangle_{X\times Y} = \phi^{-1}\left(\complement\triangle_X \times (Y\times Y)\right) \cup \phi^{-1}\left((X\times X)\times \complement\triangle_Y\right)$$

is I-open in $(X\times Y)\times(X\times Y)$, and thus $X\times Y$ is I-separated.

Now the question is, what happens with arbitrary products of I-separated spaces? Although the previous theorem deals with the finite case, the central idea works for the general case.

Theorem 89. *The notion of I-separated is closed under arbitrary products. Let Λ be an index set , $\{X_\lambda\}_{\lambda\in\Lambda}$ an indexed family of spaces in* **Top** *and $I := (i_X)_{X\in\mathbf{Top}}$ an interior operator on* **Top**. *If $\{X_\lambda\}_{\lambda\in\Lambda}$ is an indexed family of I-separated spaces, then $\prod_{\lambda\in\Lambda} X_\lambda$ is I-separated.*

Proof. The following function is defined as follows

$$\begin{cases} \phi : \prod\limits_{\lambda\in\Lambda} X_\lambda \times \prod\limits_{\lambda\in\Lambda} X_\lambda \longrightarrow \prod\limits_{\lambda\in\Lambda}(X_\lambda\times X_\lambda) \\ \quad \phi\left(\left((x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}\right)\right) := \left((x_\lambda, x_\lambda)\right)_{\lambda\in\Lambda}. \end{cases}$$

Then ϕ is a bijective function. Since $\{X_\lambda\}_{\lambda\in\Lambda}$ is an indexed family of topological spaces, ϕ is a homeomorphism. It is enough to take a subbasis element of $\prod_{\lambda\in\Lambda}(X_\lambda\times X_\lambda)$ of the form

$$\pi_\mu^{-1}(U_\mu\times V_\mu),$$

with $\mu\in\Lambda$, and U_μ, V_μ are open sets in X_μ. Then,

$$\begin{aligned} \phi^{-1}\left(\pi_\mu^{-1}(U_\mu\times V_\mu)\right) &= \phi^{-1}\left(\prod_{\lambda\in\Lambda}(W_\lambda\times G_\lambda)\right) \\ &= \prod_{\lambda\in\Lambda} W_\lambda \times \prod_{\lambda\in\Lambda} G_\lambda, \end{aligned}$$

where

$$W_\lambda := \begin{cases} U_\mu, & \text{if } \lambda = \mu \\ X_\lambda, & \text{if } \lambda \neq \mu \end{cases}, \quad G_\lambda := \begin{cases} V_\mu, & \text{if } \lambda = \mu \\ X_\lambda, & \text{if } \lambda \neq \mu \end{cases}.$$

But $\prod_{\lambda\in\Lambda} W_\lambda$, $\prod_{\lambda\in\Lambda} G_\lambda$ are basis elements in $\prod_{\lambda\in\Lambda} X_\lambda$, so that $\phi^{-1}\left(\pi_\mu^{-1}(U_\mu \times V_\mu)\right)$ is an open set in $\prod_{\lambda\in\Lambda} X_\lambda \times \prod_{\lambda\in\Lambda} X_\lambda$ and hence ϕ is continuous. To see that ϕ^{-1} is continuous, the concrete type of subbasis element of $\prod_{\lambda\in\Lambda} X_\lambda \times \prod_{\lambda\in\Lambda} X_\lambda$ of the form $\pi_1^{-1}\left(\pi_\mu^{-1}(U_\mu)\right)$ [or the form $\pi_2^{-1}\left(\pi_\mu^{-1}(U_\mu)\right)$] is considered, where $\mu \in \Lambda$, U_μ is an open set in X_μ, and $\pi_1 : \prod_{\lambda\in\Lambda} X_\lambda \times \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow \prod_{\lambda\in\Lambda} X_\lambda$ is the projection on the first coordinate [$\pi_2 : \prod_{\lambda\in\Lambda} X_\lambda \times \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow \prod_{\lambda\in\Lambda} X_\lambda$ is the projection on the second coordinate, respectively]. Since

$$\pi_1^{-1}\left(\pi_\mu^{-1}(U_\mu)\right) = \prod_{\lambda\in\Lambda} V_\lambda \times \prod_{\lambda\in\Lambda} X_\lambda$$

$$\left[\pi_2^{-1}\left(\pi_\mu^{-1}(U_\mu)\right) = \prod_{\lambda\in\Lambda} X_\lambda \times \prod_{\lambda\in\Lambda} V_\lambda \text{ , respectively}\right]$$

where

$$V_\lambda := \begin{cases} U_\mu, & \text{if } \lambda = \mu \\ X_\lambda, & \text{if } \lambda \neq \mu \end{cases},$$

follows that

$$\begin{aligned} \left(\phi^{-1}\right)^{-1}\left(\pi_1^{-1}\left(\pi_\mu^{-1}(U_\mu)\right)\right) &= \phi\left(\pi_1^{-1}\left(\pi_\mu^{-1}(U_\mu)\right)\right) \\ &= \prod_{\lambda\in\Lambda} (V_\lambda \times X_\lambda), \end{aligned}$$

$$\left[\begin{aligned} \left(\phi^{-1}\right)^{-1}\left(\pi_2^{-1}\left(\pi_\mu^{-1}(U_\mu)\right)\right) &= \phi\left(\pi_2^{-1}\left(\pi_\mu^{-1}(U_\mu)\right)\right) \\ &= \prod_{\lambda\in\Lambda} (X_\lambda \times V_\lambda), \text{ resp.} \end{aligned}\right]$$

with

$$V_\lambda \times X_\lambda = \begin{cases} U_\mu \times X_\mu, & \text{if } \lambda = \mu \\ X_\lambda \times X_\lambda, & \text{if } \lambda \neq \mu \end{cases}. \quad \left[X_\lambda \times V_\lambda = \begin{cases} X_\mu \times U_\mu, & \text{if } \lambda = \mu \\ X_\lambda \times X_\lambda, & \text{if } \lambda \neq \mu \end{cases}, \text{ resp.}\right]$$

Therefore $\left(\phi^{-1}\right)^{-1}\left(\pi_1^{-1}\left(\pi_\mu^{-1}\left(U_\mu\right)\right)\right)$ [$\left(\phi^{-1}\right)^{-1}\left(\pi_2^{-1}\left(\pi_\mu^{-1}\left(U_\mu\right)\right)\right)$, resp.] is an open set in $\prod_{\lambda\in\Lambda} X_\lambda \times X_\lambda$, and thus ϕ^{-1} is continuous.

For the following argument the notation that is going to be used is

$$\begin{aligned}\triangle_\Pi &:= \triangle_{\prod_{\lambda\in\Lambda} X_\lambda}\\ \triangle_\lambda &:= \triangle_{X_\lambda},\end{aligned}$$

the identity

$$\phi\left(\triangle_\Pi\right) = \prod_{\lambda\in\Lambda} \triangle_\lambda,$$

and the set

$$\complement\triangle_\Pi = \bigcup\left\{M \subset \left(\prod_{\lambda\in\Lambda} X_\lambda\right) \times \left(\prod_{\lambda\in\Lambda} X_\lambda\right) : M \cap \triangle_\Pi = \emptyset\right\}.$$

Note that

$$\begin{aligned}\left(\left(x_\lambda, y_\lambda\right)\right)_{\lambda\in\Lambda} \in \phi\left(\complement\triangle_\Pi\right) &\Leftrightarrow \left(\left(x_\lambda\right)_{\lambda\in\Lambda}, \left(y_\lambda\right)_{\lambda\in\Lambda}\right) \in \complement\triangle_\Pi\\ &\Leftrightarrow \left(\left(x_\lambda\right)_{\lambda\in\Lambda}, \left(y_\lambda\right)_{\lambda\in\Lambda}\right) \notin \triangle_\Pi\\ &\Leftrightarrow \left(\left(x_\lambda, y_\lambda\right)\right)_{\lambda\in\Lambda} \notin \phi\left(\triangle_\Pi\right)\\ &\Leftrightarrow \left(\left(x_\lambda, y_\lambda\right)\right)_{\lambda\in\Lambda} \notin \prod_{\lambda\in\Lambda} \triangle_\lambda\\ &\Leftrightarrow \left(\exists\ \mu \in \Lambda\right),\ \left(x_\mu, y_\mu\right) \notin \triangle_\mu\\ &\Leftrightarrow \left(\exists\ \mu \in \Lambda\right),\ \left(x_\mu, y_\mu\right) \in \complement\triangle_\mu\\ &\Leftrightarrow \left(\exists\ \mu \in \Lambda\right),\ \left(\left(x_\lambda, y_\lambda\right)\right)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} C_\lambda\left(\mu\right),\\ &\qquad \text{where } C_\lambda\left(\mu\right) := \begin{cases}\complement\triangle_\mu, & \text{if } \lambda = \mu\\ X_\lambda \times X_\lambda, & \text{if } \lambda \neq \mu\end{cases}\\ &\Leftrightarrow \left(\left(x_\lambda, y_\lambda\right)\right)_{\lambda\in\Lambda} \in \bigcup_{\mu\in\Lambda}\prod_{\lambda\in\Lambda} C_\lambda\left(\mu\right).\end{aligned}$$

Hence,

$$\phi\left(\complement\triangle_{\Pi}\right)=\bigcup_{\mu\in\Lambda}\prod_{\lambda\in\Lambda}C_{\lambda}\left(\mu\right),$$

with

$$C_{\lambda}\left(\mu\right)=\begin{cases}\complement\triangle_{\mu}, & \text{if } \lambda=\mu\\ X_{\lambda}\times X_{\lambda}, & \text{if } \lambda\neq\mu\end{cases}.$$

Since $\phi:\prod_{\lambda\in\Lambda}X_{\lambda}\times\prod_{\lambda\in\Lambda}X_{\lambda}\longrightarrow\prod_{\lambda\in\Lambda}X_{\lambda}\times X_{\lambda}$ is a homeomorphism, and for every $\mu\in\Lambda$ the projection

$$\pi_{\mu}:\prod_{\lambda\in\Lambda}\left(X_{\lambda}\times X_{\lambda}\right)\longrightarrow X_{\mu}\times X_{\mu}$$

is continuous, for each $\mu\in\Lambda$ the composition

$$\pi_{\mu}\circ\phi:\prod_{\lambda\in\Lambda}X_{\lambda}\times\prod_{\lambda\in\Lambda}X_{\lambda}\longrightarrow X_{\mu}\times X_{\mu}$$

is a continuous function. Denote by i_{Π^2} the interior operator working on the topological space $\prod_{\lambda\in\Lambda}X_{\lambda}\times\prod_{\lambda\in\Lambda}X_{\lambda}$; in symbols

$$i_{\Pi^2}:=i_{\left(\prod_{\lambda\in\Lambda}X_{\lambda}\right)^2}.$$

Now, fixing $\mu\in\Lambda$ and using the continuity of the interior operator I,

$$\left(\pi_{\mu}\circ\phi\right)^{-1}\left(i_{\mu}\left(\complement\triangle_{\mu}\right)\right)\subseteq i_{\Pi^2}\left(\left(\pi_{\mu}\circ\phi\right)^{-1}\left(\complement\triangle_{\mu}\right)\right),$$

and since X_{μ} is I-open,

$$\phi^{-1}\left(\pi_{\mu}^{-1}\left(\complement\triangle_{\mu}\right)\right)\subseteq i_{\Pi^2}\left(\phi^{-1}\left(\pi_{\mu}^{-1}\left(\complement\triangle_{\mu}\right)\right)\right).$$

Taking into account the definition of $C_{\lambda}\left(\mu\right)$ it can written that
$\pi_{\mu}^{-1}\left(\complement\triangle_{\mu}\right)=\prod_{\lambda\in\Lambda}C_{\lambda}\left(\mu\right)$, so that

$$\phi^{-1}\left(\prod_{\lambda\in\Lambda}C_{\lambda}\left(\mu\right)\right)\subseteq i_{\Pi^2}\left(\phi^{-1}\left(\prod_{\lambda\in\Lambda}C_{\lambda}\left(\mu\right)\right)\right),$$

and by the monotonicity property of I,

$$\phi^{-1}\left(\prod_{\lambda\in\Lambda} C_\lambda(\mu)\right) \subseteq i_{\Pi^2}\left(\bigcup_{\mu\in\Lambda} \phi^{-1}\left(\prod_{\lambda\in\Lambda} C_\lambda(\mu)\right)\right).$$

But μ was fixed, and the previous relation holds for arbitrary $\mu \in \Lambda$, so that

$$(\forall\ \mu \in \Lambda), \quad \phi^{-1}\left(\prod_{\lambda\in\Lambda} C_\lambda(\mu)\right) \subseteq i_{\Pi^2}\left(\bigcup_{\mu\in\Lambda} \phi^{-1}\left(\prod_{\lambda\in\Lambda} C_\lambda(\mu)\right)\right),$$

and therefore

$$\bigcup_{\mu\in\Lambda} \phi^{-1}\left(\prod_{\lambda\in\Lambda} C_\lambda(\mu)\right) \subseteq i_{\Pi^2}\left(\bigcup_{\mu\in\Lambda} \phi^{-1}\left(\prod_{\lambda\in\Lambda} C_\lambda(\mu)\right)\right),$$

or that it is the same

$$\phi^{-1}\left(\bigcup_{\mu\in\Lambda}\prod_{\lambda\in\Lambda} C_\lambda(\mu)\right) \subseteq i_{\Pi^2}\left(\phi^{-1}\left(\bigcup_{\mu\in\Lambda}\prod_{\lambda\in\Lambda} C_\lambda(\mu)\right)\right). \qquad (\star)$$

But it was obtained that

$$\phi\left(\mathsf{C}\triangle_\Pi\right) = \bigcup_{\mu\in\Lambda}\prod_{\lambda\in\Lambda} C_\lambda(\mu),$$

or equivalently

$$\mathsf{C}\triangle_\Pi = \phi^{-1}\left(\bigcup_{\mu\in\Lambda}\prod_{\lambda\in\Lambda} C_\lambda(\mu)\right),$$

and using this identity in $(\star)$,

$$\mathsf{C}\triangle_\Pi \subseteq i_{\Pi^2}\left(\mathsf{C}\triangle_\Pi\right),$$

so that $\prod_{\lambda\in\Lambda} X_\lambda$ is I-separated. □

It is concluded by observing that the proofs in Theorems 87, 88 and 89 are of categorical nature and consequently, they can be generalized to an arbitrary category.

CHAPTER 5
CONCLUSION AND FUTURE WORK

A notion of separation with respect to an interior operator on **Top** was introduced and it was proved that it is closed with respect to subspaces and products, as the classical interior operator induced by the topology.

For every subcategory $\mathcal{A}$ of **Top**, the existence of an interior operator which makes all the spaces in $\mathcal{A}$ separated is guaranteed by a Galois connection between the collection of all Topological Spaces and the collection of all Interior Operators on **Top**.

There are concrete examples of interior operators with known collections of separated spaces. These examples provide an appropiate motivation for the notion of separation with respect to an interior operator on **Top**.

A definition that is equivalent to the notion of separation with respect to an interior operator on **Top** is given. Its purpose is to generalise the notion of separation to other categories of non-topological nature, since the first definition of separation is not susceptible to do it. There are results that prove that this new definition is closed with respect to subspaces and products.

A first step toward the extension to other categories consists in studying the modified notion of separation with respect to an interior operator and test it in other categories, in particular in Algebra. Another open problem is to obtain an explicit characterization of $Sep\,(Q)$, the interior operator considered in Example 46 and Example 59.

REFERENCE LIST

[1] G. Castellini. *Categorical Closure Operators.* Birkhäuser Boston, first edition, 2003.

[2] G. Castellini; J. Ramos. Interior operators and topological connectedness. *Quaestiones Mathematicae*, 33:290–304, 2010.

[3] L. Steen and J. Seebach Jr. *Counterexamples in Topology.* Holt, Rinehart and Winston, Inc., first edition, 1970.

[4] J. Munkres. *Topología.* Prentice Hall, second edition, 2002.

[5] I. James. *Topological and Uniform Spaces.* Springer, first edition, 1987.

[6] E. Moise. *Geometric Topology in Dimensions 2 and 3.* Springer, first edition, 1977.

THE NOTION OF SEPARATION FOR INTERIOR OPERATORS IN TOPOLOGY

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In the branch of mathematics called Topology there is a notion of separation. Many relevant ideas are related to this notion. This thesis introduces a definition of separation with respect to an interior operator in Topology, and studies a way to generalise it in order to deal with separation in other branchs of mathematics, like Algebra. After giving some concrete examples that show that the modified notion is useful, a mathematical treatment is done to obtain some properties. In the last part of the work, a modification of the notion of separation is done that promises to be more convenient for the initial purpose of generalization.