INTERIOR OPERATORS IN THE CATEGORY OF GROUPS

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A previously introduced notion of categorical interior operator is studied in the category of groups. The main purpose of this research is to try to find out how many of the general results that hold for a categorical interior operator in topology can be proved in the category of groups, paying particular attention to the notions of connectedness and disconnectedness.

Some general properties of interior operators in groups are studied and the notions of discrete, indiscrete, connected and disconnected groups with respect to an interior operator are introduced. The main objective of this work is to discover whether by means of the above notions, a commutative diagram of Galois connections previously presented in the category of topological spaces, can be reconstructed in the group environment. However, unlike the topological case, the lack of commutativity between inverse images and suprema created a big obstacle that, for the time being, could be overcome only by means of two conjectures. Examples are provided.

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Departamento: Ciencias Matemáticas

Una noción introducida previamente de operador de interior de categorías es estudiado en la categoría de grupos. El propósito principal de esta investigación es intentar hallar cuántos de los resultados generales que se tienen para un operador de interior de categorías en topología pueden ser demostrados en la categoría de grupos, prestando especial atención a las nociones de conexidad y desconexidad.

Algunas propiedades generales de los operadores de interior en grupos son estudiadas y las nociones de grupos discretos, indiscretos, conexos y desconexos son introducidas con respecto a un operador de interior. El objetivo principal de este trabajo es descubrir si por medio de las nociones mencionadas, un diagrama conmutativo de conexiones de Galois previamente presentado en la categoría de espacios topológicos, puede reconstruirse en el entorno de los grupos. Sin embargo, a diferencia del caso topológico, la falta de conmutatividad entre las imágenes inversas y supremos creó un gran obstáculo que, por el momento, podría ser superado sólo por medio de dos conjeturas. Se proporcionan ejemplos.

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Contents

1 Introduction

A topological space is a set X together with a topology, that is a family of subsets of X that include X, $\emptyset$ and that is closed under arbitrary unions and finite intersections. These subsets are normally called open (cf. [M], [W]). Every topology yields the concepts of interior and closure. Given a subset M of X, the interior of M is defined as the union of all open subsets of X contained in M. By setting a subset closed if its complement is open one can define the closure of a subset M as the intersection of all closed subsets of X containing M. Finally, the topology of the set X can be reconstructed in an equivalent way either via its associated interior or via its associated closure. Due to this equivalence, in topology one is normally free to choose either of the two concepts according to which one is more convenient to use for a specific problem.

In the early 80's an attempt to introduce a more general version of the above mentioned closure in topology appeared in [DG_1] and it was followed by [DG_2], where a generalization to an arbitrary category was presented. This generalization was called Categorical Closure Operator. The aim of this new notion was to introduce notions of topological nature in categories that do not have a topology, like for instance the category of Groups. This approach was very successful and produced a fairly high number of papers. Most of the theory of Categorical Closure Operators can be found in [C_1], [DT].

At a certain point the following questions arose. Why was the notion of closure used for the above mentioned generalization and not the notion of interior? Since they are equivalent in topology would they also be equivalent in an arbitrary category? Is there any chance that one of the two would outperform the other in an arbitrary category? In order to answer all these questions, the study of a general notion of interior operator was started. A categorical notion of interior operator was introduced in [V]. Subsequently, a few papers have been published in topology ([CR], [CM]) and in an arbitrary category ([C_{2-3}], [HS], [RH]). After these early works on the subject, it became clear that unlike the topological case, the behavior of interior operators was quite different form the one of closure operator in an arbitrary category. Consequently, in order to gain more insight into this aspect, we decided to approach in the category of groups, the same kind of problem that was successfully approached in [CR] in the category of topological spaces. In other words, our main aim is to introduce notions of connectedness and disconnectedness with respect to an interior operator in the category of groups and trying to see how far the theory developed in [CR] can be reconstructed in this different setting.

2 Preliminaries

In this section we will include all the basic definitions and some results of group theory that will be used throughout this work.

Definition 2.1. A *group*, denoted by $(G,\cdot)$, is an algebraic structure that consists of a set and a binary operation $(\cdot)$ which satisfies the following four conditions:

(a) $\forall a,b \in G, a\cdot b \in G$

(b) $\forall a,b,c \in G, a\cdot(b\cdot c) = (a\cdot b)\cdot c$

(c) $\exists e \in G : \forall a \in G, e\cdot a = a = a\cdot e$

(d) $\forall a \in G, \exists b \in G : a\cdot b = e = b\cdot a$

The group $(G,\cdot)$ is usually simply denoted as G and the product of two elements $a\cdot b$ is often denoted by ab, when no confusion is possible. The unit or identity element of a group G will be denoted by e_G whenever necessary to avoid confusion.

Definition 2.2. Let $(G,\cdot)$ be a group. Then the algebraic structure $(H,\cdot)$ is a *subgroup of* G if and only if $(H,\cdot)$ is a group and H is a subset of G. This is denoted as $H \leq G$.

Definition 2.3. Let $(G,\cdot)$ and $(H,*)$ be groups. Let $\phi\colon G \to H$ be a function such that for all $a, b \in G$, $\phi(a\cdot b) = \phi(a)*\phi(b)$. Then $\phi\colon (G,\cdot) \to (H,*)$ is called a *homomorphism* and $\phi^{-1}(\{e_H\}) = \{g \in G | \phi(g) = e_H\}$ is called the *kernel of* ϕ. It is denoted as $Ker\phi$.

Definition 2.4. Let $(G,\cdot)$ and $(H,*)$ be groups, and let $\phi\colon (G,\cdot) \to (H,*)$ be a homomorphism.

(a) If ϕ is injective, that is for all $a, b \in G$, $\phi(a) = \phi(b)$ implies that $a = b$, then ϕ is called a *monomorphism*.

(b) If ϕ is surjective, that is for all $a \in H$, there exists $b \in G$ such that $\phi(b) = a$, then ϕ is called an *epimorphism*.

In what follows, **Grp** is used to denote the class that consists of all groups. **Ab** is used to denote the subclass of **Grp** that consists of all abelian groups, that is groups whose operation is commutative.

Definition 2.5. Let $X \in \mathbf{Grp}$ and let A be a subset of X. Then the *subgroup generated by* A, denoted by $\langle A \rangle$, is the smallest subgroup of X containing every element of A, that is

$\langle A \rangle = \bigcap_{j \in J} \{H_j : A \subseteq H_j \leq X\}$

Proposition 2.6. Let $G \in \mathbf{Grp}$ and $S \subseteq G$. Then $\langle S \rangle = \{s_1^{\varepsilon_1} s_2^{\varepsilon_2} \ldots s_n^{\varepsilon_n} : n \in \mathbb{N}, s_i \in S,$ and $\varepsilon_i = \pm 1$ for all $1 \leq i \leq n\}$, that is $\langle S \rangle$ consists of all finite products of elements of S and their inverses.

Proof: Let $G \in \mathbf{Grp}$ and $S \subseteq G$. From Definition 2.5 we have that $\langle S \rangle = \bigcap_{j \in J} \{H_j : S \subseteq H_j \leq G\}$. Let $\overline{S} = \{s_1^{\varepsilon_1} s_2^{\varepsilon_2} \ldots s_n^{\varepsilon_n} : n \in \mathbb{N}, s_i \in S,$ and $\varepsilon_i = \pm 1$ for all $1 \leq i \leq n\}$. Since e_G can be considered as an empty product or as ss^{-1} for some $s \in S$, then $e_G \in \overline{S}$ and $\overline{S}$ is not empty. Let $x,\ y \in \overline{S}$ and elements $b_i,\ c_j \in S,\ \varepsilon_i,\ \delta_j \in \{\pm 1\}$ such that $x = b_1^{\varepsilon_1} b_2^{\varepsilon_2} \ldots b_k^{\varepsilon_k}$, and $y = c_1^{\delta_1} c_2^{\delta_2} \ldots c_l^{\delta_l}$. Note that $y^{-1} = c_l^{-\delta_l} \ldots c_2^{-\delta_2} c_1^{-\delta_1}$. Thus $xy^{-1} = b_1^{\varepsilon_1} b_2^{\varepsilon_2} \ldots b_k^{\varepsilon_k} c_l^{-\delta_l} \ldots c_2^{-\delta_2} c_1^{-\delta_1}$, which has the form of an element of $\overline{S}$. Thus, by the subgroup criterion (c.f. [DF]), we have that $\overline{S}$ is a subgroup of G. Since $S \subseteq \overline{S} \leq G$, then $\overline{S} \in \{H_j : \forall j \in J, S \subseteq H_j \leq G\}$. Then $\langle S \rangle = \bigcap_{j \in J} \{H_j : S \subseteq H_j \leq G\} \subseteq \overline{S}$. Since every subgroup of G containing S must contain all finite products of elements of S and their inverses, then $\overline{S} \subseteq H_j$ for every $j \in J$ such that $S \subseteq H_j \leq G$. Then $\overline{S} \subseteq \bigcap_{j \in J} \{H_j : S \subseteq H_j \leq G\} = \langle S \rangle$. Hence we conclude that $\langle S \rangle = \{s_1^{\varepsilon_1} s_2^{\varepsilon_2} \ldots s_n^{\varepsilon_n} : n \in \mathbb{N}, s_i \in S,$ and $\varepsilon_i = \pm 1$ for all $1 \leq i \leq n\}$.

For any group X, the set $S(X)$ that consists of all subgroups of X ordered by inclusion is a complete lattice. This implies that any family $\{H_i\}_{i \in I}$ of subgroups of X has a supremum (which is the subgroup generated by the union of the H_i's) denoted by $\bigvee$ and an infimum (which is the intersection of the H_i's) denoted by $\bigwedge$.

Definition 2.7. Let G be a group. Let N be a subgroup of G. N is called a *normal subgroup of* G if $\forall g \in G : gNg^{-1} = N$. By gNg^{-1} we mean the set $\{gng^{-1} : n \in N\}$. It is denoted as $N \trianglelefteq G$.

To clarify some terminology that will be used throughout the thesis, we say that a *trivial group*, usually denoted by $\{e\}$, is a group with only one element, a *simple group* is a group that has only itself and the trivial group as normal subgroups, a *Dedekind group* is a group X such that every subgroup of X is normal, and a *torsion group* is a group such that every element has finite order.

Definition 2.8. Let G be a group, and $H \leq G$. The set $aH = \{y \in G : \exists h \in H : y = ah\}$ is called a *left coset of H*.

If H is a normal subgroup of G, every left coset is also a right coset and in this case is simply called a *coset*. It can then be proved that for a normal subgroup H, the set that consists of every coset of H, denoted as G/H, is a group (cf. [DF]). The group operation is defined as $(aH)(bH) = (ab)H$. G/H is called the *quotient group of G by H*. The associated homomorphism q: $G \to G/H$ defined by $q(x) = xH$ is called the *quotient homomorphism*.

Lemma 2.9. Let $\{H_j\}_{j\in J}$ be an indexed family of subgroups of the group G. Then for a given $g \in G$ we have that $\bigcap\limits_{j\in J} gH_jg^{-1} = g(\bigcap\limits_{j\in J} H_j)g^{-1}$.

Proof : Let $\{H_j\}_{j\in J}$ be an indexed family of subgroups of the group G, where J is an arbitrary index set and let $g \in G$. Let $x \in \bigcap\limits_{j\in J} gH_jg^{-1}$ be arbitrary. Then $x \in gH_jg^{-1}$ for every $j \in J$. Then $x = gh_jg^{-1}$, where $h_j \in H_j$ for every $j \in J$. Then $gh_jg^{-1} = gh_ig^{-1}$, where $h_j \in H_j$ and $h_i \in H_i$ for every j, $i \in J$. By the Cancellation Laws, $h_j = h_i$, where $h_j \in H_j$ and $h_i \in H_i$ for every j, $i \in J$. Let $h = h_j$ for every $j \in J$. Then $x = ghg^{-1}$, where $h \in H_j$ for all $j \in J$. Then $x = ghg^{-1}$, where $h \in \bigcap\limits_{j\in J} H_j$, that is $x \in g(\bigcap\limits_{j\in J} H_j)g^{-1}$. Hence we conclude that $\bigcap\limits_{j\in J} gH_jg^{-1} \subseteq g(\bigcap\limits_{j\in J} H_j)g^{-1}$. Now let $x \in g(\bigcap\limits_{j\in J} H_j)g^{-1}$ be arbitrary. Then $x = ghg^{-1}$, where $h \in \bigcap\limits_{j\in J} H_j$ and so $x = ghg^{-1}$, where $h \in H_j$ for all $j \in J$. Then $x \in gH_jg^{-1}$ for every $j \in J$ and so $x \in \bigcap\limits_{j\in J} gH_jg^{-1}$. Hence we conclude that $\bigcap\limits_{j\in J} gH_jg^{-1} \supseteq g(\bigcap\limits_{j\in J} H_j)g^{-1}$. This proves that $\bigcap\limits_{j\in J} gH_jg^{-1} = g(\bigcap\limits_{j\in J} H_j)g^{-1}$.

Proposition 2.10. Let $G \in \mathbf{Grp}$. For an indexing set I, if $\{N_i\}_{i\in I}$ is a set of normal subgroups of G, then $\bigvee\{N_i\}_{i\in I} = \langle\bigcup\{N_i : i \in I\}\rangle$ is a normal subgroup of G.

Proof : Let $\{N_i\}_{i\in I}$ be a set of normal subgroups of the group G. By Definition 2.5, $\langle\bigcup\{N_i : i \in I\}\rangle \leq G$, and $\langle\bigcup\{N_i : i \in I\}\rangle = \cap\{H_j : j \in J \text{ and } \forall i \in I, N_i \subseteq H_j \leq G\}$. Let g be a fixed element of G. Let $H_j \in \{H_j : j \in J \text{ and } \forall i \in I, N_i \subseteq H_j \leq G\}$. Since all the N_i's are normal in G, then $N_i = gN_ig^{-1} \subseteq gH_jg^{-1}$ for all $i \in I$. Thus $gH_jg^{-1} \in \{H_j : j \in J \text{ and } \forall i \in I, N_i \subseteq H_j \leq G\}$. Hence we have that $\{H_j : j \in J \text{ and } \forall i \in I, N_i \subseteq H_j \leq G\} \supseteq \{gH_jg^{-1} : j \in J \text{ and } \forall i \in I, N_i \subseteq H_j \leq G\}$. Now consider the set $g^{-1}H_jg$. Since all the N_i's are normal in G, then $N_i = g^{-1}N_ig \subseteq g^{-1}H_jg$ for all $i \in I$. Thus $g(g^{-1}H_jg)g^{-1} \in \{gH_jg^{-1} : j \in J$ and $\forall i \in I, N_i \subseteq H_j \leq G\}$, and so $H_j \in \{gH_jg^{-1} : j \in J$ and $\forall i \in I, N_i \subseteq H_j \leq G\}$. Hence we have that $\{H_j : j \in J$ and $\forall i \in I, N_i \subseteq H_j \leq G\} \subseteq \{gH_jg^{-1} : j \in J$ and $\forall i \in I, N_i \subseteq H_j \leq G\}$. Then we obtain that $\{H_j : j \in J$ and $\forall i \in I, N_i \subseteq H_j \leq G\} = \{gH_jg^{-1} : j \in J$ and $\forall i \in I, N_i \subseteq H_j \leq G\}$. This implies that $\cap\{H_j : j \in J$ and $\forall i \in I, N_i \subseteq H_j \leq G\} = \cap\{gH_jg^{-1} : j \in J$ and $\forall i \in I, N_i \subseteq H_j \leq G\} = g(\cap\{H_j : j \in J$ and $\forall i \in I, N_i \subseteq H_j \leq G\})g^{-1}$, as shown in the previous lemma. Hence we conclude that $\langle\{N_i : i \in I\}\rangle = \cap\{H_j : j \in J$ and $\forall i \in I, N_i \subseteq H_j \leq G\}$ is a normal subgroup of G.

Proposition 2.11. Let $f\colon G \to H$ be a group homomorphism and let $P \trianglelefteq H$. Then $f^{-1}(P)$ is a normal subgroup of G.

Proof : Let $f\colon G \to H$ be a group homomorphism and let $P \trianglelefteq H$. Let $F = f^{-1}(P)$. Since P is a subgroup, $e_H \in P$ and thus $e_G \in F$. Then F is not empty. Suppose that $a, b \in F$ and let $x, y \in P$ such that $f(a) = x$ and $f(b) = y$. Since $P \leq H$, we have that $f(ab^{-1}) = f(a)f(b^{-1}) = f(a)f(b)^{-1} = xy^{-1} \in P$. Then $ab^{-1} \in F$. By the subgroup criterion (c.f. [DF]), we conclude that $F \leq G$. Since $P \trianglelefteq H$, by Definition 2.7, $hPh^{-1} \subseteq P$ for all $h \in H$. Let $k \in F$ and $g \in G$ be arbitrary. Then we have $f(gkg^{-1}) = f(g)f(k)f(g)^{-1} \in P$, since f is a homomorphism and $P \trianglelefteq H$. Then $gkg^{-1} \in F$. Since k and g are arbitrary, then $gFg^{-1} \subseteq F$. Hence we conclude that $F = f^{-1}(P) \trianglelefteq G$.

Lemma 2.12. Let $X,Y \in \mathbf{Grp}$ and let $K \trianglelefteq Y$. Let $f\colon X \to Y$ be a homomorphism. Let $\varphi : X/f^{-1}(K) \to Y/K$ be a function defined by $\varphi(xf^{-1}(K)) = f(x)K$. Then φ is a monomorphism.

Proof: Let $X,Y \in \mathbf{Grp}$ and let $K \trianglelefteq Y$. Let $f\colon X \to Y$ be a homomorphism. By Proposition 2.11 it follows that $f^{-1}(K) \trianglelefteq X$. Consider the function $\varphi : X/f^{-1}(K) \to Y/K$ defined by $\varphi(xf^{-1}(K)) = f(x)K$. Let $xf^{-1}(K)$, $yf^{-1}(K) \in X/f^{-1}(K)$ be such that $xf^{-1}(K) = yf^{-1}(K)$. Then $y^{-1}x \in f^{-1}(K)$, and so $f(y^{-1}x) \in K$. Consequently $(f(y))^{-1}f(x) \in K$, that is, $f(x)K = f(y)K$, also $\varphi(xf^{-1}(K)) = \varphi(yf^{-1}(K))$. Hence, we conclude that φ is well defined.

Now let $xf^{-1}(K)$, $yf^{-1}(K) \in X/f^{-1}(K)$ be arbitrary. Then $\varphi(xf^{-1}(K) \cdot yf^{-1}(K)) =$

$\varphi(xyf^{-1}(K)) = f(xy)K = f(x)f(y)K = f(x)K \cdot f(y)K = \varphi\ (xf^{-1}(K)) \cdot \varphi(yf^{-1}(K))$. Hence, we conclude that φ is a homomorphism.

Notice that $Ker\varphi = \{xf^{-1}(K) : \varphi(xf^{-1}(K)) = K\}$. Since $\varphi(xf^{-1}(K)) = K \Leftrightarrow f(x)K = K \Leftrightarrow f(x) \in K \Leftrightarrow x \in f^{-1}(K) \Leftrightarrow xf^{-1}(K) = f^{-1}(K)$, then $Ker\varphi = \{f^{-1}(K)\}$. Hence, we conclude that φ is a monomorphism.

Lemma 2.13. Let $X \in \mathbf{Grp}$ and let H, $K \leq X$ be two normal subgroups of X such that $H \leq K$. Let $\psi : X/H \to X/K$ be a function defined by $\psi(xH) = xK$. Then ψ is an epimorphism.

Proof: Let $X \in \mathbf{Grp}$ and let H,$K \leq X$ be such that $H \leq K$. Consider the function $\psi : X/H \to X/K$ defined by $\psi(xH) = xK$. Let xH, $yH \in X/H$ be such that $xH = yH$. Then, $y^{-1}x \in H \leq K$. Consequently, $xK = yK$, and so $\psi(xH) = \psi(yH)$. Hence, we conclude that ψ is well defined.

Let xH, $yH \in X/H$ be arbitrary. Then $\psi(xH \cdot yH) = \psi(xyH) = xyK = xK \cdot yK = \psi(xH) \cdot \psi(yH)$. Hence we conclude that ψ is a homomorphism.

Let $xK \in X/K$ be arbitrary. Then there exists $xH \in X/K$ and $\psi(xH) = xK$. Hence we conclude that ψ is an epimorphism.

The concept of Galois connection plays a crucial role in this work, as a consequence we include its definition and some relevant results here.

Definition 2.14. For partially ordered classes $\mathbf{X} = (X, \sqsubseteq)$ and $\mathbf{Y} = (Y, \sqsubseteq)$, a *Galois connection* $\mathbf{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{Y}$ consists of order preserving functions F and G that satisfy $x \sqsubseteq G(F(x))$ for every $x \in X$ and $F(G(y)) \sqsubseteq y$ for every $y \in Y$. If $x \in X$ and $y \in Y$ are such that $F(x) = y$ and $G(y) = x$, then x and y are said to be corresponding *fixed points* of the Galois connection $(\mathbf{X}, F, G, \mathbf{Y})$.

Proposition 2.15. Let $\mathbf{X}$ and $\mathbf{Y}$ be two partially ordered classes and assume that suprema exist in $\mathbf{X}$. Let f: $\mathbf{X} \to \mathbf{Y}$ be an order preserving function that preserves suprema. Define g: $\mathbf{Y} \to \mathbf{X}$ as follows: for every $y \in Y$, $g(y) = \bigvee\{x \in X : f(x) \sqsubseteq y\}$. Then, $\mathbf{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathbf{Y}$ is a Galois connection.

Proof: Let $x_1 \sqsubseteq x_2 \in X$. Since f is an order preserving function, we have that $f(x_1) \sqsubseteq f(x_2)$. Let $y_1 \sqsubseteq y_2 \in Y$. Clearly we have that $\{x \in X : f(x) \sqsubseteq y_1\} \subseteq \{x \in X : f(x) \sqsubseteq y_2\}$. Thus, by taking the supremum we obtain that $g(y_1) \sqsubseteq g(y_2)$, that is g is an order preserving function. Now, let $x' \in X$. By applying the definition of g we obtain that $g(f(x')) = \bigvee\{x \in X : f(x) \sqsubseteq f(x')\}$. Since $x' \in \{x \in X : f(x) \sqsubseteq f(x')\}$, then $g(f(x')) = \bigvee\{x \in X : f(x) \sqsubseteq f(x')\} \sqsupseteq x'$. Finally, let $y' \in Y$. Since f preserves suprema, we have that $f(g(y')) = f(\bigvee\{x \in X : f(x) \sqsubseteq y'\}) = \bigvee\{f(x) \in Y : f(x) \sqsubseteq y'\} \sqsubseteq y'$. Hence we conclude that $\mathbf{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathbf{Y}$ is a Galois connection.

The proof of the following result is symmetric to the previous one, so we omit it.

Proposition 2.16. Let $\mathbf{X}$ and $\mathbf{Y}$ be two partially ordered classes and assume that infima exist in $\mathbf{Y}$. Let g: $\mathbf{Y} \to \mathbf{X}$ be an order preserving function that preserves infima. Define f: $\mathbf{X} \to \mathbf{Y}$ as follows: for every $x \in X$, $f(x) = \bigwedge\{y \in Y : g(y) \sqsupseteq x\}$. Then, $\mathbf{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathbf{Y}$ is a Galois connection.

A classical example of Galois connection arises as follows. Let f: $\mathbf{X} \to \mathbf{Y}$ be a function between sets and let $S(\mathbf{X})$ and $S(\mathbf{Y})$ denote the powersets of $\mathbf{X}$ and $\mathbf{Y}$, respectively. Consider the functions $F : S(\mathbf{X}) \to S(\mathbf{Y})$ and $G : S(\mathbf{Y}) \to S(\mathbf{X})$ defined by $F(A) = f(A)$ for $A \subseteq \mathbf{X}$ and $G(B) = f^{-1}(B)$ for $B \subseteq \mathbf{Y}$, respectively. Let $U_1, U_2 \in \mathbf{X}$ and $V_1, V_2 \in \mathbf{Y}$. Since $U_1 \subseteq U_2 \Rightarrow f(U_1) \subseteq f(U_2)$ and $V_1 \subseteq V_2 \Rightarrow f^{-1}(V_1) \subseteq f^{-1}(V_2)$, then F and G are order preserving functions. Since f is a function, we have that $U \subseteq f^{-1}(f(U))$ for every $U \subseteq \mathbf{X}$ and $f(f^{-1}(V)) \subseteq V$ for every $V \subseteq \mathbf{Y}$. Hence, it follows that the diagram $S(\mathbf{X}) \underset{G}{\overset{F}{\rightleftarrows}} S(\mathbf{Y})$ is a Galois connection.

Proposition 2.17. The composition of two Galois connections is a Galois connection.

Proof: Let $\mathbf{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathbf{Y}$ and $\mathbf{Y} \underset{k}{\overset{h}{\rightleftarrows}} \mathbf{Z}$ be two Galois connections. Since the composition of order preserving functions is order preserving, we have that $h \circ f$ and $g \circ k$ are order preserving. Let $x \in X$. Then $f(x) \sqsubseteq k(h(f(x)))$ and so $x \sqsubseteq g(f(x)) \sqsubseteq g(k(h(f(x)))) = (g \circ k)((h \circ f)(x))$. Now, let $z \in Z$. We have that $f(g(k(z))) \sqsubseteq k(z)$ and so $h(f(g(k(z)))) \sqsubseteq h(k(z)) \sqsubseteq z$. Hence, $\mathbf{X} \underset{g \circ k}{\overset{h \circ f}{\rightleftarrows}} \mathbf{Z}$ is a Galois connection.

Proposition 2.18. Let $\mathbf{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathbf{Y}$ be a Galois connection between partially ordered classes $\mathbf{X}$ and $\mathbf{Y}$. Then, the functions f and g uniquely determine each other.

Proof: Let $g'\colon \mathbf{Y} \to \mathbf{X}$ be a function such that $\mathbf{X} \underset{g'}{\overset{f}{\rightleftarrows}} \mathbf{Y}$ is also a Galois connection. Let $y \in Y$. By applying g' to $f(g(y)) \sqsubseteq y$, we obtain that $g(y) \sqsubseteq g'(f(g(y))) \sqsubseteq g'(y)$. Moreover, by applying g to $f(g'(y)) \sqsubseteq y$, we obtain that $g'(y) \sqsubseteq g(f(g'(y))) \sqsubseteq g(y)$. Hence we conclude that $g(y) = g'(y)$, for every $y \in Y$. The proof of the uniqueness of f is similar.

To clarify some terminology used throughout the thesis, we include the following.

Definition 2.19. Let $\mathscr{H} \subseteq \mathbf{Grp}$, that is $\mathscr{H}$ is a subclass of $\mathbf{Grp}$.

(a) $\mathscr{H}$ is *closed under subgroups* if and only if $\forall P \in \mathscr{H}, Q \leq P \Rightarrow Q \in \mathscr{H}$.

(b) $\mathscr{H}$ is *closed under suprema* if and only if $\forall \mathscr{S} \subseteq \mathscr{H}, \bigvee \mathscr{S} \in \mathscr{H}$.

(c) $\mathscr{H}$ is *closed under quotients* if and only if $\forall P \in \mathscr{H}, Q \trianglelefteq P \Rightarrow P/Q \in \mathscr{H}$. Equivalently, $\mathscr{H}$ is *closed under quotients* if and only if for all quotient homomorphisms $f\colon P \to Q$, $P \in \mathscr{H} \Rightarrow Q \in \mathscr{H}$.

(d) $\mathscr{H}$ is *closed under inverse images via homomorphisms* if and only if for all group homomorphisms $f\colon G \to G'$ and $P \leq G'$, $P \in \mathscr{H} \Rightarrow f^{-1}(P) \in \mathscr{H}$.

(e) $\mathscr{H}$ is *closed under products* if and only if given a family of groups $\{A_i\}_{i\in I}$, if each $A_i \in \mathscr{H}$, then also the product of the A_i's belongs to $\mathscr{H}$.

3 Basic definitions and results

The following definition was introduced by S.J.R. Vorster in an arbitrary category ([V]). We specialize it here to the category **Grp** of groups.

Definition 3.1. An *interior operator* I on **Grp** is a family $\{i_X\}_{X\in\mathbf{Grp}}$ of functions on the subgroup lattices of **Grp** with the following properties that hold for each $X \in \mathbf{Grp}$:

(a) [*contractiveness*] $i_X(M) \leq M$, for every subgroup $M \leq X$;

(b) [*order-preservation*] $M \leq N$ implies that $i_X(M) \leq i_X(N)$ for every pair of subgroups M,N of X;

(c) [*continuity*] For every homomorphism $f : X \to Y$ and subgroup $N \leq Y$, $f^{-1}(i_Y(N)) \leq i_X(f^{-1}(N))$, i.e., the inverse image of the interior of N is less than or equal to the interior of the inverse image of N.

Definition 3.2. Given an interior operator I, we say that a subgroup M of a group X is *I-open* if $M = i_X(M)$. M is called *I-isolated* if $i_X(M) = \{e_X\}$. We call I idempotent provided that $i_X(M)$ is *I-open* for every $M \leq X$.

Notice that the subscript in the symbol $i_X(M)$ could be omitted whenever no confusion is possible.

Consider two subgroups N, M of $X \in \mathbf{Grp}$ such that $N \leq M \leq X$. Then for the subgroup N we can consider its interior with respect to M $(i_M(N))$ and its interior with respect to X $(i_X(N))$. The following result clarifies the relationship between $i_M(N)$ and $i_X(N)$.

Proposition 3.3. Let I be an interior operator and let $N \leq M \leq X \in \mathbf{Grp}$. Then we have that $i_X(N) \leq i_X(M) \cap i_M(N)$. In particular, $i_X(N) \leq i_M(N)$.

Proof: Let m: $M \to X$ denote the inclusion of M into X. By Definition 3.1(c), we have that $i_M(m^{-1}(N)) \geq m^{-1}(i_X(N))$. Then $i_X(M) \cap i_M(N) = i_X(M) \cap i_M(m^{-1}(N)) \geq i_X(M) \cap m^{-1}(i_X(N))$ $= i_X(M) \cap i_X(N) = i_X(N)$. This implies as a consequence that $i_X(N) \leq i_M(N)$.

Proposition 3.4.

(a) Let $f\colon X \to Y$ be a group homomorphism and let I be an interior operator on **Grp**. If $N \leq Y$ is I-open, then so is $f^{-1}(N)$.

(b) Let X be a group and let I be an interior operator. Consider the subgroup H of X defined by $H = \bigvee\{M \leq X : i_X(M) = M\} = \langle \cup\{M \leq X : i_X(M) = M\}\rangle$. Then H is I-open, in other words, the supremum of the family of I-open subgroups of X is I-open.

(c) Let X be a group and let I be an interior operator. Consider the subgroup H of X defined by $H = \cap\{M \leq X : i_X(M) = \{e_X\}\}$. Then H is I-isolated, in other words, the infimum of a family of I-isolated subgroups is I-isolated.

Proof: (a). Since N is I-open, by Definition 3.1(c), we have that $f^{-1}(N) = f^{-1}(i_Y(N)) \leq i_X(f^{-1}(N))$. Let $M = f^{-1}(N)$, so $M \leq i_X(M)$. By Definition 3.1(a), $i_X(M) \leq M$. Hence we conclude that $M = i_X(M)$, that is $M = f^{-1}(N)$ is I-open.

(b). Let $\mathscr{A} = \{M \leq X : i_X(M) = M\}$, that is the class of I-open subgroups of X. Then $H = \langle \cup \mathscr{A} \rangle$. For every $M_j \in \mathscr{A}$ ($j \in J$), $M_j \leq H$ because $M_j \subseteq H, M_j \leq X$, and $H \leq X$. Then by Definition 3.1(b), $(\forall j \in J), i_X(M_j) \leq i_X(H)$. Then $(\forall j \in J), M_j \leq i_X(H)$ because $M_j = i_X(M_j)$. Since $\cup_{j\in J} M_j \subseteq i_X(H)$, $i_X(H) \leq X$, and $\langle \cup_{j\in J} M_j\rangle$ is the smallest subgroup of X that contains $\cup_{j\in J} M_j$, then $H = \langle \cup_{j\in J} M_j \rangle \leq i_X(H)$. By Definition 3.1(a), $i_X(H) \leq H$. Hence we conclude that $H = i_X(H)$, that is H is I-open.

(c). Let $\mathscr{B} = \{M \leq X : i_X(M) = \{e_X\}\}$, that is the class of I-isolated subgroups of X. Then $H = \cap \mathscr{B}$. Since H is the intersection of subgroups of X, then $H \leq X$. For every $M_j \in \mathscr{B}$ ($j \in J$), $M_j \geq H$ because $M_j \supseteq H, M_j \leq X$, and $H \leq X$. Then by Definition 3.1(b), $(\forall j \in J), i_X(H) \leq i_X(M_j)$. Then $(\forall j \in J), i_X(H) \leq \{e_X\}$ because $i_X(M_j) = \{e_X\}$. Hence we conclude that $i_X(H) = \{e_X\}$, that is H is I-isolated.

Examples 3.5.

(a) The assignment D defined by $d_x(M) = M$ for every $M \leq X$, $X \in$ **Grp** is an interior operator on **Grp** called the *discrete operator*.

(b) The assignment T defined by $t_X(M) = \{e_X\}$ for every $M \leq X$, $X \in \mathbf{Grp}$ is an interior operator on $\mathbf{Grp}$ called the *trivial operator*.

(c) Consider the assignment P defined by $p_X(M) = \bigvee\{K \leq M : K \trianglelefteq X\}$, for every $M \leq X$ and every $X \in \mathbf{Grp}$. We want to show that P is an interior operator. Let $X \in \mathbf{Grp}$ and let $M \leq X$ be arbitrary.

(i) By definition of P, $p_X(M) = \bigvee\{K \leq M : K \trianglelefteq X\} \leq M$.

(ii) Let $N \leq X$ be such that $M \leq N$. Then $p_X(M) \leq M$ and $p_X(N) \leq N$. Then $p_X(M) \leq N$. Since $p_X(M)$ is normal in X (cf. Proposition 2.10), then $p_X(M) \in \{K \leq N : K \trianglelefteq X\}$. Then $p_X(M) \leq \bigvee\{K \leq N : K \trianglelefteq X\} = p_X(N) \leq N$.

(iii) Let $Y \in \mathbf{Grp}$. Let $N \leq Y$ be arbitrary and let $f \colon X \to Y$ be a homomorphism. Since $i_Y(N) \trianglelefteq Y$, then $f^{-1}(i_Y(N)) \trianglelefteq X$ (cf. Proposition 2.11). Since $f^{-1}(i_Y(N)) \trianglelefteq X$ and $f^{-1}(i_Y(N)) \leq f^{-1}(N)$, then $f^{-1}(i_Y(N)) \in \{K \leq f^{-1}(N) : K \trianglelefteq X\}$. It follows that $f^{-1}(i_Y(N)) \leq \bigvee\{K \leq f^{-1}(N) : K \trianglelefteq X\}$ $= i_X(f^{-1}(N))$. Hence, we conclude that $f^{-1}(i_Y(N)) \leq i_X(f^{-1}(N))$.

This operator will be called the *normal interior operator*.

(d) Consider the assignment $I_{\mathbf{Ab}}$ defined by $i_{\mathbf{Ab}}(M) = \bigvee\{K \leq M : K \trianglelefteq X \text{ and } X/K \in \mathbf{Ab}\}$, for every $M \leq X$ and every $X \in \mathbf{Grp}$. We want to show that $I_{\mathbf{Ab}}$ is an interior operator. Let $X \in \mathbf{Grp}$ and let $M \leq X$ be arbitrary.

(i) By definition of $I_{\mathbf{Ab}}$, $i_{\mathbf{Ab}}(M) = \bigvee\{K \leq M : K \trianglelefteq X \text{ and } X/K \in \mathbf{Ab}\} \leq M$.

(ii) Let $N \leq X$ be such that $M \leq N$. We observe that $K \leq M$ that occurs in the construction of $i_{\mathbf{Ab}}(M)$ satisfies $K \leq M$, K is normal in X and $X/K \in \mathbf{Ab}$. Due to the fact that $K \leq M \leq N$, K also occurs in the construction of $i_{\mathbf{Ab}}(N)$. Consequently, we have that $i_{\mathbf{Ab}}(M) \leq i_{\mathbf{Ab}}(N)$.

(iii) Let $Y \in \mathbf{Grp}$ and $N \leq Y$ be arbitrary. Let $f \colon X \to Y$ be a homomorphism. Since $i_Y(N) \trianglelefteq Y$ as a supremum of normal subgroups (cf. Proposition 2.10), then by Proposition 2.11 it follows that $f^{-1}(i_Y(N)) \trianglelefteq X$. Let $K \in \{K \leq i_Y(N) : K \trianglelefteq Y \text{ and } Y/K \text{ is abelian}\}$. Since $K \trianglelefteq Y$, then by Proposition 2.11 it follows that $f^{-1}(K) \trianglelefteq X$. Let $\varphi : X/f^{-1}(K) \to Y/K$ be a map defined by

$\varphi(xf^{-1}(K)) = f(x)K$. Since Y/K is abelian and φ is a monomorphism (cf. Lemma 2.12), then $X/f^{-1}(K)$ is isomorfic to a subgroup of Y/K. Hence, $X/f^{-1}(K)$ is abelian. Let $H = f^{-1}(K)$ and $P = f^{-1}(i_Y(N))$. Then $H \leq P$. Let $\psi : X/H \to X/P$ defined by $\psi(xH) = xP$. Since ψ is an epimorphism (cf. Lemma 2.13) and $X/f^{-1}(K)$ is abelian, then $X/f^{-1}(i_Y(N))$ is abelian. Since $f^{-1}(i_Y(N)) \trianglelefteq X$, $f^{-1}(i_Y(N)) \leq f^{-1}(N)$, and $X/f^{-1}(i_Y(N))$ is abelian, then $f^{-1}(i_Y(N)) \in \{K \leq f^{-1}(N) : K \trianglelefteq X$ and X/K is abelian$\}$. Consequently, $f^{-1}(i_Y(N)) \leq \bigvee\{K \leq f^{-1}(N) : K \trianglelefteq X$ and X/K is an abelian group$\} = i_X(f^{-1}(N))$. Hence, we conclude that $f^{-1}(i_Y(N)) \leq i_X(f^{-1}(N))$.

(e) Since in part (d) the crucial properties of **Ab** that are used are that subgroups and quotients of abelian groups are also abelian, the interior operator construction in part (d) can be generalized to any subclass $\mathscr{C}$ of groups that is closed under subgroups and quotients. Precisely, for every $M \leq X \in \mathbf{Grp}$, the assignment $i_{\mathscr{C}}(M) = \bigvee \{K \leq M : K \trianglelefteq X$ and $X/K \in \mathscr{C}\}$ defines an interior operator on **Grp**. Examples of subclasses of groups closed under subgroups and quotients are for instance, torsion groups, finite groups, and Dedekind groups.

The following result provides a way to construct an interior operator by means of a subclass of subgroups satisfying certain conditions.

Proposition 3.6. Let $\mathscr{H}$ be a class of subgroups closed under suprema and inverse images under homomorphisms. Let $M \leq X \in \mathbf{Grp}$. Then the expression $i^{\mathscr{H}}(M) = \bigvee\{F \leq M : F \in \mathscr{H}\} = \langle\bigcup\{F \leq M : F \in \mathscr{H}\}\rangle$ is an interior operator. Moreover, $I^{\mathscr{H}}$ is idempotent.

Proof:

(i) Since $i^{\mathscr{H}}(M)$ is defined by the supremum of some subgroups of M, then $i^{\mathscr{H}}(M) \leq M$.

(ii) Let $N \leq X$ be such that $M \leq N$. Then $i^{\mathscr{H}}(M) \leq M \leq N$. Since $\mathscr{H}$ is a class of subgroups closed under suprema, then $i^{\mathscr{H}}(M) = \bigvee\{F \leq M : F \in \mathscr{H}\} \in \mathscr{H}$. Since $i^{\mathscr{H}}(M) \in \mathscr{H}$ and $i^{\mathscr{H}}(M) \leq N$, then $i^{\mathscr{H}}(M) \leq \bigvee\{F \leq N : F \in \mathscr{H}\} = i^{\mathscr{H}}(N)$.

(iii) Let $f\colon X \to Y$ be a homomorphism and $N \leq Y$. Since $\mathscr{H}$ is closed under suprema, $\bigvee\{F \leq N : F \in \mathscr{H}\} \in \mathscr{H}$. Since $\mathscr{H}$ is closed under inverse images via homomorphism, $f^{-1}(\bigvee\{F \leq N :$

$F \in \mathscr{H}\}) \in \mathscr{H}$. Then $f^{-1}(\bigvee\{F \leq N : F \in \mathscr{H}\}) \in \{F \leq f^{-1}(N) | F \in \mathscr{H}\}$. Then $f^{-1}(\bigvee\{F \leq N : F \in \mathscr{H}\}) \leq \bigvee\{F \leq f^{-1}(N) | F \in \mathscr{H}\}$. Hence we conclude that $f^{-1}(i^{\mathscr{H}}(N)) = f^{-1}(\bigvee\{F \leq N | F \in \mathscr{H}\}) \leq \bigvee\{F \leq f^{-1}(N) | F \in \mathscr{H}\} = i^{\mathscr{H}}(f^{-1}(N))$.

To show that $I^{\mathscr{H}}$ is idempotent it suffices to show that for all $X \in$ **Grp** and $M \leq X$, $i^{\mathscr{H}}(M) \leq i^{\mathscr{H}}(i^{\mathscr{H}}(M))$. Let $X \in$ **Grp** and $M \leq X$ be arbitrary. Since $i^{\mathscr{H}}(M) = \bigvee\{N \in \mathscr{H} : N \leq M\}$, then $i^{\mathscr{H}}(i^{\mathscr{H}}(M)) = \bigvee\{N \in \mathscr{H} : N \leq i^{\mathscr{H}}(M)\}$. Since $\mathscr{H}$ is closed under suprema, then $i^{\mathscr{H}}(M) \in \mathscr{H}$. Since $i^{\mathscr{H}}(M) \in \mathscr{H}$ and $i^{\mathscr{H}}(M) \leq i^{\mathscr{H}}(M)$, then $i^{\mathscr{H}}(M) \in \{N \in \mathscr{H} : N \leq i^{\mathscr{H}}(M)\}$. Hence we conclude that $i^{\mathscr{H}}(M) \leq \bigvee\{N \in \mathscr{H} : N \leq i^{\mathscr{H}}(M)\} = i^{\mathscr{H}}(i^{\mathscr{H}}(M))$.

Proposition 3.7. Let I be an interior operator, let $X \in$ **Grp**, and let $\mathscr{H} = \{F \leq X : i(F) = F\}$.

(a) $i^{\mathscr{H}}(M) \leq i(M)$, for every $M \leq X \in$ **Grp**.

(b) If I is idempotent, then $I = I^{\mathscr{H}}$.

Proof : Since $\mathscr{H}$ is closed under inverse images and suprema (cf. Proposition 3.4(b)), then by Proposition 3.6 we have that $I^{\mathscr{H}}$ defines an interior operator.

(a) We have that $i^{\mathscr{H}}(M) = \bigvee\{F \leq X : F \leq M \text{ and } i(F) = F\} \leq \bigvee\{F \leq X : i(F) \leq i(M) \text{ and } i(F) = F\} = i(M)$.

(b) Suppose that I is idempotent. By definition of idempotent interior operator, we have that $i(M) = i(i(M))$, $\forall M \leq X$. Then $i(M) \in \{F \leq M : i(F) = F\}$, $\forall M \leq X$. Thus $i(M) \leq \bigvee\{F \leq M : F \in \mathscr{H}\} = i^{\mathscr{H}}(M)$, $\forall M \leq X$. Hence we have that $I = I^{\mathscr{H}}$.

We denote the collection of all the interior operators on **Grp** by IN(**Grp**) ordered as follows: for $I, J \in IN$(**Grp**), $I \sqsubseteq J$ if $i_X(M) \leq j_X(M)$ for all $M \leq X \in$ **Grp**.

Proposition 3.8. Let $(I_k)_{k \in K} \subseteq IN$(**Grp**).

(a) For every subgroup M of $X \in$ **Grp**, define $\wedge_K I_k$ as follows: $i_{\wedge I_k}(M) = \cap_{k \in K} i_k(M)$. Then $\wedge_K I_k$ belongs to IN(**Grp**) and is the infimum of the family $(I_k)_{k \in K}$.

(b) There exists an interior operator $\vee_K I_k$ in **Grp** that is the supremum of the family $(I_k)_{k \in K}$. Moreover, for every $M \leq X$, $\bigvee i_k(M) \leq i_{\vee I_k}(M)$.

Proof:

(a)Let $M \leq X$. Since $i_k(M) \leq M$ for each $k \in K$, then we also have that $i_{\wedge I_k}(M) = \cap_{k\in K} i_k(M) \leq M$.

Let $M,N \leq X$ such that $M \leq N$, then $i_k(M) \leq i_k(N)$ for every $k \in K$. Thus $i_{\wedge I_k}(M) = \cap_{k\in K} i_k(M) \leq i_k(M) \leq i_k(N)$ for every $k \in K$ and so $i_{\wedge I_k}(M) = \cap_{k\in K} i_k(M) \leq \cap_{k\in K} i_k(N) = i_{\wedge I_k}(N)$.

Let $f \colon X \to Y$ be a group homomorphism and let N be a subgroup of Y. Then $f^{-1}(i_{\wedge I_k}(N)) = f^{-1}(\cap_{k\in K} i_k(N)) = \cap_{k\in K} f^{-1}(i_k(N)) \leq \cap_{k\in K} i_k(f^{-1}(N)) = i_{\wedge I_k}(f^{-1}(N))$. Notice that here we have used the fact that each I_k satisfies condition (c) of Definition 3.1 and that inverse images and intersections commute. Thus all the conditions of Definition 3.1 are satisfied and so $\wedge_K I_k$ belongs to $IN(\mathbf{Grp})$

Notice that $i_{\wedge I_k}(M) = \cap_{k\in K} i_k(M) \leq i_k(M)$ for every $k \in K$. Hence $\wedge_K I_k \sqsubseteq I_k$ for every $k \in K$. Moreover, if $I \in IN(\mathbf{Grp})$ satisfies that $i(M) \leq i_k(M)$ for every $k \in K$, then $i(M) \leq \cap_{k\in K} i_k(N) = i_{\wedge I_k}(M)$. Hence $I \sqsubseteq \wedge_K I_k$. We conclude that $\wedge_K I_k$ is the infimum of the family $(I_k)_{k\in K}$.

(b) Let $\mathscr{U} \subseteq IN(\mathbf{Grp})$ be the set of every upper bound of $(I_k)_{k\in K}$. Then by part (a) there exists the infimum, $I_{\wedge \mathscr{U}}$, of $\mathscr{U}$. Then $I_{\wedge \mathscr{U}}$ is an upper bound of $(I_k)_{k\in K}$, since if $I_k \leq I$ for each $I \in \mathscr{U}$, then $I_k \leq I_{\wedge \mathscr{U}}$. Moreover, for every upper bound, $I \in \mathscr{U}$, of $(I_k)_{k\in K}$, we have that $\forall M \leq X, i_{\wedge \mathscr{U}}(M) \leq i(M)$ Hence we conclude that $I_{\wedge \mathscr{U}} = I_{\vee_K I_k}$.

Notice that by definition of $\vee_K I_k$ for every $M \leq X$ and $k \in K$, $i_k(M) \leq i_{\vee I_k}(M)$. Then by Definition 2.5, for every $M \leq X$, $\bigvee i_k(M) \leq i_{\vee I_k}(M)$.

Remark 3.9. It is important to observe that even though in the above proposition the existence of $\vee_K I_k$ is proved, we do not have a practical description of it. However, if inverse images and suprema would commute in **Grp**, then an expression for $\vee_K I_k$ similar to the one of $\wedge_K I_k$ could be found.

Lemma 3.10. Let $(I_k)_{k\in K} \subseteq IN(\mathbf{Grp})$. For every subgroup M of X we have:

(a) M is $\wedge_K I_k$-open if and only if M is I_k-open for every $k \in K$;

(b) M is $\vee_K I_k$-isolated implies that M is I_k-isolated for every $k \in K$.

Proof:

(a) If each subgroup M of X is I_k-open for each $k \in K$, then $i_k(M) = M$ for every $k \in K$ and consequently, $i_{\wedge I_k}(M) = \cap_{k\in K} i_k(M) = \cap_{k\in K} M = M$. Hence, M is $\wedge_K I_k$-open. Now, let M be $\wedge_K I_k$-open. Then, since for every $k \in K$ we have that $i_{\wedge I_k}(M) \leq i_k(M) \leq M$, we conclude that $i_k(M) = M$ for every $k \in K$. Hence, M is I_k-open for every $k \in K$.

(b) Let $M \leq X$ be $\vee_K I_k$-isolated. Then $i_{\vee I_k}(M) = \{e_X\}$. Since $\{e_X\} \leq i_k(M) \leq i_{\vee I_k}(M)$ for every $k \in K$, then $i_k(M) = \{e_X\}$ for every $k \in K$. Hence, M is I_k-isolated for every $k \in K$.

Definition 3.11. Given an interior operator I, we say that

(a) $X \in$ **Grp** is I-discrete if every subgroup M of X is I-open.

(b) $X \in$ **Grp** is I-indiscrete if every proper subgroup M of X is I-isolated.

Proposition 3.12. Let I be an interior operator. Let $\mathscr{X}$ be the subclass of **Grp** that consists of all I-discrete groups and let $\mathscr{Y}$ be the subclass of **Grp** that consists of all I-indiscrete groups. Then we have that:

(a) $\mathscr{X}$ is closed under subgroups.

(b) $\mathscr{Y}$ is closed under quotients.

Proof:

(a) Let $G \in \mathscr{X}$, that is H is I-discrete, and let $H \leq G$. Then $i_G(M) = M$ for every $M \leq G$. Let $Q \leq H$ arbitrary. From Proposition 3.3, $i_G(Q) \leq i_H(Q) \leq Q$ and so $i_G(Q) = Q$ implies that $i_H(Q) = Q$. Then $H \in \mathscr{X}$, that is H is I-discrete. Hence, we conclude that $\mathscr{X}$ is closed under subgroups.

(b) Let $G \in \mathscr{Y}$, that is G is I-indiscrete, and let q: $G \to Q$ be a quotient homomorphism. If $M \lneqq Q$, then the continuity condition of interior operator implies that $q^{-1}(i_Q(M)) \leq i_G(q^{-1}(M))$. Since G is I-indiscrete and $q^{-1}(M) \neq G$, then $i_G(q^{-1}(M)) = \{e_G\}$. Consequently, $q^{-1}(i_Q(M)) = \{e_G\}$. This implies that $i_Q(M) = \{e_Q\}$, that is $Q \in \mathscr{Y}$. Hence, we conclude that $\mathscr{Y}$ is closed under quotients.

Examples 3.13.

(a) For the interior operator D every group is D-discrete by the definition of the discrete operator. We also have that the groups that do not have nontrivial proper subgroups are D-indiscrete, for example, the trivial group and the groups of prime order.

(b) For the interior operator T every group is T-indiscrete by the definition of the indiscrete operator and the trivial group is the only one that is T-discrete.

(c) For the normal interior operator P we have that X is P-discrete $\Leftrightarrow \forall M \leq X,\ i(M) = M \Leftrightarrow \forall M \leq X,\ M \trianglelefteq X \Leftrightarrow X$ is a Dedekind group. We also have that X is P-indiscrete $\Leftrightarrow \forall M \lneqq X$, $i(M) = \{e_X\} \Leftrightarrow \forall M \lneqq X,\ \nexists K \leq M$ such that $K \neq \{e_X\}$ and $K \trianglelefteq X \Leftrightarrow X$ is a simple group.

(d) Consider the interior operator $I_{\mathbf{Ab}}$ defined for every $M \leq X \in \mathbf{Grp}$ by $i_{\mathbf{Ab}}(M) = \bigvee \{K \leq M : K \trianglelefteq X \text{ and } X/K \in \mathbf{Ab}\}$. Then $X \in \mathbf{Grp}$ is $I_{\mathbf{Ab}}$-discrete $\Leftrightarrow \forall M \leq X,\ i_{\mathbf{Ab}}(M) = M \Leftrightarrow \forall M \leq X,\ M \trianglelefteq X$ and $X/M \in \mathbf{Ab}$. Notice that the class of $I_{\mathbf{Ab}}$-discrete groups is contained in the class of Dedekind groups and contains all abelian groups. We also have that $X \in \mathbf{Grp}$ is $I_{\mathbf{Ab}}$-indiscrete $\Leftrightarrow \forall M \lneqq X$, $i_{\mathscr{C}}(M) = \{e_X\} \Leftrightarrow \forall M \lneqq X,\ \nexists K \leq M$ such that $K \neq \{e_X\}$, $K \trianglelefteq X$ and $X/K \in \mathbf{Ab}$, in other words, X does not have any non-trivial proper normal subgroup K such that X/K is abelian. Notice that groups of prime order and simple groups are $I_{\mathbf{Ab}}$-indiscrete.

(e) Consider the interior operator $I_{\mathscr{C}}$ defined by $i_{\mathscr{C}}(M) = \bigvee \{K \leq M : K \trianglelefteq X \text{ and } X/K \in \mathscr{C}\}$, for every $M \leq X \in \mathbf{Grp}$, where $\mathscr{C}$ is a subclass of groups that is closed under subgroups and quotients. Then X is $I_{\mathscr{C}}$-discrete $\Leftrightarrow \forall M \leq X,\ i_{\mathscr{C}}(M) = M \Leftrightarrow \forall M \leq X,\ M \trianglelefteq X$ and $X/M \in \mathscr{C}$. Notice that if $\mathscr{C}$ is the class of finite groups, then the class of $I_{\mathscr{C}}$-discrete contains the class of Dedekind finite groups. We also have that $\mathbb{Z}$ is an $I_{\mathscr{C}}$-discrete group if $\mathscr{C}$ is the class of finite groups. If $\mathscr{C}$ is the class of Dedekind groups, then the class of $I_{\mathscr{C}}$-discrete is the class of Dedekind groups (cf. Proposition 3.28). If $\mathscr{C}$ is the class of torsion groups, then the class of $I_{\mathscr{C}}$-discrete contains the class of Dedekind torsion groups.

A homomorphism $f\colon X \to Y$ is constant if and only if $f(x) = e_Y$, for all $x \in X$.

Definition 3.14. Given an interior operator I, we say that

(a) $X \in \mathbf{Grp}$ is I-connected if every homomorphism from X into any I-discrete group Y is constant.

(b) $X \in \mathbf{Grp}$ is I-disconnected if every homomorphism from any I-indiscrete group Y into X is constant.

Examples 3.15.

(a) For the operator D the groups that do not have any subgroup of prime order are D-disconnected and the trivial group is the only one that is D-connected.

(b) For the operator T every group is T-connected and the trivial group is the only one that is T-disconnected.

(c) For the normal operator P we have:

X is P-connected $\Leftrightarrow \forall f\colon X \to Y$, where f is a homomorphism and Y is an P-discrete group, f is a constant function $\Leftrightarrow \forall$ homomorphism $f\colon X \to Y$, where Y is Dedekind, we have that $f(x) = e_Y$, $\forall x \in X$.

Any simple group that is not of prime order is P-connected.

Let X be a simple group that is not of prime order. Let Y be a Dedekind group, and let $f\colon X \to Y$ be a group homomorphism. Consider $f^{-1}(\{e_Y\}) \leq X$. Then, by Proposition 2.11, since $\{e_Y\} \trianglelefteq Y$, we have that $f^{-1}(\{e_Y\}) \trianglelefteq X$. Since X has only X and $\{e_X\}$ as normal subgroups, then $f^{-1}(\{e_Y\}) = X$ or $f^{-1}(\{e_Y\}) = \{e_X\}$. Suppose that $f^{-1}(\{e_Y\}) = \{e_X\}$. Then f is a monomorphism. This implies that X is isomorphic to a subgroup of Y. Since Y is a Dedekind group and X is isomorphic to a subgroup of Y, by Proposition 3.28, we have that X is a Dedekind subgroup. Since X is Dedekind and simple, then X has no proper subgroup. Then X is of prime order (c.f. [DF]), which is a contradiction. Then $f^{-1}(\{e_Y\}) = X$, that is, f is a constant function. Hence, we conclude that X is P-connected.

X is P-disconnected $\Leftrightarrow \forall f : Y \to X$, where f is a homomorphism and Y is an P-indiscrete group, f is a constant function $\Leftrightarrow \forall$ homomorphism $f : Y \to X$, where Y is a simple group, we have that $f(y) = e_X, \forall y \in Y$.

Any group that is not simple and that does not have any non-trivial simple subgroup
is P-disconnected.

Let X be a group that is not simple and that does not have any non-trivial simple subgroup. Let Y be a simple group, and let f: $Y \to X$ be a group homomorphism. Since simple groups are closed under quotients (c.f. [DF]), then $f(Y)$ is a simple subgroup of X. By assumption on X, $f(Y) \neq X$ and so $f(Y) = \{e_X\}$, that is, f is a constant function.

Next we provide some properties of I-connected and I-disconnected groups.

Proposition 3.16. Let $(M_i)_{i\in I}$ be a family of subgroups of $X \in$ **Grp**. If each M_i is I-connected, then so is $\bigvee M_i = \langle \cup_{i\in I} M_i \rangle$.

Proof: Let f: $\langle \cup_{i\in I} M_i \rangle \to Y$ be a homomorphims with Y I-discrete. Since each M_i is I-connected, the restriction of f to each M_i is constant, i.e. $f(x) = e_Y, \forall x \in M_i, \forall i \in I$. Then $f(x) = e_Y$, $\forall x \in \cup_{i\in I} M_i$. We want to prove that $f(x) = e_Y$, $\forall x \in \langle \cup_{i\in I} M_i \rangle$. Y is I-discrete implies that $i_Y(N) = N$, $\forall N \leq Y$. Let $a \in \langle \cup_{i\in I} M_i \rangle - \cup_{i\in I} M_i$. We want to prove that $f(a) = e_Y$. Notice that $a = x_1x_2x_3...x_n$, where $x_j \in \cup_{i\in I} M_i$ for $j \in \{1,2,3,...,n\}$. Then $f(a) = f(x_1x_2x_3...x_n) = f(x_1)f(x_2)$ $f(x_3)...f(x_n) = e_Ye_Ye_Y...e_Y = e_Y$ because f is a homomorphism from $\langle \cup_{i\in I} M_i \rangle$ to Y. Then $f(a) = e_Y$, $\forall a \in \langle \cup_{i\in I} M_i \rangle - \cup_{i\in I} M_i$. Consequently, we have that $f(x) = e_Y$, $\forall x \in \langle \cup_{i\in I} M_i \rangle$, that is f is constant. Hence, we conclude that $\bigvee M_i = \langle \cup_{i\in I} M_i \rangle$ is I-connected.

Proposition 3.17. Let X, $Y \in$ **Grp** and let $f : X \to Y$ be an epimorphism. If X is I-connected, then Y is I-connected.

Proof: Let X, $Y \in$ **Grp** such that X is I-connected and let $Z \in$ **Grp** be I-discrete. Since X is I-connected, $g(x) = e_Z$, $\forall x \in X$ for every homomorphism g: $X \to Z$. Let f: $X \to Y$ be an epimorphism and let h: $Y \to Z$ be an arbitrary homomorphism. Then $h \circ f$ is constant, i.e. $h(f(x)) = e_Z$, $\forall x \in X$. Since f is surjective, $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$. Then $h(y) = h(f(x)) = e_Z$, $\forall y \in Y$. Then h is constant. Hence, we conclude that Y is I-connected.

Proposition 3.18. Let M be a subgroup of $X \in \mathbf{Grp}$. If X is I-disconnected, then so is M.

Proof: Let m: $M \to X$ denote the inclusion homomorphism of M into X. Let f: $Y \to M$ be a homomorphism with Y I-indiscrete. The fact that $m \circ f$ is constant (since X is I-disconnected) and that m is injective implies that f is constant. Hence, we conclude that M is I-disconnected.

Proposition 3.19. The product of a family of I-disconnected groups is I-disconnected.

Proof: Let $(X_i)_{i \in I}$ be a family of I-disconnected groups. Let f: $Y \to \Pi_{i \in I} X_i$ be a homomorphism with Y I-indiscrete and let $(\pi_i)_{i \in I}$ denote the usual projections. Since the X_i's are I-disconnected groups, then $\pi_i \circ f$ is constant for every $i \in I$, that is $\pi_i(f(y)) = \{e_{X_i}\}$ for all $y \in Y$ and $i \in I$. Then $f(y) = (e_{X_i})_{i \in I}$, $\forall y \in Y$, that is f is constant. Hence, we conclude that $\Pi_{i \in I} X_i$ is I-disconnected.

Let $S(\mathbf{Grp})$ denote the conglomerate of all subclasses of objects of $\mathbf{Grp}$, ordered by inclusion. $S(\mathbf{Grp})^{op}$ will denote the same conglomerate with the order reversed.

Proposition 3.20. The function $D : IN(\mathbf{Grp}) \to S(\mathbf{Grp})$ defined by $D(I) = \{X \in \mathbf{Grp}\colon\ X$ is I-discrete$\} = \{X \in \mathbf{Grp} :$ every subgroup of X is I-open$\}$ preserves infima.

Proof: Let $(I_k)_{k \in K} \subseteq IN(\mathbf{Grp})$. We wish to show that $D(\wedge_K I_k) = \cap_{k \in K} D(I_k)$. Let $M \subseteq X \in \mathbf{Grp}$. From Lemma 3.10(a) we have that M is $\bigwedge_{k \in K} I_k$-open if and only if M is I_k-open for every $k \in K$. Consequently, $X \in D(\bigwedge_K I_k)$ if and only if $X \in D(I_k)$ for every $k \in K$ if and only if $X \in \bigcap_{k \in K} D(I_k)$. This proves our assertion.

Thus, as a direct consequence of Proposition 2.16 we obtain the following:

Theorem 3.21. Consider the function D: $IN(\mathbf{Grp}) \to S(\mathbf{Grp})$ defined for any interior operator I by: $D(I) = \{X \in \mathbf{Grp}\colon X$ is I-discrete$\}$. Then, there is a function T: $S(\mathbf{Grp}) \to IN(\mathbf{Grp})$ defined for any subclass $\mathscr{A}$ of groups by $T(\mathscr{A}) = \bigwedge\{I \in IN(\mathbf{Grp}) : D(I) \supseteq \mathscr{A}\}$ such that

$S(\mathbf{Grp}) \underset{D}{\overset{T}{\rightleftarrows}} IN(\mathbf{Grp})$ is a Galois connection.

The following result is a direct consequence of the fact that $S(\mathbf{Grp}) \underset{D}{\overset{T}{\rightleftarrows}} IN(\mathbf{Grp})$ is a Galois connection but for reference purposes we state it as a separate lemma.

Lemma 3.22. $T(\mathscr{A})$ satisfies that $D(T(\mathscr{A})) \supseteq \mathscr{A}$.

Our next aim is to try to obtain a more practical description of the function T. We first observe that for every homomorphism $f\colon X \to Y$ with Y I-discrete, one has that any subgroup $N \leq Y$ is I-open and consequently, so is $f^{-1}(N)$. As a consequence we make the following. Let $\mathscr{A}$ be any subclass of groups.

Conjecture 3.23. For any subgroup $M \leq X \in \mathbf{Grp}$, $i_{T(\mathscr{A})}(M) = \bigvee\{f^{-1}(N) | \ f\colon X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq M\}$.

In what follows we provide a reason for such a conjecture. Consider the assignment $I_{(\mathscr{A})}$ defined by $i_{(\mathscr{A})}(M) = \bigvee\{f^{-1}(N) | \ f\colon X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq M\}$, for any subgroup $M \leq X \in \mathbf{Grp}$. Then we have the following result.

Proposition 3.24. For every $X \in \mathbf{Grp}$ and $M \leq X$, we have that $i_{T(\mathscr{A})}(M) \geq i_{(\mathscr{A})}(M)$.

Proof: Since, from Lemma 3.22, every $Y \in \mathscr{A}$ is $T(\mathscr{A})$-discrete, it follows that any $N \leq Y$ is $T(\mathscr{A})$-open and so, from Proposition 3.4a, $f^{-1}(N)$ is also $T(\mathscr{A})$-open. Since $f^{-1}(N)$ is $T(\mathscr{A})$-open and $f^{-1}(N) \leq M$, by Definition 3.1c, it follows that $f^{-1}(N) = i_{T(\mathscr{A})}(f^{-1}(N)) \leq i_{T(\mathscr{A})}(M)$. Since this is true for every $N \leq Y \in \mathscr{A}$, we have that $i_{(\mathscr{A})}(M) = \bigvee\{f^{-1}(N) | \ f : X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq M\} \leq i_{T(\mathscr{A})}(M)$.

Remark 3.25. We would like to show that $I_{(\mathscr{A})}$ is an interior operator. For this let $M \leq X \in \mathbf{Grp}$.

Contractiveness of $I_{(\mathscr{A})}$ is clear since, by construction, $i_{(\mathscr{A})}(M)$ is a supremum of subgroups of M and so $i_{(\mathscr{A})}(M) \leq M$.

Next we would like to show order-preservation of $I_{(\mathscr{A})}$. For this let $Q \leq X$ such that $M \leq Q$. Then we have that $i_{(\mathscr{A})}(M) \leq Q$. We want to show that $i_{(\mathscr{A})}(M) \leq i_{(\mathscr{A})}(Q)$. Let $K \in \{f^{-1}(N) |$ $f : X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq M\}$. Then $K = f^{-1}(N)$, where $f : X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $K \leq M$. Since $K \leq M \leq Q$, then $K \in \{f^{-1}(N) |$ $f : X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq Q\}$. Consequently, $\{f^{-1}(N) |$ $f : X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq M\} \subseteq \{f^{-1}(N) | \ f : X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq Q\}$. Hence we conclude that $i_{(\mathscr{A})}(M) = \bigvee\{f^{-1}(N) |$ $f : X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq M\} \leq \bigvee\{f^{-1}(N) | \ f : X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq Q\} = i_{(\mathscr{A})}(Q)$.

Now, let g: $Z \to X$ be a continuous function. If we could show that $g^{-1}(i_{(\mathscr{A})}(M)) = g^{-1}(\bigvee\{f^{-1}\ (N) | \ f$: $X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq$ $M\}) = \bigvee(\{g^{-1}(f^{-1}(N)) | \ f$: $X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq M\})$ $= \bigvee\{(f \circ g)^{-1}(N) | \ f$: $X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $f^{-1}(N) \leq M\}$, then it could be shown that $g^{-1}(i_{(\mathscr{A})}(M)) \leq \bigvee\{h^{-1}(N) | \ h$: $Z \to Y$ is a homomorphism, $Y \in \mathscr{A}$, $N \leq Y$, and $h^{-1}(N) \leq g^{-1}(M)\} = i_{(\mathscr{A})}(g^{-1}(M))$.

Unfortunately, this could not be proved because it is not a fact that inverse images and suprema do necessarily commute in **Grp** and we could not find a way to go around this problem.

Even though proving that $I_{(\mathscr{A})}$ is an interior operator for every $\mathscr{A} \in S(\mathbf{Grp})$ could not be accomplished, for some specific examples of the subclass $\mathscr{A}$ of groups, $I_{(\mathscr{A})}$ yields an interior operator, as it will be shown later. Consequently, we give the following.

Proposition 3.26. If the assignment $I_{(\mathscr{A})}$ defined for $M \leq X \in \mathbf{Grp}$ by $i_{(\mathscr{A})}(M) = \bigvee\{f^{-1}(N) |$ $f : X \to Y$ is a homomorphism, $Y \in \mathscr{A}$, and $f^{-1}(N) \leq M\}$ defines an interior operator, then $I_{(\mathscr{A})} = I_{T(\mathscr{A})}$.

Proof: In the previous proposition it was proved that $I_{(\mathscr{A})} \sqsubseteq I_{T(\mathscr{A})}$. Now let $M \leq X \in \mathscr{A}$. The existence of the identity function of X, id_X implies that $i_{(\mathscr{A})}(M) = M$ that is X is $I_{(\mathscr{A})}$-discrete.

Hence $I_{(\mathscr{A})}$ satisfies that $D(I_{(\mathscr{A})}) \supseteq \mathscr{A}$. By definition of $I_{T(\mathscr{A})}$ we conclude that $I_{T(\mathscr{A})} \sqsubseteq I_{(\mathscr{A})}$, that is $I_{T(\mathscr{A})} = I_{(\mathscr{A})}$.

Next we show that if $\mathscr{A} = \mathbf{Ab}$, then $I_{(\mathbf{Ab})}$ defines an interior operator.

Proposition 3.27. For $\mathscr{A} = \mathbf{Ab}$, the assignment defined for $M \leq X \in \mathbf{Grp}$ by

$i_{(\mathbf{Ab})}(M) = \bigvee\{f^{-1}(N) | \; f : X \to Y$ is a homomorphism, $Y \in \mathbf{Ab}$, $N \leq Y$, and $f^{-1}(N) \leq M\}$

defines an interior operator.

Proof: We will prove our statement by showing that $i_{(\mathbf{Ab})}(M) = i_{\mathbf{Ab}}(M) = \bigvee\{K \leq M\colon K \trianglelefteq X$ and $X/K \in \mathbf{Ab}\}$ and this last expression defines an interior operator as shown in Examples 3.5(d). First we observe that every subgroup $K \leq M$ such that $K \trianglelefteq X$ and $X/K \in \mathbf{Ab}$ satisfies $K = q^{-1}(0)$, where $q\colon X \to X/K$ is the quotient homomorphism and 0 is the identity of X/K (or the equivalence class of the subgroup K). Hence any subgroup K that occurs in the construction of $i_{\mathbf{Ab}}(M)$ can be seen as a subgroup $f^{-1}(N)$ that occurs in the construction of $i_{(\mathbf{Ab})}(M)$. Consequently, we have that $i_{\mathbf{Ab}}(M) \leq i_{(\mathbf{Ab})}(M)$. Now, take a subgroup $f^{-1}(N)$ that occurs in the construction of $i_{(\mathbf{Ab})}(M)$. Since $Y \in \mathbf{Ab}$, then $N \trianglelefteq Y$ (cf. [DF]) and so, by Proposition 2.11, we have that $f^{-1}(N) \trianglelefteq X$. From Lemma 2.12, the function $\varphi\colon X/f^{-1}(N) \to Y/N$ is a monomorphism. Since $\mathbf{Ab}$ is closed under subgroups and quotients, $X/f^{-1}(N) \in \mathbf{Ab}$ as a subgroup of Y/N. Since $f^{-1}(N)$ satisfies $f^{-1}(N) \leq M$, $f^{-1}(N) \trianglelefteq X$, and $X/f^{-1}(N) \in \mathbf{Ab}$, then $f^{-1}(N)$ occurs in the construction of $i_{\mathbf{Ab}}(M)$. This implies that $i_{(\mathbf{Ab})}(M) \leq i_{\mathbf{Ab}}(M)$. Finally we conclude that $i_{(\mathbf{Ab})}(M) = i_{\mathbf{Ab}}(M)$.

If we look carefully at the proof of Proposition 3.27, we conclude that the crucial properties used were the facts that every subgroup of an abelian group is normal and that abelian groups are closed under subgroups and quotients. This suggests that the above result can be obtained for any class of groups that has the above mentioned properties. A further example is provided by the class of Dedekind groups, as the following result shows.

Proposition 3.28. Dedekind groups are closed under subgroups and quotients.

Proof: Let X be a Dedekind group. Then, by definition, $M \trianglelefteq X$ for all $M \leq X$. Let $N \leq X$ and let $Q \leq N$. Then $Q \leq X$. Since X is a Dedekind group, then $Q \trianglelefteq X$. Since $Q \leq N \leq X$ and $Q \trianglelefteq X$, then $Q \trianglelefteq N$. Consequently, N is a Dedekind group. Hence we conclude that Dedekind groups are closed under subgroups.

Let $P \trianglelefteq X$. Consider the quotient homomorphism q: $X \to X/P$. Let $A \leq X/P$. Then $q^{-1}(A) \leq X$. Since X is a Dedekind group, then $q^{-1}(A) \trianglelefteq X$. Then, since q is a surjective function, $A = q(q^{-1}(A)) \trianglelefteq X/P$, that is X/P is a Dedekind group. Hence we conclude that Dedekind groups are closed under quotients.

As a consequence of the above proposition and Example 3.5(e) we obtain the following corollary.

Corollary 3.29. If $\mathscr{B}$ is the class of Dedekind groups, then $\forall M \leq X \in \mathbf{Grp}$, $i_{(\mathscr{B})}(M) = i_{\mathscr{B}}(M)$ and so $I_{\mathscr{B}}$ defines an interior operator.

Remark 3.30. The results in Proposition 3.27 and Corollary 3.29 show that for $\mathscr{A}$ consisting of either **Ab** or all Dedekind groups, $I_{(\mathscr{A})}$ is an interior operator and from Proposition 3.26 for these two cases $i_{(\mathscr{A})} = i_{T(\mathscr{A})}$. This shows that at least for $\mathscr{A} = \mathbf{Ab}$ or $\mathscr{A} =$ Dedekind groups, Conjecture 3.23 is correct.

Our next aim is to try to build a new Galois connection similar to the one in Theorem 3.21, but by means of the notion of I-indiscrete groups.

Consider the function C: $IN(\mathbf{Grp}) \to S(\mathbf{Grp})^{op}$ defined by $C(I) = \{X \in \mathbf{Grp}$ such that X is I-indiscrete$\}$. We would like to show that there is a function G: $S(\mathbf{Grp})^{op} \to IN(\mathbf{Grp})$ such that $IN(\mathbf{Grp}) \underset{G}{\overset{C}{\rightleftarrows}} S(\mathbf{Grp})^{op}$ is a Galois connection.

Conjecture 3.31. For every class $\mathscr{B}$ of groups the assignment $I_{G(\mathscr{B})}$ that to each subgroup $M \leq Y \in \mathbf{Grp}$ associates $i_{G(\mathscr{B})}(M) = \bigvee\{N \leq M : \forall$ homomorphism f: $X \to Y$ with $X \in \mathscr{B}$ and $f^{-1}(M) \neq X$ we have that $N \cap f(X) = \{e_Y\}\}$ is an interior operator.

Remark 3.32. The reason for this conjecture will be made clear later. Unfortunately, we can only prove two of the three conditions of Definition 3.1. We include their proofs here.

(a) Since $i_{G(\mathscr{B})}(M)$ is defined as the group generated by some subgroups of M, then clearly $i_{G(\mathscr{B})}(M) \leq M$.

(b) Let $H \leq Y \in \mathbf{Grp}$ be such that $M \leq H$. Let $N_0 \leq M$ be such that $N_0 \in \{N \leq M : \forall$ homomorphism $f : X \to Y$ with $X \in \mathscr{B}$ and $f^{-1}(M) \neq X$ we have that $N \cap f(X) = \{e_Y\}\}$. Then $N_0 \leq H$. Let f_0: $X \to Y$ be a homomorphism that satisfies that $X \in \mathscr{B}$ and $f_0^{-1}(H) \neq X$. Then $f_0^{-1}(M) \neq X$ and so $N_0 \cap f(X) = \{e_Y\}$. Consequently, $N_0 \in \{N \leq H : \forall$ homomorphism $f : X \to Y$ with $X \in \mathscr{B}$ and $f^{-1}(H) \neq X$ we have that $N \cap f(X) = \{e_Y\}\}$. Hence we conclude that $i_{G(\mathscr{B})}(M) \leq i_{G(\mathscr{B})}(H)$.

(c) Let g: $Y \to Z$ be a homomorphism and let $M \leq Z$. We have that $i_{G(\mathscr{B})}(g^{-1}(M)) = \bigvee\{H \leq g^{-1}(M)$: $\forall$ homomorphism h: $X \to Y$ with $X \in \mathscr{B}$ and $h^{-1}(g^{-1}(M))) \neq X$, $H \cap h(X) = \{e_Y\}\} \geq \bigvee\{g^{-1}(N)$: $N \leq M$ and $\forall$ homomorphism h: $X \to Y$ with $X \in \mathscr{B}$ and $h^{-1}(g^{-1}(M)) \neq X$, $g^{-1}(N) \cap h(X) = \{e_Y\}\}$. Again, as in previous occasions, the fact it has not been proved that inverse images and suprema commute in **Grp** prevents to carry on with the proof.

Theorem 3.33. Under Conjecture 3.31 the diagram $IN(\mathbf{Grp}) \underset{G}{\overset{C}{\rightleftarrows}} S(\mathbf{Grp})^{op}$ is a Galois connection.

Proof: Let $\mathscr{B} \subseteq \mathbf{Grp}$. We need to show that $C(G(\mathscr{B})) \preceq \mathscr{B}$ in $S(\mathbf{Grp})^{op}$, that is $C(G(\mathscr{B})) \supseteq \mathscr{B}$ in $S(\mathbf{Grp})$. Let $Y \in \mathscr{B}$ and let M be a proper subgroup of Y. Consider id_Y: $Y \to Y$. Clearly $id_Y^{-1}(M) \neq Y$. Now, the only $N \leq M$ that satisfies $N \cap f(Y) = \{e_Y\}$ for $f = id_Y$ is $\{e_Y\}$. Hence, $i_{G(\mathscr{B})}(M) = \{e_Y\}$, that is Y is $G(\mathscr{B})$-indiscrete. Consequently, $\mathscr{B} \subseteq C(G(\mathscr{B}))$, that is $C(G(\mathscr{B})) \preceq \mathscr{B}$ in $S(\mathbf{Grp})^{op}$.

Let $M \leq Y$, let $X \in C(I)$ and let f: $X \to Y$ be a homomorphism such that $f^{-1}(M) \neq X$. Then $f^{-1}(i_Y(M)) \leq i_X(f^{-1}(M))$ by Definition 3.1(c). Since $f^{-1}(M) \lneqq X$, then $i_X(f^{-1}(M)) = \{e_X\}$. Then $f^{-1}(i_Y(M)) = \{e_X\}$. Now, let $z \in i_Y(M) \cap f(X)$. Then $\exists x \in X$ such that $z = f(x)$ and $x \in f^{-1}(i_Y(M)) = \{e_X\}$. Then $x = e_X$ and so $z = f(x) = f(e_X) = e_Y$. Thus we have that $i_Y(M) \cap f(X) = \{e_Y\}$. Hence we conclude that $i_Y(M)$ is one of the subgroups N's in the construction of $G(C(I))$ and so $i_Y(M) \leq i_{G(C(I))}(M)$, that is $I \sqsubseteq G(C(I))$.

To show that C: $IN(\mathbf{Grp}) \to S(\mathbf{Grp})^{op}$ is order-preserving let $I, J \in IN(\mathbf{Grp})$ such that $I \sqsubseteq J$. Let $X \in C(J)$. Then $\forall K \lneqq X$, $j_X(K) = \{e_x\}$ and so $i_X(K) \leq j_X(K) = \{e_x\}$ for every $K \lneqq X$. Thus $i_X(K) = \{e_x\}$ for every $K \lneqq X$, that is $X \in C(I)$. We conclude that $C(J) \subseteq C(I)$, that is $C(I) \preceq C(J)$.

To show that G: $S(\mathbf{Grp})^{op} \to IN(\mathbf{Grp})$ is order-preserving let $\mathscr{A}$, $\mathscr{B} \in S(\mathbf{Grp})^{op}$ such that $\mathscr{A} \preceq \mathscr{B}$, that is $\mathscr{A} \supseteq \mathscr{B}$ and let $M \leq X \in \mathbf{Grp}$. It is easily seen that if N occurs in the construction of $i_{G(\mathscr{A})}(M)$, then it also occurs in the construction of $i_{G(\mathscr{B})}(M)$. Then $i_{G(\mathscr{A})}(M) \leq i_{G(\mathscr{B})}(M)$, that is $G(\mathscr{A}) \sqsubseteq G(\mathscr{B})$.

Hence we proved that $IN(\mathbf{Grp}) \underset{G}{\overset{C}{\rightleftarrows}} S(\mathbf{Grp})^{op}$ is a Galois connection.

Let consider the functions $S(\mathbf{Grp}) \underset{F^*}{\overset{F_*}{\rightleftarrows}} S(\mathbf{Grp})^{op}$ defined for every $\mathscr{A}$, $\mathscr{B} \subseteq \mathbf{Grp}$ by $F_*(\mathscr{B}) = \{Y \in \mathbf{Grp} :$ every homomorphism $h : X \to Y$ with $X \in \mathscr{B}$ is constant$\}$ and $F^*(\mathscr{A}) = \{X \in \mathbf{Grp} :$ every homomorphism $h : X \to Y$ with $Y \in \mathscr{A}$ is constant$\}$. It is easy to verify that $S(\mathbf{Grp}) \underset{F^*}{\overset{F_*}{\rightleftarrows}} S(\mathbf{Grp})^{op}$ forms a Galois connection and since it is a special case of a more general one that appeared in [H], we omit its proof. We observe that in Topology a similar Galois connection was used to define notions of connectedness and disconnectedness with respect to subclasses of topological spaces (cf. [AW]) and in Algebra was used to define torsion theories (cf. [L]).

Under Conjectures 3.23 and 3.31 we can prove that the Galois connection $S(\mathbf{Grp}) \underset{F^*}{\overset{F_*}{\rightleftarrows}} S(\mathbf{Grp})^{op}$ factors via the two Galois connections $S(\mathbf{Grp}) \underset{C}{\overset{G}{\rightleftarrows}} IN(\mathbf{Grp})^{op}$ and $IN(\mathbf{Grp})^{op} \underset{T}{\overset{D}{\rightleftarrows}} S(\mathbf{Grp})^{op}$. This is shown in the following.

Theorem 3.34. Under Conjectures 3.23 and 3.31 we have the following commutative diagram of Galois connections:

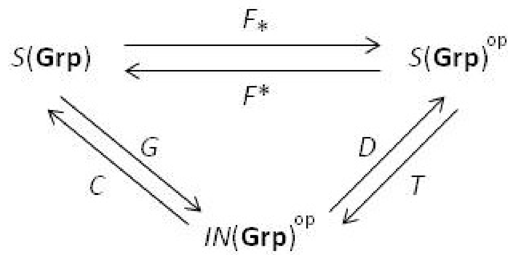

Figure 1: Commutative Diagram of Galois Connections

Proof: We start by showing that $F^* = C \circ T$. Let $\mathscr{A} \subseteq \mathbf{Grp}$ and let $X \in F^*(\mathscr{A})$. If X does not have any proper subgroup $M \neq \{e_X\}$ then X trivially belongs to $C(T(\mathscr{A}))$. Now, let $M \lneqq X$. Since every homomorphism $f\colon X \to Y$ with $Y \in \mathscr{A}$ is constant, from Conjecture 3.23, the family $\{f^{-1}(N)|$ $f\colon X \to Y$ homomorphism, $Y \in \mathscr{A}$, $N \leq Y$ and $f^{-1}(N) \leq M\}$ is empty and so its supremum $i_{T(\mathscr{A})}(M) = \{e_X\}$. Hence, $X \in C(T(\mathscr{A}))$ and so $F^*(\mathscr{A}) \subseteq C(T(\mathscr{A}))$.

Conversely, let $X \in C(T(\mathscr{A}))$ and let $f\colon X \to Y$ be a homomorphism with $Y \in \mathscr{A}$. If X does not have any proper subgroup $M \lneqq X$, then $X = \{e_X\}$ and so f is constant, that is $X \in F^*(\mathscr{A})$. Now, let $M \lneqq X$. Since $X \in C(T(\mathscr{A}))$, $i_{T(\mathscr{A})}(M) = \{e_X\}$. This implies that the family $\mathscr{U} = \{f^{-1}(N)|$ $f\colon X \to Y$ homomorphism, $Y \in \mathscr{A}$, $N \leq Y$ and $f^{-1}(N) \leq M\}$ is either empty or not.

Case 1: $\mathscr{U} = \emptyset$. For $M = Kerf$ we have that $i_{T(\mathscr{A})}(Kerf) = Kerf$. This is a contradiction with $\mathscr{U} = \emptyset$ unless $Kerf = X$, that is f is constant and so $X \in F^*(\mathscr{A})$.

Case 2: $\mathscr{U}$ is not empty. Since $i_{T(\mathscr{A})}(M) = \bigvee \mathscr{U} = \{e_X\}$, then for every $N \leq Y$, $Y \in \mathscr{A}$ we have that $f^{-1}(N) = \{e_X\}$. This implies that $f^{-1}(\{e_Y\}) = \{e_X\}$, that is f is a monomorphism. Consequently $M = f^{-1}(f(M))$ and so $i_{T(\mathscr{A})}(M) = M$, that is a contradiction with $i_{T(\mathscr{A})}(M) = \{e_X\}$. Hence $C(T(\mathscr{A})) \subseteq F^*(\mathscr{A})$. This together with the previous inclusion yields $F^* = C \circ T$.

To prove that $F_* = \triangle \circ G$, it is enough to observe that since the composition of two Galois connections is a Galois connection (cf. Proposition 2.17) and we already proved that $F^* = C \circ T$, then from Proposition 2.18 we conclude that $F_* = \triangle \circ G$. This completes the proof.

4 Conclusions and Future Work

The main aim of this work was to apply to the category of groups the notion of interior operator introduced in [V] with the purpose of trying to assess how this notion performs in this specific environment compared to the one of closure operator. Particular emphasis was given to the notions of connectedness and disconnectedness introduced in [CR]. Our main research project was to see how much of the theory presented in [CR] could be exported to the group environment.

The notion of interior operator in groups was introduced and a few examples presented. The existence of infima and suprema of interior operators was proved. Notions of indiscrete and discrete groups of an interior operator were introduced, followed by the notions of connected and disconnected groups with respect to an interior operator. Finally, three Galois connections were presented, even though the existence of one of them was only guaranteed through an assumption. Under two assumptions, a commutative diagram of Galois connections was constructed.

The conclusion we can draw from this work is the following. We have had a certain degree of difficulty while trying to recreate in the group environment the work already done in topology in [CR]. In particular, a specific expression for the supremum of a family of interior operators could not be found (cf. Proposition 3.8 and Remark 3.9). A practical description of the interior operator $T(\mathscr{A})$ in Theorem 3.21 could not be found and, as a consequence, Assumption 3.23 was made. Furthermore, we were forced to make the further Assumption 3.31 that would guarantee the existence of the Galois connection in Theorem 3.33. After these considerations we feel like concluding that the notion of interior operator in groups does not seem to perform as well as the same notion in topology and it does not perform as well as the notion of closure operator in groups either (cf. [C_1]).

We have been able to identify one of the main reasons for the lack of performance of interior operators in algebra. Precisely, in topology inverse images and suprema commute since the supremum of a family of subsets is just their union. Unfortunately, in algebra this does not seem to be true due to the fact that the supremum of a family of subgroups is not just the union of them but it is the subgroup generated by their union. This is a main inconvenience that prevented us to find a concrete description of the supremum of a family of interior operators and as a consequence, forced us to make Assumptions 3.23 and 3.31. So, based on the above considerations, possible future work on this subject would be to analyze in more details this apparent lack of commutativity between inverse images and suprema. In other words one should either try to prove that they do commute or find a counterexample that would show the opposite. The first scenario would allow to transform the two assumptions made in this work into actual results. However, we are more inclined to think that the second option is more likely to be true.

5 References

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