HYPERBOLICITY AND GENUINE NONLINEARITY CONDITIONS FOR CERTAIN P-SYSTEMS OF CONSERVATION LAWS, WEAK SOLUTIONS AND THE ENTROPY CONDITION

By

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We consider a p-system of conservation laws that emerges in one dimensional elasticity theory. Such system is determined by a function W, called strain-energy function. We consider four forms of W which are known in the literature. These are St.Venant-Kirchhoff, Ogden, Kirchhoff modified, Blatz-Ko-Ogden forms. In each of those cases we determine the conditions for the parameters ρ_0, μ and λ, under which the corresponding system is hyperbolic and genuinely nonlinear.

We establish what it means a weak solution of an initial and boundary value problem. Next we concentrate on a particular problem whose weak solution is obtained in a linear theory by means of D'Alembert's formula. In cases under consideration the p-systems are nonlinear, so we solve them employing Rankine-Hugoniot conditions. Finally we ask if such solutions satisfy the entropy condition. For a standard entropy function we provide a complete answer, except of the Blatz-Ko-Ogden case. For a general strictly convex entropy function the result is that for the initial value of velocity function near zero these solutions satisfy the entropy condition, under the assumption of hyperbolicity and genuine nonlinearity.

Resumen de Disertación Presentado a Escuela Graduada
de la Universidad de Puerto Rico como requisito parcial de los
Requerimientos para el grado de Maestría en Ciencias

CONDICIONES DE HIPERBOLICIDAD Y NOLINEALIDAD GENUINA PARA CIERTOS P-SISTEMAS DE LEYES DE CONSERVACIÓN, SOLUCIONES DÉBILES Y CONDICIÓN DE ENTROPIA

Por

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Mayo 2010

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Nosotros consideramos un p-sistema de leyes de conservación que surge de la teoría de elasticidad unidimensional. Tal sistema es determinado por una función W, llamada función de energía. Consideramos cuatro formas de W cuales son conocidas en la literatura. Estos son los modelos de St.Venant-Kirchhoff, Ogden, Kirchhoff modificado, Blatz-Ko-Ogden. En cada uno de estos casos determinamos las condiciones para los parámetros μ, λ y f, bajo el cual el correspondiente sistema es hiperbólico y genuinamente no lineal.

Establecemos que significa una solución débil de un problema de valor inicial y de frontera. Después nos concentramos en un problema particular cuya solución débil es obtenida en teoría lineal por medio de la formula de Alembert. En nuestro caso bajo consideración los p-sistemas son no lineales, así que empleamos las condiciones de Rankine-Hugoniot para solucionarlos. Finalmente nos preguntamos si tales soluciones satisfacen la condición de entropía. Para una función de entropía estándar probamos una completa respuesta, excepto del caso de Blatz-Ko-Ogden. Para una

función de entropía general estrictamente convexa, el resultado es que para el valor inicial de la función velocidad cerca de cero estas soluciones satisfacen la condición de entropía, bajo la restricción de hiperbolicidad y no linealidad genuina.

I dedicate this thesis to:

my parents Edgardo and Nicolasa,

my sisters Maura and Odila

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IBVP	Initial and boundary value problem.

LIST OF SYMBOLS

X^A	Position of a material point of a body in reference configuration.
$\phi^i(X,t)$	Position of the same material point at an instant t.
$\mathbf{U}$	Displacement vector.
F^i_A	Deformation gradient
C	Deformation tensor
I_1, I_2, I_3	Invariants of C.
W	Strain-energy function.
S^{AB}	The second Piola-Kirchhoff tensor.
P^{iA}	The first Piola tensor.
ρ_0	Mass density in reference configuration
λ, μ	The Lamé constants.
f, β	Parameters in the model of Blatz-Ko and Ogden.
δ_{ij}	Kronecker delta.
$\mathbb{M}^{m\times n}$	The space of real $m \times n$ matrices with m rows and n columns.
Φ, Ψ	Entropy/entropy flux pair.
$\mathbb{R}$	Set of real numbers.
$\mathbb{R}^m$	Set of m-tuples of real numbers.
Df	Gradient matrix of f.
D^2f	Hessian matrix of f.
C_0^∞	Space of infinitely differentiable functions with compact support.

CHAPTER 1
INTRODUCTION

The mathematical theory of hyperbolic systems of conservation laws were started by Eberhardt Hopf in 1950, followed in a series of studies by Olga Oleinik, Peter D. Lax and James Glimm [9] . The class of conservation laws is a very important class of partial differential equations because as their name indicates, they include those equations that model conservation laws of physics (mass, momentum, energy, etc.).

A conservation law stipulates that the time rate of change in the amount of an extensive quantity stored inside any subdomain of the body is balanced by the rate of flux of this quantity through the boundary of the subdomain together with the rate of its production inside the subdomain. In the absence of production, a balanced extensive quantity is conserved.

The special feature that renders continuum physics amenable to analytical treatment is that, under quite natural assumptions, statements of gross balance, as above, reduce to field equations, i.e., partial differential equations in divergence form [16].

The collection of balance laws or “conservation laws” in force demarcates and identifies particular continuum theories, such as mechanics, thermomechanics, electrodynamics, and so on.

Systems of balance laws may be elliptic, typically in statics; hyperbolic, in dynamics, for media with “elastic” response; mixed elliptic-hyperbolic, in statics or dynamics, when the medium undergoes phase transitions; parabolic or mixed parabolic hyperbolic in the presence of viscosity, heat conductivity or other diffusive mechanisms.

As important examples of hyperbolic systems of balance laws arising in continuum physics we have: Euler's equations for compressible gas flow, the one dimensional shallow water equations [1], Maxwell's equations in nonlinear dielectrics, Lundquist's equations of magnetohydrodynamics and Boltzmann equation in thermodynamics [16] and equations of elasticity [7].

One of the main motivations of the theory of hyperbolic systems is that they describe for the most part real physical problems, because they are consistent with the fact that the physical signals have a finite propagation speed [7]. Such systems even with smooth initial conditions may fail to have a solution for all time, in such cases we have to extend the concept of classical solutions to the concept of a weak solution or generalized solution [1].

In the case of hyperbolic systems, the notion of weak solution based on distributions does not guarantee uniqueness, and it is necessary to devise admissibility criteria that will hopefully single out a unique weak solution. Several such criteria have indeed been proposed, motivated by physical and/or mathematical considerations. It seems that a consensus has been reached on this issue for such solutions, they are called entropy conditions [5]. Nevertheless, to the question about existence and uniqueness of generalized solutions subject to the entropy conditions, the answer is, in general, open. For the scalar conservation law, the questions existence and uniqueness are basically settled [1]. For genuinely nonlinear systems, existence (but not uniqueness) is known for initial data of small total variation [2]. Some of the main contributors to the field are Lax , Glimm , DiPerna, Tartar, Godunov, Liu, Smoller and Oleinik [4], [12], [13] .

All of this motivates us to study systems of conservation laws that emerge in the theory of elasticity. These systems are determined by constitutive relations between the stress and strain. For hyperelastic materials, the constitutive relations can be written in a simpler form. Now the stress is determined by a scalar function of the strain called the strain-energy function W. A further simplification of a stress-strain relation is obtained for isotropic materials.

In applications some specific strain-energy functions are used; in our work we consider four different forms of W. In all our studies we restrict ourselves to the case of one dimensional elasticity.

The first important question that arises is the following: given the function W, is the corresponding system of PDE's hyperbolic? By answering it we can assess how good the model corresponding to that particular W is.

There exists also another important condition called genuine nonlinearity condition, which is related to the entropy condition, [2]. According to our previous remarks the entropy condition can be considered a physical one. This implies an importance of genuine nonlinearity condition as well. For that reason our second question is about the validity of that particular condition for the models under study.

Our third important question is how manageable is the entropy condition. To put it differently, given a weak solution of the elasticity system, can we conclude if it is or not an entropy solution? In general, except of the linear case, it is not easy to answer that question, because in the entropy condition there appear two functions: entropy and entropy-flux, which satisfy a given nonlinear system of PDE's, the first of them is convex and otherwise they are arbitrary.

For this reason we restrict ourselves to study the entropy condition for a relatively simple weak solutions, which correspond to a well understood physical situation of what can be called a compression shock. Such solutions are obtained easily in linear case by means of D'Alembert's formula and by analogy in nonlinear case, employing the Rankine-Hugoniot conditions. If for a given model (W function) such solution does not satisfy the entropy condition, we can consider the model as inadequate to describe the compression shock.

In this work we give answers to all mentioned above questions. The obtained results do not appear in the reviewed literature.

It has to be added also that the concept of a weak solution is well known in the literature. For example in [1] one can find a definition of a weak solution of an initial value problem for a system of conservation laws in two variables. Using a general idea of that concept we define what it means to be a weak solution of an initial and boundary value problem for p-system. This definition does not appear explicitly in the reviewed literature.

We discuss the topics in the following order. In Chapter 2 we present some important concepts that are required in the remaining chapters of this work. This includes the concepts of conservation laws, hyperbolic system, weak solution, Rankine-Hugoniot condition, genuine nonlinearity, entropy/entropy-flux pair. Next we give a brief presentation of basic concepts of the theory of elasticity, such as deformation gradient, deformation tensor, second Piola-Kirchhoff stress tensor, first Piola-tensor and a system of partial differential equations for elasticity. We also present four forms of W (strain-energy function) appearing in the theory of elasticity, to model a behavior of certain materials. We refer to them as: St.Venant-Kirchhoff, Kirchhoff modified, Ogden and Blatz-Ko-Ogden functions.

In Chapter 3 we consider one dimensional reduction of the system of partial differential equations for elasticity, which depends on the strain-energy function W and results in a p-system. Also we introduce the notions of hyperbolicity, no interpenetration of matter and genuine nonlinearity.

In Chapter 4 we provide the concept of weak solutions for various versions of an IBVP (initial and boundary value problem) for a p-system, including a particular case of IBVP, $IBVP_{V_0}$ and we find its solutions employing the Rankine-Hugoniot conditions, we denote such solution by $S(V_0)$.

In Chapter 5 we discuss also the notions of an entropy/entropy-flux pair for a p-system, entropy condition, entropy condition for a solution of $IBVP_{V_0}$ and standard entropy function. We also establish the importance of the requirements of hyperbolicity(strict) and genuine nonlinearity, as being essential in proving if a weak solution is an entropy solution.

It is important to note that for the linear case one can obtain an entropy/entropy-flux pairs and verify that $S(V_0)$ satisfies the entropy condition. This motivates us to ask if the same is true for nonlinear case. In Chapters 6, 7 and 8 we show the results concerning hyperbolicity and genuine nonlinearity for the models under consideration and the entropy condition corresponding to a standard entropy function for a solution of $IBVP_{V_0}$.

In the course of this work we use extensively the Maple software to perform symbolic computation and to study the graphs of functions.

CHAPTER 2
PRELIMINARIES

2.1 Conservation laws and related concepts

We begin this chapter with some essential definitions, that we will use in the course of this work.

Definition 2.1.1. *A conservation law asserts that the change in the total amount of a physical entity contained in any bounded region $G \subset \mathbb{R}^n$ of space is due to the flux of that entity across the boundary of G. In particular, the rate of change is*

$$\frac{d}{dt}\int_G \mathbf{u}dX = -\int_{\partial G} \mathbf{F}(\mathbf{u})\mathbf{n}dS, \tag{2.1}$$

where $\mathbf{u} = \mathbf{u}(X,t) = (u^1(X,t), \dots, u^m(X,t))$ $(X \in \mathbb{R}^n, t \geq 0)$ measures the density of the physical entity under discussion, the vector $\mathbf{F} : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ describes its flux and $\mathbf{n}$ is the outward normal to the boundary ∂G of G. Here $\mathbf{u}$ and $\mathbf{F}$ are C^1 functions. Rewriting (2.1), we deduce

$$\int_G \mathbf{u}_t dX = -\int_{\partial G} \mathbf{F}(\mathbf{u})\mathbf{n}dS = -\int_G div\mathbf{F}(\mathbf{u})dX. \tag{2.2}$$

As the region $G \subset \mathbb{R}^n$ was arbitrary, we derive from (2.2) this initial-value problem for a general system of conservation laws:

$$\begin{cases} \mathbf{u}_t + div\mathbf{F}(\mathbf{u}) = 0 & in\ \mathbb{R}^n \times (0,\infty) \\ \mathbf{u} = g & on\ \mathbb{R}^n \times \{t=0\} \end{cases} \tag{2.3}$$

where $g = (g^1, \ldots, g^m)$ *is a given function describing the initial distribution of* $\mathbf{u} = (u^1, \ldots, u^m)$. *In particular, the initial-value problem for a system of conservation laws in one space dimension,* $n = 1$, *takes the following form*

$$\mathbf{u}_t + \mathbf{F}(\mathbf{u})_X = 0 \quad \textit{in } \mathbb{R} \times (0, \infty) \tag{2.4}$$

with initial condition given by

$$\mathbf{u}(X,t) = g \quad \textit{on } \mathbb{R} \times \{t = 0\} \tag{2.5}$$

where $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ *and* $g : \mathbb{R} \to \mathbb{R}^m$ *are given and* $\mathbf{u} : \mathbb{R} \times [0, \infty) \to \mathbb{R}^m$ *is the unknown,* $\mathbf{u} = \mathbf{u}(X,t)$ *[1].*

For C^1 functions the conservation law (2.4) is equivalent to

$$\mathbf{u}_t + \mathbf{B}(\mathbf{u})\mathbf{u}_X = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \tag{2.6}$$

where $\mathbf{B} : \mathbb{R}^m \to \mathbb{M}^{m \times m}$ is given by $B(z) = DF(z)$, for $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$, where

$$D\mathbf{F}(z) = \begin{pmatrix} F^1_{z_1} & \cdots & F^1_{z_m} \\ \vdots & \ddots & \vdots \\ F^m_{z_1} & \cdots & F^m_{z_m} \end{pmatrix}. \tag{2.7}$$

Definition 2.1.2. *If for each* $z \in \mathbb{R}^m$ *the eigenvalues of* $\mathbf{B}(z)$ *are real and distinct, we call the system (2.6) strictly hyperbolic [1].*

Definition 2.1.3. *A system of conservation laws (2.6) is said to be genuinely non-linear in a region* $\Omega \subseteq \mathbb{R}^n$ *if*

$$\nabla \lambda_k \cdot \mathbf{r}_k \neq 0,$$

for $k = 1, 2, \ldots, n$ *at all points in* Ω, *where* $\lambda_k(z)$ *are the eigenvalues of* $\mathbf{B}(z)$, *with corresponding eigenvectors* $\mathbf{r}_k(z)$, *[2].*

Definition 2.1.4. *The p-system is a conservation law being this collection of two equations:*

$$\begin{cases} u_t^2 - p(u^1)_X = 0 \ \textit{(Newton's law)} \\ \\ \quad u_t^1 - u_X^2 = 0 \ \textit{(compatibility condition)} \end{cases} \tag{2.8}$$

in $\mathbb{R} \times (0, \infty)$, *where* $p : \mathbb{R} \to \mathbb{R}$ *is given. Here* $F(z) = (-p(z_1), -z_2)$ *for* $z = (z_1, z_2)$ *[1].*

Definition 2.1.5. *A weak solution of (2.4) is a function* $\mathbf{u} \in L^\infty(\mathbb{R} \times (0, \infty); \mathbb{R}^m)$ *such that*

$$\int_0^\infty \int_{-\infty}^\infty (\mathbf{u} \cdot \phi_t + \mathbf{F}(\mathbf{u}) \cdot \phi_X) dX dt + \int_{-\infty}^\infty (\mathbf{g} \cdot \phi)|_{t=0} \, dX = 0$$

for every smooth $\phi : \mathbb{R} \times [0, \infty) \to \mathbb{R}^m$, *with compact support [1].*

Definition 2.1.6. ***(Rankine-Hugoniot condition)*** *Let* N *be an open neighborhood in the open upper half of the* Xt*-plane, and suppose a smooth curve* $C : t \to \hat{X}(t)$ *divides* N *into two pieces,* N^l *and* N^r, *lying to the left and right of the curve, respectively. Let* $\mathbf{u}$ *be a weak solution of (2.4) such that*

1. $\mathbf{u}$ *is a classical solution of (2.4) in both* N^l *and* N^r,
2. $\mathbf{u}$ *undergoes a jump discontinuity* $[\mathbf{u}] := \mathbf{u_l} - \mathbf{u_r}$ *at the curve* C,

   and
3. *the jump* $[\mathbf{u}]$ *is continuous along* C.

For any $\mathbf{p} \in C$, *where* $\mathbf{p} = \hat{X}(t)$, *let* $\sigma := \hat{X}'(t)$ *be the slope of* C *at* $\mathbf{p}$. *Then the following relation, called Rankine-Hugoniot condition, holds between the curve and the jumps:*

$$[\mathbf{F}(\mathbf{u})] = \sigma[\mathbf{u}], \tag{2.9}$$

where $[\mathbf{F}(\mathbf{u})] := F(u_l) - F(u_r)$ *[2].*

Definition 2.1.7. *Let* C *be a set in a real or complex vector space.* C *is said to be convex if, for all* x *and* y *in* C *and all* t *in the interval* $[0, 1]$, *the point* $(1 - t)x + ty$ *is in* C.

Definition 2.1.8. *Let $K \subset \mathbb{R}^m$ be a convex set. Then we say a mapping $f : K \to \mathbb{R}$ is convex if for every $u, v \in K$, we have*

$$f(\theta u + (1-\theta)v) \leq \theta f(u) + (1-\theta)f(v),$$

for all $\theta \in [0,1]$ [1].

Definition 2.1.9. *Let $K \subset \mathbb{R}^m$ be a convex set. Then we say a mapping $f : K \to \mathbb{R}$ is strictly convex if for every $u, v \in K$, $u \neq v$, we have*

$$f(\theta u + (1-\theta)v) < \theta f(u) + (1-\theta)f(v),$$

for all $\theta \in (0,1)$.

Remark 1. *If f is C^2, then f is convex if and only if $D^2 f \geq 0$(positive semidefinite), where $D^2 f$ is the Hessian matrix of f,*

$$D^2 f(z) = \begin{pmatrix} \frac{\partial^2 f}{\partial z_1^2} & \cdots & \frac{\partial^2 f}{\partial z_1 \partial z_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial z_m \partial z_1} & \cdots & \frac{\partial^2 f}{\partial z_m^2} \end{pmatrix},$$

for all $z = (z_1, \ldots, z_m) \in K$.

Remark 2. *If f is C^2, then f is strictly convex if and only if $D^2 f > 0$(positive definite).*

Definition 2.1.10. *Two smooth functions $\Phi, \Psi : \mathbb{R}^m \to \mathbb{R}$ comprise an entropy/ entropy-flux pair for the conservation law (2.4) provided Φ is convex and*

$$D\Phi(z)DF(z) = D\Psi(z)$$

for $z \in \mathbb{R}^m$ [1].

Remark 3. *Here $D\Phi(z)$ and $D\Psi(z)$ must be understood as row vectors.*

2.2 Basic notions of Elasticity Theory

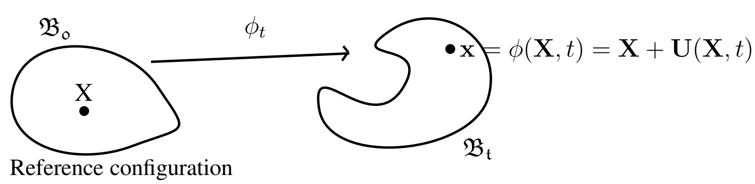
2.2.1 Motion and Configurations

We consider a continuous body which occupies a connected open subset of a three-dimensional Euclidean point space, and we refer to such a subset as a configuration of the body. We identify an arbitrary configuration as a reference configuration and denote this by $\mathfrak{B}_o$. Let points in $\mathfrak{B}_o$ be labelled by their position vectors $\mathbf{X} = (X^1, X^2, X^3)$, where X^1, X^2 and X^3 are coordinates relative to an arbitrary chosen Cartesian orthogonal coordinate system. Now suppose that the body is deformed from $\mathfrak{B}_o$ so that it occupies a new configuration, which is denoted by $\mathfrak{B}_t$. We refer to $\mathfrak{B}_t$ as the deformed configuration of the body. The deformation is represented by the mapping $\phi_t : \mathfrak{B}_o \to \mathfrak{B}_t$ which takes points $\mathbf{X}$ in $\mathfrak{B}_o$ to points $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathfrak{B}_t$, where x_1, x_2 and x_3 are coordinates relative to the same Cartesian orthogonal coordinate system as X^1, X^2 and X^3. Thus, the position vector of the point $\mathbf{X}$ in $\mathfrak{B}_t$, which is denoted by $\mathbf{x}$, is

$$\mathbf{x} = \phi(\mathbf{X}, \mathbf{t}) \equiv \phi_{\mathbf{t}}(\mathbf{X}).$$

The mapping ϕ is called the deformation from $\mathfrak{B}_o$ to $\mathfrak{B}_t$. We require ϕ_t to be sufficiently smooth, orientation preserving and invertible. The last two requirements mean physically, that no interpenetration of matter occurs. Now we define the displacement vector $\mathbf{U}$, of any point $\mathbf{X} \in \mathfrak{B}_o$, as

$$\mathbf{U}(\mathbf{X}, t) = \mathbf{x} - \mathbf{X}.$$



2.2.2 Deformation gradient, deformation tensor, strain-energy function and time evolution of an elastic body

Now, we introduce some basic definitions of Elasticity theory, namely: deformation gradient, deformation tensor, second Piola-Kirchhoff stress tensor, first Piola-tensor [7]. We restrict our discussion to hyperelastic, homogeneous and isotropic materials.

Definition 2.2.3. *Deformation gradient, F^a_A where $a, A \in \{1,2,3\}$, is*

$$F^a_A(X,t) = \frac{\partial \phi^a}{\partial X^A}.$$

Definition 2.2.4. *Deformation tensor (Green tensor), C is defined by*

$$C = F^T F$$

or componentwise by

$$C_{AB} = \delta_{ij} F^i_A F^j_B = F^1_A F^1_B + F^2_A F^2_B + F^3_A F^3_B,$$

where $A, B \in \{1,2,3\}$.

Definition 2.2.5. *Principal invariants of C are the following functions of C:*

$$I_1 = tr(C),$$

$$I_2 = (\det(C))tr(C^{-1}),$$

$$I_3(C) = \det(C).$$

These invariants are understood in the following way: they remain unchanged under similarity transformations of C.

Definition 2.2.6. *(stress) Stress is a measure of the average force per unit area of a surface within a deformable body on which internal forces act. For the particular case of hyperelastic and isotropic materials, there exists a constitutive relation between stress, called the Second Piola-Kirchhoff stress tensor S^{AB}, and the deformation* *tensor C (i.e., the strain) [7], given by*

$$S^{AB} = 2\left\{\frac{\partial W}{\partial I_1}G^{AB} + \left(\frac{\partial W}{\partial I_2}I_2 + \frac{\partial W}{\partial I_3}I_3\right)C^{-1} - \frac{\partial W}{\partial I_2}I_3C^{-2}\right\},$$

where G^{AB} is Kronecker's delta and W is the strain-energy function. W is a measure of the energy stored in the material as a result of deformation and is therefore sometimes called the elastic stored energy function. For hyperelastic, homogeneous and isotropic materials, W depends on C only through the principal invariants I_1, I_2, I_3, [18].

We consider the following four forms of W, [7]:

- St.Venant-Kirchhoff

$$W = \frac{\lambda}{8}(I_1 - 3)^2 + \frac{\mu}{4}(I_1^2 - 2I_2 - 2I_1 + 3) \tag{2.10}$$

- Kirchhoff modified

$$W = \frac{\lambda}{8}(\ln I_3)^2 + \frac{\mu}{4}(I_1^2 - 2I_2 - 2I_1 + 3) \tag{2.11}$$

- Ogden

$$W = \frac{\mu}{2}(I_1 - 3 - 2\ln(\sqrt{I_3})) + \frac{\lambda}{2}(\sqrt{I_3} - 1)^2 \tag{2.12}$$

- Blatz-Ko-Ogden

$$W = f\frac{\mu}{2}[(I_1 - 3) + \frac{1}{\beta}(I_3^{-\beta} - 1)] + (1 - f)\frac{\mu}{2}\left[\frac{I_2}{I_3} - 3 + \frac{1}{\beta}(I_3^{\beta} - 1)\right], \tag{2.13}$$

We can see that the functions (2.10)-(2.12) depend on two parameters: Lamé moduli λ and μ, where $\lambda, \mu > 0$. In (2.13) $\beta = \frac{\lambda}{2\mu}$ and this W depends also on a parameter f restricted by $0 < f < 1$.

Definition 2.2.7. *The first Piola-tensor*

$$P^{iA} = F^i_B S^{BA} = F^i_1 S^{1A} + F^i_2 S^{2A} + F^i_3 S^{3A},$$

where $i, A, B \in \{1, 2, 3\}$.

Finally, the components of the mapping

$$\phi(\mathbf{X}, t) = (\phi^1(\mathbf{X}, t), \phi^2(\mathbf{X}, t), \phi^3(\mathbf{X}, t))$$

are subject to the following system of PDE's, describing the evolution of an elastic body:

$$\rho_0 \frac{\partial^2 \phi^i}{\partial t^2} = \frac{\partial P^{iA}}{\partial X^A}. \tag{2.14}$$

Here $\rho_0 = \rho_0(\mathbf{X})$ is the mass density in reference configuration assumed further to be constant.

CHAPTER 3
ONE DIMENSIONAL REDUCTIONS FOR CERTAIN MODELS OF ELASTIC MATERIALS

Assuming that there is a motion of particles only in the direction of X^1-axis, we obtain

$$\begin{cases} \phi^1(\mathbf{X}, t) = X^1 + U(X^1, t) \\ \phi^2(\mathbf{X}, t) = X^2 \\ \phi^3(\mathbf{X}, t) = X^3. \end{cases} \tag{3.1}$$

Then $F^i_A, C_{AB}, C^{-1}_{AB}, I_1, I_2$ and I_3 become

$$F^i_A = \begin{pmatrix} \frac{\partial\phi^1}{\partial X^1} & \frac{\partial\phi^1}{\partial X^2} & \frac{\partial\phi^1}{\partial X^3} \\ \frac{\partial\phi^2}{\partial X^1} & \frac{\partial\phi^2}{\partial X^2} & \frac{\partial\phi^2}{\partial X^3} \\ \frac{\partial\phi^3}{\partial X^1} & \frac{\partial\phi^3}{\partial X^2} & \frac{\partial\phi^3}{\partial X^3} \end{pmatrix} = \begin{pmatrix} \frac{\partial\phi^1}{\partial X^1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

$$C_{AB} = \begin{pmatrix} (F^1_1)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.3}$$

$$C^{-1}_{AB} = \begin{pmatrix} 1/(F^1_1)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.4}$$

$$I_1 = 2 + (F^1_1)^2 \tag{3.5}$$

$$I_2 = 2(F^1_1)^2 + 1 \tag{3.6}$$

$$I_3 = (F^1_1)^2. \tag{3.7}$$

Therefore the system (2.14) becomes

$$\rho_0 \frac{\partial^2 \phi^i}{\partial t^2} = \frac{\partial (P^{iA})}{\partial X^A}.$$

More specifically,

$$\begin{aligned} \rho_0 \frac{\partial^2 \phi^1}{\partial t} &= P,^{11}_1 + P,^{12}_2 + P,^{13}_3 \\ \rho_0 \frac{\partial^2 \phi^2}{\partial t} &= P,^{21}_1 + P,^{22}_2 + P,^{23}_3 \\ \rho_0 \frac{\partial^2 \phi^3}{\partial t} &= P,^{31}_1 + P,^{32}_2 + P,^{33}_3. \end{aligned} \tag{3.8}$$

Here

$$P^{11} = F^1_1 S^{11} + F^1_2 S^{21} + F^1_3 S^{31} = F^1_1 S^{11}$$

$$P^{12} = P^{21} = P^{13} = P^{31} = P^{23} = P^{32} = 0$$

$$P^{22} = S^{22}$$

$$P^{33} = S^{33}$$

and

$$\frac{\partial \phi^1}{\partial t} = \frac{\partial U}{\partial t},$$
$$\frac{\partial \phi^2}{\partial t} = 0,$$
$$\frac{\partial \phi^3}{\partial t} = 0.$$

Consequently (3.8) is reduced to one equation, which after denoting X^1 by X and putting

$$P = \frac{P^{11}}{\rho_0}, \tag{3.9}$$

reads

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial P}{\partial X}. \tag{3.10}$$

Setting

$$V = \frac{\partial U}{\partial t}$$

and

$$\Gamma = \frac{\partial U}{\partial X}$$

one obtains a p-system of first order PDE's:

$$\begin{cases} V_t - (P(\Gamma))_X & = 0 \\ \Gamma_t - V_X & = 0 \end{cases} \tag{3.11}$$

Remark 4. *Under the assumption (3.1) the requirement of no interpenetration of matter means that $\phi_X > 0$, i.e., $1 + \mathbf{U}_X > 0$.*

Next, we obtain explicit forms of the P functions for every model of study, (2.10)-(2.13). According to (3.9), we need to find P^{11}.

For the St.Venant Kirchhoff model, (2.10), we get

$$S^{11} = \left(\frac{\lambda + \mu}{2}\right)\left((F_1^1)^2 - 1\right),$$

so that

$$P^{11} = F_1^1 S^{11} = \left(\frac{\lambda + \mu}{2}\right)\left((F_1^1)^3 - F_1^1\right).$$

Therefore

$$P = \left[\left(\frac{\lambda + 2\mu}{2\rho_0}\right)\left(\left(1 + \frac{\partial U}{\partial X}\right)^3 - \left(1 + \frac{\partial U}{\partial X}\right)\right)\right]$$

and finally

$$P(\Gamma) = \left(\frac{\lambda + 2\mu}{2\rho_0}\right)(1+\Gamma)(2+\Gamma)\Gamma.$$

Similarly one obtains P for the other models. Here are the results.

St.Venant-Kirchhoff

$$P(\Gamma) = (\frac{\lambda + 2\mu}{2\rho_0})(1+\Gamma)(2+\Gamma)\Gamma \tag{3.12}$$

Modified Kirchhoff

$$P(\Gamma) = \frac{1}{\rho_0}\big(\mu(1+\Gamma)^3 - \mu(1+\Gamma) + \lambda\frac{\ln(1+\Gamma)}{(1+\Gamma)}\big) \tag{3.13}$$

Ogden

$$P(\Gamma) = \frac{1}{\rho_0}\big(\lambda\Gamma + \mu\frac{(2+\Gamma)\Gamma}{\Gamma+1}\big) \tag{3.14}$$

Blatz-Ko and Ogden

$$P(\Gamma) = \frac{\mu(1+\Gamma)}{\rho_0}\left\{f\left[1-(1+\Gamma)^{-2\beta-2}\right] + \frac{(1-f)}{(1+\Gamma)^4}\left[(1+\Gamma)^{2\beta+2}-1\right]\right\} \tag{3.15}$$

Proposition 3.1. *The linearization of the models (3.12)-(3.15) is given by*

$$P(\Gamma) = \frac{(\lambda+2\mu)}{\rho_0}\Gamma, \tag{3.16}$$

assuming Γ *to be near zero.*

Proof.

1. Kirchhoff

   If $|\Gamma| \ll 1$, $(1+\Gamma)(2+\Gamma)\Gamma \approx 2\Gamma$. Thus (3.16) results.

2. Modified Kirchhoff

   If $|\Gamma| \ll 1$, then

$$\mu(1+\Gamma)^3 - \mu(1+\Gamma) + \lambda\frac{\ln(1+\Gamma)}{(1+\Gamma)} \approx \mu(1+3\Gamma) - \mu(1+\Gamma) + \lambda(1-\Gamma)\Gamma$$
$$\approx (\lambda+2\mu)\Gamma,$$

   so that again (3.16) follows.

3. Ogden

   If $|\Gamma| \ll 1$, then

$$\frac{1}{\rho_0}\left(\lambda\Gamma + \mu\frac{\Gamma^2+2\Gamma}{\Gamma+1}\right) \approx \frac{1}{\rho_0}(\lambda\Gamma + 2\mu\Gamma(1-\Gamma)) \approx \frac{(\lambda+2\mu)}{\rho_0}\Gamma.$$

4. Blatz-Ko and Ogden

   If $|\Gamma| \ll 1$, then

$$(1+\Gamma)\Big[f\mu + (1-f)\mu(1+\Gamma)^{2\beta-2} - f\mu(1+\Gamma)^{-2\beta-2} - (1-f)\mu\frac{1}{(1+\Gamma)^4}\Big]$$
$$\approx (1+\Gamma)f\mu + (1-f)\mu(1+(2\beta-1)\Gamma) - f\mu(1-(1+2\beta)\Gamma) - (1-f)\mu(1-3\Gamma)$$
$$\approx 2\Gamma\mu(1+\beta).$$

   Now because of $\beta = \frac{\lambda}{2\mu}$, one obtains (3.16).

□

Definition 3.1. *If $P(\Gamma) = \frac{(\lambda+2\mu)}{\rho_0}\Gamma$, the model is called linear model.*

3.1 Hyperbolicity, no interpenetration of matter and genuine nonlinearity requirements for a p-system

The p-system (3.11) can be rewritten as

$$\mathbf{u}_t + \mathbf{B}(\mathbf{u})\mathbf{u}_X = 0 \tag{3.17}$$

where $\mathbf{u} = (V, \Gamma)$ and

$$\mathbf{B} = \begin{pmatrix} 0 & -P'(\Gamma) \\ -1 & 0 \end{pmatrix}. \tag{3.18}$$

Now the eigenvalues of $\mathbf{B}$ are $\lambda_1 = -\sqrt{P'(\Gamma)}$ and $\lambda_2 = \sqrt{P'(\Gamma)}$ with corresponding eigenvectors

$$\mathbf{r}_1 = \begin{pmatrix} \sqrt{P'(\Gamma)} \\ 1 \end{pmatrix}$$

and

$$\mathbf{r}_2 = \begin{pmatrix} -\sqrt{P'(\Gamma)} \\ 1 \end{pmatrix}$$

Remark 5. *Note that for our case of a p-system, no interpenetration of matter condition, $\phi_X > 0$, is equivalent to $\Gamma > -1$, since $\phi_X = 1 + U_X$.*

According to the *definitions* (2.1.2) and (2.1.3) we arrive at the following assertions.

1. The p-system (3.17) is strictly hyperbolic if $P' > 0$, everywhere in the domain of $P(\Gamma)$.
2. The p-system (3.17) is genuinely nonlinear in a region Ω of the domain of $P(\Gamma)$ if $P'' \neq 0$ everywhere in Ω.

   indeed it is so since

$$-\nabla\lambda_1 \cdot \mathbf{r}_1 = \nabla\lambda_2 \cdot \mathbf{r}_2 = \frac{P''(\Gamma)}{2\sqrt{P'(\Gamma)}}.$$

   By continuity of $P''(\Gamma)$ genuine nonlinearity means that $P''(\Gamma)$ is of constant sign in Ω. However we will call a p-system (3.17) genuinely nonlinear if $P'' < 0$, since this requirement plays an important role in studying entropy inequality.

We remark also that hyperbolicity condition is an essential physical requirement, since it guarantees that particles have a finite propagation speed.

CHAPTER 4
WEAK SOLUTION OF AN IBVP FOR A P-SYSTEM

In this chapter we give the concept of weak solutions for various versions of an IBVP (initial and boundary value problem), for a p-system, including a particular case of IBVP, $IBVP_{V_0}$.

We also provide notions of an entropy/entropy-flux pair and entropy condition for a solution of $IBVP_{V_0}$.

First we consider a p-system (3.11), with the following initial and boundary conditions:

$$\begin{cases} V(X,0) = f(X) \\ \Gamma(X,0) = g(X) \\ V(0,t) = h(t). \end{cases} \tag{4.1}$$

To define a weak solution of such IBVP in the first quadrant of the Xt-plane we use arbitrary C^1 functions φ, ψ and χ,

$$\varphi, \psi, \chi : [0,\infty) \times [0,\infty) \to \mathbb{R}$$

of compact supports. We refer to those functions as test functions.

Proposition 4.1. *Let f, g and h be C^1 functions on $[0,\infty)$, and let $V(X,t)$, $\Gamma(X,t)$ be C^1 functions on $[0,\infty)^2$, such that $P(\Gamma(X,t))$ is C^1 on $[0,\infty)^2$. Then the pair (V,Γ) is a classical solution of IBVP (3.11), (4.1), if and only if for all φ and ψ, with φ satisfying the condition $\varphi(0,t) = 0$, it holds*

$$\int_0^\infty \int_0^\infty (V\varphi_t - P(\Gamma)\varphi_X)dtdX + \int_0^\infty f(X)\varphi(X,0)dX = 0 \tag{4.2}$$

and

$$\int_0^\infty \int_0^\infty (\Gamma\psi_t - V\psi_X)dtdX - \int_0^\infty h(t)\psi(0,t)dt + \int_0^\infty g(X)\psi(X,0)dt = 0. \tag{4.3}$$

Proof. Indeed, assuming that (V, Γ) is a classical solution of IBVP (3.11), (4.1), we multiply the first equation in (3.11) by φ and next integrate by parts. This results in

$$-\int_0^\infty f(X)\varphi(X,0)dX - \int_0^\infty \int_0^\infty V(X,t)\varphi_t(X,t)dtdX + \int_0^\infty \int_0^\infty P(\Gamma(0,t))\varphi(0,t)dt \\ + \int_0^\infty \int_0^\infty P(\Gamma)\varphi_X(X,t)dXdt = 0, \tag{4.4}$$

since $\varphi(0,t) = 0$ the equation (4.4) is equivalent to

$$\int_0^\infty \int_0^\infty (V\varphi_t - P(\Gamma)\varphi_X)dtdX + \int_0^\infty f(X)\varphi(X,0)dX = 0.$$

which is (4.2).

Similarly, multiplying the second equation in (3.11) by ψ and integrating by parts one arrives at

$$\int_0^\infty \int_0^\infty (\Gamma_t - V_X)\psi(X,t)dXdt = 0,$$

$$-\int_0^\infty g(X)\psi(X,0)dX - \int_0^\infty \int_0^\infty \Gamma(X,t)\psi_t(X,t)dtdX + \int_0^\infty h(t)\psi(0,t)dt \\ + \int_0^\infty \int_0^\infty V(X,t)\psi_X(X,t)dXdt = 0$$

and

$$\int_0^\infty \int_0^\infty (\Gamma\psi_t - V\psi_X)dtdX - \int_0^\infty h(t)\psi(0,t)dt + \int_0^\infty g(X)\psi(X,0)dt = 0,$$

which is (4.3).

Now it remains to verify that if (V, Γ) satisfies (4.2) and (4.3) for all φ and ψ, then (V, Γ) is a classical solution of the IBVP (3.11), (4.1).

Integrating by parts the equation (4.2) and employing the condition $\varphi(0, t) = 0$, we obtain

$$-\int_0^\infty f(X)\varphi(X,0)dX + \int_0^\infty V(X,0)\varphi(X,0)dX + \int_0^\infty \int_0^\infty V_t(X,t)\varphi(X,t)dtdX \\ - \int_0^\infty \int_0^\infty P(\Gamma)_X \varphi(X,t)dXdt = 0$$

or equivalently

$$\int_0^\infty \int_0^\infty (V_t(X,t) - (P(\Gamma))_X)\varphi(X,t)dXdt + \int_0^\infty (V(X,0) - f(X))\varphi(X,0) = 0. \quad (4.5)$$

Now if we assume that φ has compact support in $(0, \infty) \times (0, \infty)$, the second term in the left-hand sized of (4.5) is zero, so that

$$\int_0^\infty \int_0^\infty (V_t(X,t) - (P(\Gamma))_X)\varphi(X,t)dXdt = 0.$$

Since that equation holds for all φ, with compact support in $(0, \infty) \times (0, \infty)$, and the integrand function is continuous, we conclude that

$$V_t(X,t) - (P(\Gamma))_X = 0.$$

in $[0, \infty) \times [0, \infty)$. Thus (4.5) is reduced to

$$\int_0^\infty (V(X,0) - f(X))\varphi(X,0)dX = 0.$$

Now because the function $V(X, 0) - f(X)$ is continuous we infer that

$$V(X,0) = f(X).$$

Similarly, integrating by parts the equation (4.3), we obtain:

$$\int_0^\infty \int_0^\infty (\Gamma_t(X,t) - V_X(X,t))\psi(X,t)dXdt + \int_0^\infty (h(t) - V(0,t))\psi(0,t)dt + \int_0^\infty (\Gamma(X,0) - g(X))\psi(X,0)dX = 0. \quad (4.6)$$

Then employ any ψ with compact support in $(0,\infty)\times(0,\infty)$, we obtain

$$\int_0^\infty \int_0^\infty (\Gamma_t(X,t) - V_X(X,t))\psi(X,t)dXdt = 0$$

and consequently

$$\Gamma_t(X,t) - V_X(X,t) = 0.$$

in $[0,\infty)\times[0,\infty)$. Then the equation (4.6) is reduced to

$$\int_0^\infty (h(t) - V(0,t))\psi(0,t)dt + \int_0^\infty (\Gamma(X,0) - g(X))\psi(X,0)dX = 0. \quad (4.7)$$

Employing any ψ with compact support, containing points on the t-axis, but not on the X-axis, we obtain

$$\int_0^\infty (h(t) - V(0,t))\psi(0,t)dt = 0.$$

As this holds for all such functions ψ, we conclude that

$$h(t) = V(0,t).$$

After that (4.7) takes the form

$$\int_0^\infty (\Gamma(X,0) - g(X))\psi(X,0)dX = 0.$$

Since this holds also for all functions ψ, with compact support, containing points of the X-axis, we infer that

$$\Gamma(X,0) = g(X).$$

Therefore (V,Γ) is a classical solution of the IBVP (3.11), (4.1). $\square$

Proposition 4.1 suggests the following definition.

Definition 4.0.1. *Let $f, g, h \in L^\infty([0,\infty))$. We say that the pair $(V,\Gamma) \in L^\infty([0,\infty)^2)$, such that $P(\Gamma(X,t)) \in L^\infty([0,\infty)^2)$, is a weak solution of IBVP (3.11), (4.1), provided (4.2) and (4.3) hold for all test functions φ and ψ, with φ restricted by $\varphi(0,t) = 0$.*

Remark 6. *An interesting question arises if (V,Γ) satisfies (4.1) in the sense of traces, [2], [1]. To answer that question we need to prove that the traces, of (V,Γ) on the positive part of the X axis and of V on the positive part of the t axis exist and are equal to $f(X)$, $g(X)$ and $h(t)$ respectively. That problem seems to be non trivial.*

Remark 7. *A similar argument shows that by replacing $\varphi(0,t) = 0$ by $\psi(0,t) = 0$, in the Definition 4.0.1 we can arrive to an analogous definition of a weak solution of the system (3.11), with the following initial and boundary conditions*

$$\begin{cases} V(X,0) = f(X) \\ \Gamma(X,0) = g(X) \\ P(\Gamma(0,t)) = h(t). \end{cases} \tag{4.8}$$

Now our aim is to give an answer to the question about a weak solution for an IVBP for (3.11) with these initial and boundary conditions:

$$\begin{cases} V(X,0) = f(X) \\ \Gamma(X,0) = g(X) \\ P(\Gamma(0,t)) + a(t)V(0,t) = c(t) \end{cases} \tag{4.9}$$

or

$$\begin{cases} V(X,0) = f(X) \\ \Gamma(X,0) = g(X) \\ V(0,t) + b(t)P(\Gamma(0,t)) = c(t). \end{cases} \tag{4.10}$$

Proposition 4.2. *Let f, g, a and c be C^1 functions on $[0,\infty)$, and let $V(X,t)$, $\Gamma(X,t)$ be C^1 functions on $[0,\infty)^2$, such that $P(\Gamma(X,t))$ is C^1 on $([0,\infty)^2)$. Then the pair (V,Γ) is a classical solution of IBVP (3.11),(4.9), if and only if for all φ and ψ, with ψ satisfying the condition $\psi(0,t)=0$, it holds*

$$-\int_0^\infty g(X)\psi(X,0)dX - \int_0^\infty\int_0^\infty \Gamma\psi_t dtdX + \int_0^\infty\int_0^\infty V\psi_X dXdt = 0 \tag{4.11}$$

and

$$\begin{aligned}-\int_0^\infty f(X)\varphi(X,0)dX - \int_0^\infty\int_0^\infty V\varphi_t dtdX + \int_0^\infty c(t)\varphi(0,t)dt\\ + \int_0^\infty\int_0^\infty P(\Gamma)\varphi_X dXdt - \int_0^\infty g(X)a(0)\varphi(X,0)dX\\ - \int_0^\infty\int_0^\infty (a(t)\varphi(X,t))_t\Gamma dtdX + \int_0^\infty\int_0^\infty a(t)\varphi_X(X,t)VdXdt = 0.\end{aligned} \tag{4.12}$$

Proof. Indeed, assuming that (V,Γ) is a classical solution of IBVP (3.11),(4.9), we multiply the first equation in (3.11) by φ and next integrate by parts. This results in

$$\begin{aligned}&-\int_0^\infty V(X,0)\varphi(X,0)dX - \int_0^\infty\int_0^\infty V(X,t)\varphi_t(X,t)dtdX\\ &+\int_0^\infty P(\Gamma(0,t))\varphi(0,t)dt\int_0^\infty\int_0^\infty P(\Gamma)\varphi_X(X,t)dXdt = 0,\end{aligned}$$

using the initial and boundary conditions (4.9), the last expression is equivalent to

$$\begin{aligned}&-\int_0^\infty f(X)\varphi(X,0)dX - \int_0^\infty\int_0^\infty V(X,t)\varphi_t(X,t)dtdX + \int_0^\infty c(t)\varphi(0,t)dt\\ &-\int_0^\infty a(t)V(0,t)\varphi(0,t)dt + \int_0^\infty\int_0^\infty P(\Gamma)\varphi_X(X,t)dXdt = 0.\end{aligned} \tag{4.13}$$

Similarly multiplying the second equation in (3.11) by a test function χ and integrating by parts results in

$$-\int_0^\infty g(X)\chi(X,0)dX - \int_0^\infty\int_0^\infty \Gamma(X,t)\chi_t(X,t)dtdX + \int_0^\infty V(0,t)\chi(0,t)dt \\ + \int_0^\infty\int_0^\infty V(X,t)\chi_X(X,t)dXdt = 0. \quad (4.14)$$

For $\chi = \psi$ the equation (4.14) is equivalent to

$$-\int_0^\infty g(X)\psi(X,0)dX - \int_0^\infty\int_0^\infty \Gamma(X,t)\psi_t(X,t)dtdX + \int_0^\infty\int_0^\infty V(X,t)\psi_X(X,t)dXdt = 0,$$

which is (4.11).

Next assuming that $\chi(X,t) = a(t)\varphi(X,t)$ the equation (4.14) becomes

$$-\int_0^\infty g(X)a(0)\varphi(X,0)dX - \int_0^\infty\int_0^\infty \Gamma(X,t)(a(t)\varphi(X,t))_t dtdX + \\ \int_0^\infty a(t)V(0,t)\varphi(0,t)dt + \int_0^\infty\int_0^\infty V(X,t)(a(t)\varphi(X,t))_X dXdt = 0. \quad (4.15)$$

Now adding (4.13) to (4.15), we get

$$-\int_0^\infty f(X)\varphi(X,0)dx - \int_0^\infty\int_0^\infty V(X,t)\varphi_t(X,t)dtdX + \int_0^\infty c(t)\varphi(0,t)dt \\ + \int_0^\infty\int_0^\infty P(\Gamma)\varphi_X(X,t)dXdt - \int_0^\infty g(X)a(0)\varphi(X,0)dX \\ - \int_0^\infty\int_0^\infty \Gamma(X,t)(a(t)\varphi(X,t))_t dtdX + \int_0^\infty\int_0^\infty V(X,t)(a(t)\varphi(X,t))_X dXdt = 0,$$

which is (4.12).

Next it remains to verify that if (V,Γ) satisfies (4.11) and (4.12) for all φ and ψ, then (V,Γ) is a classical solution of the IBVP (3.11), (4.9).

Indeed, integrating by parts the equation (4.11) we obtain

$$\int_0^\infty \psi(X,0)(\Gamma(X,0) - g(X))dX + \int_0^\infty\int_0^\infty \psi(X,t)(\Gamma_t(X,t) - V_X(X,t))dXdt = 0. \quad (4.16)$$

If in addition ψ has compact support in $(0,\infty)\times(0,\infty)$, we obtain

$$\int_0^\infty \int_0^\infty \psi(X,t)(\Gamma_t(X,t)-V_X(X,t))dXdt = 0,$$

for all such test functions ψ. Therefore we conclude

$$\Gamma_t(X,t)-V_X(X,t)=0. \tag{4.17}$$

Now since $\Gamma_t(X,t)-V_X(X,t)=0$, then (4.16) reduces to

$$\int_0^\infty \psi(X,0)(\Gamma(X,0)-g(X))dX = 0,$$

and this holds for all function ψ, with compact support in $[0,\infty)\times[0,\infty)$, containing points on the X-axis, and subject to $\psi(0,t)=0$. Hence

$$\Gamma(X,0)=g(X). \tag{4.18}$$

Similarly integrating by parts the equation (4.12), we get:

$$\begin{aligned}\int_0^\infty \varphi(X,0)(V(X,0)-f(X))dX + \int_0^\infty \varphi(0,t)(c(t)-P(\Gamma(0,t))-V(0,t)a(t))dt\\ +\int_0^\infty\int_0^\infty \varphi(X,t)(V_t(X,t)-(P(\Gamma))_X)dXdt + \int_0^\infty a(0)\varphi(X,0)(\Gamma(X,0)-g(X))dX\\ +\int_0^\infty\int_0^\infty a(t)\varphi(X,t)(\Gamma_t(X,t)-V_X(X,t))dXdt = 0,\end{aligned} \tag{4.19}$$

which because of (4.17) and (4.18) becomes

$$\begin{aligned}\int_0^\infty \varphi(X,0)(V(X,0)-f(X))dX + \int_0^\infty \varphi(0,t)(c(t)-P(\Gamma(0,t))-V(0,t)a(t))dt\\ +\int_0^\infty\int_0^\infty \varphi(X,t)(V_t(X,t)-(P(\Gamma))_X)dXdt = 0.\end{aligned} \tag{4.20}$$

If in addition φ has compact support in $(0,\infty)\times(0,\infty)$, we obtain

$$\int_0^\infty\int_0^\infty \varphi(X,t)(V_t(X,t)-(P(\Gamma))_X)dXdt = 0$$

for all such test functions φ. Therefore we conclude

$$V_t(X,t) - (P(\Gamma))_X = 0.$$

Now if $V_t(X,t) - (P(\Gamma))_X = 0$, then the equation (4.20) becomes

$$\int_0^\infty \varphi(X,0)(V(X,0)-f(X))dX + \int_0^\infty \varphi(0,t)(c(t)-P(\Gamma(0,t))-V(0,t)a(t))dt = 0. \tag{4.21}$$

Assuming that φ has compact support in $[0,\infty)\times[0,\infty)$ containing points on the X-axis, but not on the t-axis, we get

$$\int_0^\infty \varphi(X,0)(V(X,0)-f(X))dX = 0$$

and because of that

$$V(X,0) - f(X) = 0.$$

Then (4.21) reduces to

$$\int_0^\infty \varphi(0,t)(c(t)-P(\Gamma(0,t))-V(0,t)a(t))dt = 0.$$

Since this holds for all function φ with compact support in $[0,\infty)\times[0,\infty)$, and containing points on the t-axis, then it follows

$$P(\Gamma(0,t)) + a(t)V(0,t) = c(t).$$

Therefore we conclude that (V,Γ) is a classical solution for the IVBP (3.11), (4.9).

□

Proposition 4.2 suggests the following definition

Definition 4.0.2. *Let f, g and $c \in L^\infty([0,\infty))$, and $a \in C^1([0,\infty))$. We say that the pair $(V,\Gamma) \in L^\infty([0,\infty)^2)$, such that $P(\Gamma(X,t)) \in L^\infty([0,\infty)^2)$, is a weak solution* *of IBVP (3.11),(4.9), provided (4.11) and (4.12) hold for all test functions φ and ψ with ψ restricted by $\psi(0,t) = 0$.*

Next, we prove an proposition analogous to the *Proposition* 4.2, for the IBVP (3.11),(4.10).

Proposition 4.3. *Let f, g, b and c be C^1 functions on $[0,\infty)$, and let $V(X,t)$, $\Gamma(X,t)$ be C^1 functions on $[0,\infty)^2$, such that $P(\Gamma(X,t))$ is C^1 on $([0,\infty)^2$. Then the pair (V,Γ) is a classical solution of IBVP (3.11),(4.10), if and only if for all φ and ψ, with φ satisfying the condition $\varphi(0,t) = 0$ it holds*

$$-\int_0^\infty f(X)\varphi(X,0)dX - \int_0^\infty \int_0^\infty V(X,t)\varphi_t(X,t)dtdX + \int_0^\infty \int_0^\infty P(\Gamma)\varphi_X(X,t)dXdt = 0 \tag{4.22}$$

and

$$\begin{aligned} -\int_0^\infty g(X)\psi(X,0)dX - \int_0^\infty \int_0^\infty \Gamma(X,t)\psi_t(X,t)dtdX + \int_0^\infty c(t)\psi(0,t)dt \\ + \int_0^\infty \int_0^\infty V(X,t)\psi_X(X,t)dXdt - \int_0^\infty f(X)b(0)\psi(X,0)dX \\ - \int_0^\infty \int_0^\infty (b(t)\psi(X,t))_t V dtdX + \int_0^\infty \int_0^\infty b(t)\psi_X P(\Gamma)dXdt = 0. \end{aligned} \tag{4.23}$$

Proof. We proceed similarly as in *Proposition* 4.2. Indeed, assuming that (V,Γ) is a classical solution of IBVP (3.11),(4.10), we multiply the first equation in (3.11) by a test function χ and integrate by parts, which results in

$$\begin{aligned} -\int_0^\infty f(X)\chi(X,0)dX - \int_0^\infty \int_0^\infty V(X,t)\chi_t(X,t)dtdX + \int_0^\infty P(\Gamma(0,t))\chi(0,t)dt \\ + \int_0^\infty \int_0^\infty P(\Gamma)\chi_X(X,t)dXdt = 0. \end{aligned} \tag{4.24}$$

For $\chi = \varphi$ the equation (4.24) is equivalent to

$$-\int_0^\infty f(X)\varphi(X,0)dX - \int_0^\infty \int_0^\infty V(X,t)\varphi_t(X,t)dtdX \\ + \int_0^\infty \int_0^\infty P(\Gamma)\varphi_X(X,t)dXdt = 0,$$

which is (4.22).

Similarly, multiplying the second equation in (3.11) by a test function ψ and integrating by parts we get

$$\begin{aligned} -\int_0^\infty g(X)\psi(X,0)dX - \int_0^\infty \int_0^\infty \Gamma(X,t)\psi_t(X,t)dtdX + \int_0^\infty c(t)\psi(0,t)dt \\ -\int_0^\infty b(t)P(\Gamma(0,t))\psi(0,t)dt + \int_0^\infty \int_0^\infty V(X,t)\psi_X(X,t)dXdt = 0. \end{aligned} \tag{4.25}$$

Next assuming that $\chi(X,t) = b(t)\psi(X,t)$ the equation (4.24) becomes

$$-\int_0^\infty f(X)b(0)\psi(X,0)dX \\ -\int_0^\infty \int_0^\infty V(X,t)(b(t)\psi(X,t))_t dtdX + \int_0^\infty b(t)P(\Gamma(0,t))\psi(0,t)dt \\ + \int_0^\infty \int_0^\infty P(\Gamma)(b(t)\psi(X,t))_X dXdt. \tag{4.26}$$

Now adding (4.25) to (4.26), we get

$$-\int_0^\infty g(X)\psi(X,0)dX - \int_0^\infty \int_0^\infty \Gamma(X,t)\psi_t(X,t)dtdX + \int_0^\infty c(t)\psi(0,t)dt \\ + \int_0^\infty \int_0^\infty V(X,t)\psi_X(X,t)dXdt - \int_0^\infty f(X)b(0)\psi(X,0)dX \\ -\int_0^\infty \int_0^\infty V(X,t)(b(t)\psi(X,t))_t dtdX + \int_0^\infty \int_0^\infty P(\Gamma)(b(t)\psi(X,t))_X dXdt = 0,$$

which is (4.23). To verify that if (V,Γ) satisfies (4.22) and (4.23) for all φ and ψ then (V,Γ) is a classical solution of the IVBP (3.11), (4.10), we proceed as in the proof of *Proposition* 4.2. □

Proposition 4.3 suggests the following definition.

Definition 4.0.3. *Let $f, g, c \in L^\infty([0,\infty))$, and $b \in C^1([0,\infty))$. We say that the pair $(V,\Gamma) \in L^\infty([0,\infty)^2)$, such that $P(\Gamma(X,t)) \in L^\infty([0,\infty)^2)$, is a weak solution of the IBVP (3.11),(4.10), provided (4.22) and (4.23) hold for all test functions φ and ψ with φ restricted by $\varphi(0,t) = 0$.*

Remark 8. *Note that setting $b(t) = 0$ and $c(t) = h(t)$ in (4.23) we obtain (4.3). This is consistent with the fact that the initial and boundary conditions (4.10) are equivalent to (4.1) for this choice of b and c.*

Next we consider a particular case of an IBVP, (3.11), (4.1) denoted further by $IBVP_{V_0}$.

4.1 An initial and boundary value problem, $IBVP_{V_0}$, for a p-system

Here the initial and boundary conditions are

$$\begin{cases} V(X,0) = -V_0 \\ \Gamma(X,0) = 0 \\ V(0,t) = 0. \end{cases} \tag{4.27}$$

It is quite elementary to construct a solution of IBVP (3.10),(4.27), in a linear case (3.16), in which the equation for $U(X,t)$, (3.10) is a wave equation,

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial X^2} \tag{4.28}$$

where $c = \sqrt{\frac{\lambda+2\mu}{\rho_0}}$. We use a standard formula, based on D'Alembert's formula, to obtain $U(X,t)$ which satisfies the conditions

$$\begin{cases} U(X,0) = 0 \\ U_t(X,0) = -V_0 \\ U(0,t) = 0. \end{cases}$$

Then

$$U(X,t) = \begin{cases} -\frac{1}{2c}\int_{X-ct}^{X+ct} V_0 \, dy & \text{if} \quad X \geq ct \\ \\ -\frac{1}{2c}\int_{ct-X}^{ct+X} V_0 \, dy & \text{if} \quad 0 \leq X \leq ct \end{cases}$$

what simplifies to

$$U(X,t) = \begin{cases} -tV_0 & \text{if} \quad X \geq ct \\ \\ -\frac{X}{c}V_0 & \text{if} \quad X \leq ct \end{cases} \tag{4.29}$$

From that it follows

$$V(X,t) = U_t = \begin{cases} -V_0 & \text{if } X > ct \\ 0 & \text{if } X < ct \end{cases} \tag{4.30}$$

$$\Gamma(X,t) = U_X = \begin{cases} 0 & \text{if } X > ct \\ -\frac{V_0}{c} & \text{if } \ X < ct \end{cases} \tag{4.31}$$

and

$$\begin{cases} V(X,0) = U_t(X,0) = -V_0 \\ \Gamma(X,0) = U_X(X,0) = 0 \\ V(0,t) = U_t(0,t) = 0. \end{cases}$$

Notice that the requirement of nonpenetration of matter, *Remark* 4, results in $V_0 < c$. Next we construct a solution of $IBVP_{V_0}$ for a nonlinear p-system analogous to the solution (4.30) and (4.31) in a linear case:

$$V(X,t) = \begin{cases} -V_0, & \text{if } X > \sigma t \\ 0, & \text{if } X < \sigma t \end{cases} \tag{4.32}$$

$$\Gamma(X,t) = \begin{cases} 0, & \text{if } X > \sigma t \\ \Gamma_l, & \text{if } X < \sigma t \end{cases} \tag{4.33}$$

where Γ_l and σ are determined from the Rankine-Hugoniot condition in *Definition* 2.1.6.

4.2 Rankine-Hugoniot conditions for nonlinear p-system

Rewriting (3.11) in conservative form

$$\begin{pmatrix} V \\ \Gamma \end{pmatrix}_t + \begin{pmatrix} -P(\Gamma) \\ -V \end{pmatrix}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we identify $\mathbf{u}$ and $\mathbf{F}$ to be

$$\mathbf{u} = [V, \Gamma] \tag{4.34}$$

and

$$\mathbf{F}(V, \Gamma) = [-P(\Gamma), -V] \tag{4.35}$$

The Rankine-Hugoniot condition, in *Definition* 2.1.6, reads

$$\mathbf{F}(V_l, \Gamma_l) - \mathbf{F}(V_r, \Gamma_r) = \sigma\big((V_l, \Gamma_l) - (V_r, \Gamma_r)\big)$$

which can be simplified to

$$(P(\Gamma_r) - P(\Gamma_l), V_r - V_l) = \sigma(V_l - V_r, \Gamma_l - \Gamma_r)$$

and finally to:

$$\begin{cases} \sigma[|V|] & = [|P(\Gamma)|] \\ \sigma[|\Gamma|] & = [|V|]. \end{cases} \tag{4.36}$$

Now using the equations (4.32) and (4.33) the system (4.36) becomes

$$\begin{aligned} \sigma V_0 &= -P(\Gamma_l) \\ \sigma \Gamma &= -V_0, \end{aligned} \tag{4.37}$$

and that can be rewritten into

$$\begin{aligned} \Gamma_l P(\Gamma_l) &= V_0^2 \\ \sigma &= -\frac{V_0}{\Gamma_l}. \end{aligned} \tag{4.38}$$

We observe that the system (4.38) has an unique solution (σ, Γ_l) provided the first equation has an unique solution for Γ_l. We denote such solution by $S(V_0)$ and because of the relation between V_0 and Γ_l by $S(\Gamma_l)$ as well.

Concerning solvability of the first equation, we notice the following fact.

Proposition 4.4. *Let* $P(0) = 0$, $\lim_{\Gamma \to -1^+} P(\Gamma) = -\infty$ *and for all* $\Gamma \in (-1, 0)$, $P'(\Gamma) > 0$. *Then for each* $V_0 > 0$ *there exists an unique* $\Gamma_l \in (-1, 0)$ *such that* $\Gamma_l P(\Gamma_l) = V_0^2$.

Proof. Let $H(\Gamma) = \Gamma P(\Gamma)$. Then $H'(\Gamma) = P(\Gamma) + \Gamma P'(\Gamma) < 0$ for all $\Gamma \in (-1, 0)$ and $H(\Gamma)$ is decreasing. Since $H(0) = 0$ and $\lim_{\Gamma \to -1^+} P(\Gamma) = -\infty$ therefore for each $V_0 > 0$ there exists an unique $\Gamma_l \in (-1, 0)$ such that $\Gamma_l P(\Gamma_l) = V_0^2$. $\square$

From the discussion in chapter 6 it will become clear that *Proposition* 4.4 can be applied to Kirchhoff modified, Ogden, Blatz-Ko-Ogden models. For the case of St.Venant-Kirchhoff the first equation of the system (4.38) has a solution for Γ_l in $(-1, 0)$ provided $V_0^2 \leq H(\overline{\Gamma})$, where $\overline{\Gamma} = \frac{-9+\sqrt{17}}{8}$.

Now we show some typical graphs of $H(\Gamma) = \Gamma P(\Gamma)$, for every model in question, for $\rho_0 = 1, \mu = \frac{1}{4}, \lambda = \frac{1}{2}$ and $f = \frac{1}{2}$, which confirm our observations.

Figure 4–1: St.Venant-Kirchhoff

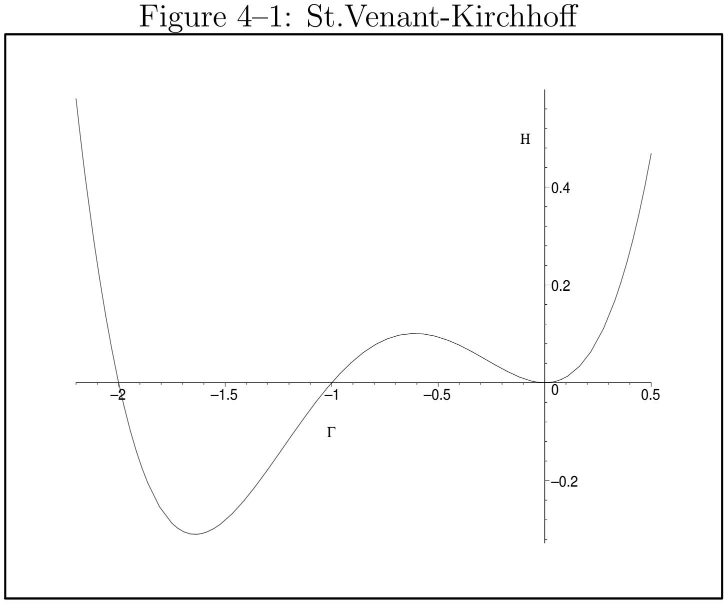


Figure 4–2: Modified Kirchhoff

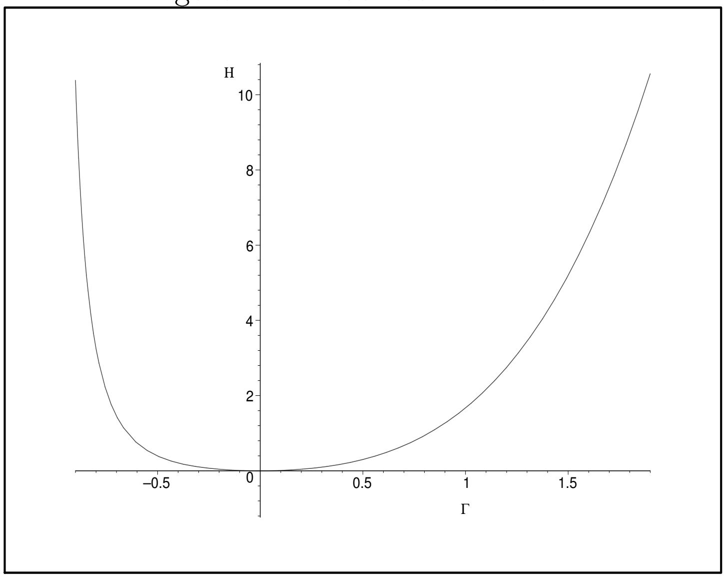


Figure 4–3: Ogden

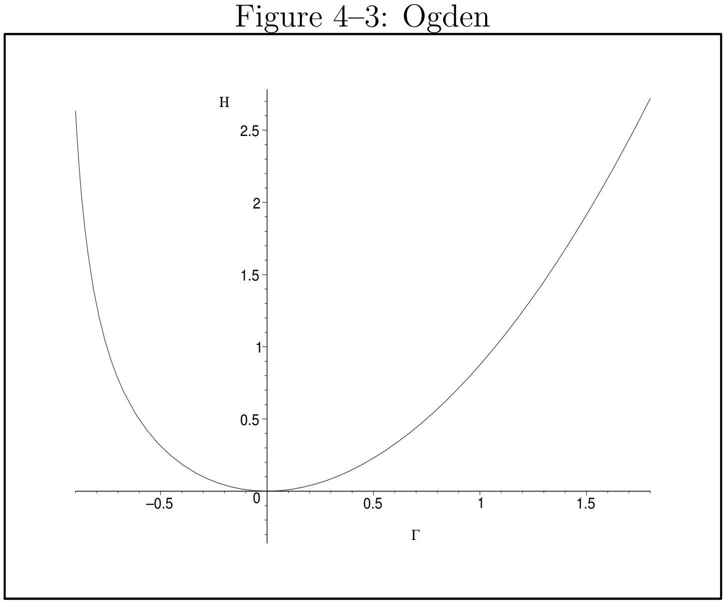
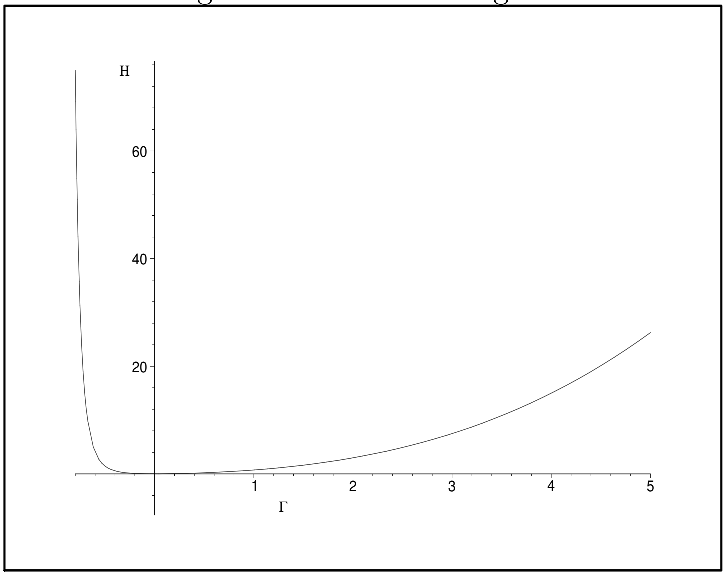


Figure 4–4: Blatz-Ko-Ogden



Finally we verify that $S(\Gamma_l)$ is indeed a weak solution of $IBVP_{V_0}$ (3.11),(4.27), i.e. we verify that $S(\Gamma_l)$ satisfies the equations (4.2) and (4.3) for all test functions φ and ψ, with φ restricted by the condition $\varphi(0,t) = 0$. We also assume that $P(0) = 0$. Now $f(X) = -V_0$, $g(X) = 0$ and $h(t) = 0$ in (4.1).

We verify (4.2) first. Its left-hand sized is:

$$\begin{aligned}
&\int_0^\infty \int_0^\infty (V\varphi_t - P(\Gamma)\varphi_X)dtdX + \int_0^\infty f(X)\varphi(X,0)dX \\
&= -V_0 \int_0^\infty \int_0^{X/\sigma} \varphi_t dtdX + \int_0^\infty \int_{X/\sigma}^\infty 0 \cdot \varphi_t dtdX - P(\Gamma_l) \int_0^\infty \int_0^{\sigma t} \varphi_X dXdt \\
&- P(0) \int_0^\infty \int_{\sigma t}^\infty \varphi_X dXdt - V_0 \int_0^\infty \varphi(X,0)dX \\
&= -V_0 \int_0^\infty \varphi(X, X/\sigma)dX + V_0 \int_0^\infty \varphi(X,0)dX - P(\Gamma_l) \int_0^\infty \varphi(\sigma t, t)dt \\
&+ P(\Gamma_l) \int_0^\infty \varphi(0,t)dt - V_0 \int_0^\infty \varphi(X,0)dX \\
&= -V_0 \int_0^\infty \varphi(X, X/\sigma)dX - \frac{P(\Gamma_l)}{\sigma} \int_0^\infty \varphi(X, X/\sigma)dX,
\end{aligned}$$

which vanishes due to (4.37), so that (4.2) holds.

Next, for the left-hand sized of (4.3) we have:

$$\int_0^\infty \int_0^\infty (\Gamma\psi_t - V\psi_X) dtdX$$
$$= \int_0^\infty \int_0^{X/\sigma} 0 \cdot \psi_t dtdX + \Gamma_l \int_0^\infty \int_{X/\sigma}^\infty \psi_t dtdX - \int_0^\infty \int_0^{\sigma t} 0 \cdot \psi_X dXdt$$
$$+ V_0 \int_0^\infty \int_{\sigma t}^\infty \psi_X dXdt$$
$$= \Gamma_l \int_0^\infty \int_{X/\sigma}^\infty \psi_t dtdX + V_0 \int_0^\infty \int_{\sigma t}^\infty \psi_X dXdt$$
$$= -\Gamma_l \int_0^\infty \psi(X, X/\sigma) dX - \frac{V_0}{\sigma} \int_0^\infty \psi(X, X/\sigma) dX,$$

which is zero due to (4.37) so that (4.3) holds.

CHAPTER 5
ENTROPY CONDITION AND ENTROPY SOLUTION FOR A P-SYSTEM

According to the *Definition* 2.1.10 an entropy/entropy-flux pair for a p-system is a pair of real valued $C^2(\mathbb{R}^2)$ functions $\Phi(V,\Gamma)$ and $\Psi(V,\Gamma)$, where Φ is convex, and such that

$$D\Phi(V,\Gamma)DF(V,\Gamma) = D\Psi(V,\Gamma) \tag{5.1}$$

with $F(V,\Gamma)$ given by (4.35). Working out that condition one obtains

$$\begin{aligned} \Psi_V &= -\Phi_\Gamma \\ \Psi_\Gamma &= -P'(\Gamma)\Phi_V. \end{aligned} \tag{5.2}$$

Now the integrability condition of the system (5.2) for Ψ is

$$\Phi_{\Gamma\Gamma} - P'(\Gamma)\Phi_{VV} = 0. \tag{5.3}$$

Given a convex function Φ that fulfills this equation we can obtain Ψ by solving the system (5.2). As an example of these entropy/entropy-flux pairs we have that for the linear case (3.16) the equation (5.3) becomes the wave equation, therefore we obtain

$$\Phi(V,\Gamma) = F(V + c\Gamma) + G(V - c\Gamma) \tag{5.4}$$

and given Φ, (5.4), we find Ψ using (5.2)

$$\Psi(V,\Gamma) = -cF(V + c\Gamma) + cG(V - c\Gamma), \tag{5.5}$$

where F and $G \in C^2$.

Here the condition of convexity for Φ implies that $F'' \geq 0$ and $G'' \geq 0$.

Next (5.2) implies that for any smooth solution (V, Γ) of the p-system, (3.11), it holds

$$\frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial X} = 0. \tag{5.6}$$

Now the new idea is to replace (5.6) with an inequality

$$\frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial X} \leq 0$$

if (V, Γ) is a weak solution of (3.11).

Such inequality has to be understood in distributional sense. We can rewrite it in an integral form, which is employed in the following definition.

Definition 5.1. *A weak solution $V(X,t)$, $\Gamma(X,t)$ of an IBVP, is an entropy solution provided for each nonnegative $\varphi \in C_0^\infty((0,\infty)\times(0,\infty))$ and for each entropy/entropy-flux pair Φ, Ψ it holds*

$$\int_0^\infty \int_0^\infty [\Phi(V,\Gamma)\varphi_t(X,t) + \Psi(V,\Gamma)\varphi_X(X,t)]dXdt \geq 0. \tag{5.7}$$

We refer to (5.7) as the entropy condition corresponding to (Φ, Ψ).

Remark 9. *If a trivial solution $(V, \Gamma) = (0, 0)$ is a solution of an IBVP then it is an entropy solution.*

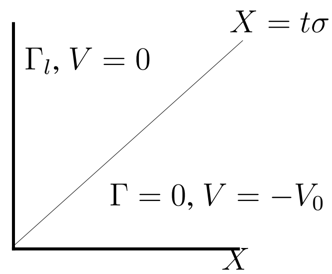
The following *Proposition* translates the entropy condition (5.7) into a jump condition for piecewise continuous weak solutions.

Proposition 5.1. *Suppose that $\mathbf{u} = (V, \Gamma)$ is a piecewise continuous weak solution of (2.4) that satisfies the entropy condition corresponding to (Φ, Ψ). Suppose $\mathbf{u}$ has a jump discontinuity along a shock curve with slope σ. Then*

$$\sigma[\Phi(\mathbf{u})] - [\Psi(\mathbf{u})] \geq 0 \tag{5.8}$$

We call (5.8) the entropy jump condition corresponding to (Φ, Ψ).

We demonstrate (5.8), for our solution, (4.32) and (4.33). In doing this the following figure is useful.



Here $\mathbf{u} = (V, \Gamma)$ satisfies the inequality (5.7), therefore we get

$$\int_0^\infty \int_0^\infty (\Phi(V,\Gamma)\varphi_t + \Psi(V,\Gamma)\varphi_X) dX dt \geq 0,$$

$$\int_0^\infty \int_0^\infty \Phi(V,\Gamma)\varphi_t dt dX + \int_0^\infty \int_0^\infty \Psi(V,\Gamma)\varphi_X dX dt \geq 0,$$

$$\int_0^\infty \int_0^{X/\sigma} \Phi(-V_0,0)\varphi_t dt dX + \int_0^\infty \int_{X/\sigma}^\infty \Phi(0,\Gamma)\varphi_t dt dX + \int_0^\infty \int_0^{t\sigma} \Psi(0,\Gamma)\varphi_X dX dt + \int_0^\infty \int_{t\sigma}^\infty \Psi(-V_0,0)\varphi_X dX dt \geq 0,$$

and

$$\Big(\int_0^\infty \varphi(X, X/\sigma) dX\Big)(\Phi(-V_0,0) - \Phi(0,\Gamma)) + \Big(\int_0^\infty \varphi(t\sigma, t) dt\Big)(\Psi(0,\Gamma) - \Psi(-V_0,0)) \geq 0.$$

Making the substitution $X = t\sigma$, in the integral with respect to t, we obtain:

$$\Big(\int_0^\infty \varphi(X, X/\sigma) dX\Big)(\Phi(-V_0,0) - \Phi(0,\Gamma)) + \Big(\frac{1}{\sigma}\int_0^\infty \varphi(X, X/\sigma) dX\Big)(\Psi(0,\Gamma) - \Psi(-V_0,0)) \geq 0,$$

which simplifies into

$$\Big(\int_0^\infty \varphi(X, X/\sigma)dX\Big)\big[(\Phi(-V_0,0)-\Phi(0,\Gamma))+\frac{1}{\sigma}((\Psi(0,\Gamma)-\Psi(-V_0,0))\big]\geq 0.$$

Here $\varphi \geq 0$, therefore this last inequality is equivalent to

$$\Phi(-V_0,0)-\Phi(0,\Gamma)\geq \frac{\Psi(-V_0,0)-\Psi(0,\Gamma)}{\sigma}, \tag{5.9}$$

whose compact form is (5.8).

Remark 10. *Notice that for the linear case (3.16) Φ and Ψ are given by (5.4) and (5.5) respectively. We verify that (5.9) holds, for these with Φ and Ψ. Indeed*

$$\Phi(-V_0,0)=F(-V_0)+G(-V_0),$$

$$\Phi(0,-V_0/c)=F(-V_0)+G(V_0),$$

$$\Psi(0,-V_0/c)=-cF(-V_0)+cG(V_0),$$

and

$$\Psi(-V_0,0)=-cF(-V_0)+cG(-V_0).$$

Therefore (5.9) becomes this identity:

$$G(-V_0)-G(V_0)+\frac{c(G(V_0)-G(-V_0))}{c}=0.$$

Thus we conclude that for the linear case (3.16), $S(V_0)$ satisfies the entropy condition (5.7).

The following *Proposition* states that in the case of genuine nonlinear systems, the entropy condition is satisfied for Γ sufficiently close to zero.

Proposition 5.2. *If $P(0)=0$, $P'(0)>0$ and $P''(0)<0$, then for each entropy/entropy-flux pair (Φ,Ψ), where Φ is strictly convex, $S(\Gamma)$ satisfies the entropy condition corresponding to (Φ,Ψ), for all Γ sufficiently close to zero and $\Gamma\leq 0$.*

Proof. We notice that for $\Gamma = 0$, $S(0) = 0$. Therefore by *Remark* 9 this solution is an entropy solution.

Now we consider $\Gamma < 0$. Let $\epsilon = -V_0$, then $\epsilon = -\sqrt{\Gamma P(\Gamma)}$, and $\sigma = \frac{\epsilon}{\Gamma}$. Define

$$E(\Gamma) = \frac{\epsilon}{\Gamma}[\Phi(0,\Gamma) - \Phi(\epsilon,0)] - [\Psi(0,\Gamma) - \Psi(\epsilon,0)].$$

Consequently, the entropy jump condition, (5.8), holds if and only if

$$E(\Gamma) \leq 0.$$

We now let a "prime" indicate differentiation with respect to Γ .

We observe that

$$\lim_{\Gamma\to 0^-}\left(\frac{\epsilon}{\Gamma}\right) = \lim_{\Gamma\to 0^-} -\frac{\sqrt{\Gamma P(\Gamma)}}{\Gamma} = \sqrt{P'(0)}.$$

Therefore

$$\lim_{\Gamma\to 0^-} E(\Gamma) = 0.$$

Now,

$$E'(\Gamma) = \left(\frac{\epsilon}{\Gamma}\right)'[\Phi(0,\Gamma) - \Phi(\epsilon,0)] + \frac{\epsilon}{\Gamma}[\Phi_\Gamma(0,\Gamma) - \Phi_\epsilon(\epsilon,0)\epsilon'] - \Psi_\Gamma(0,\Gamma) + \Psi_\epsilon(\epsilon,0)\epsilon',$$

and because of (5.2),

$$E'(\Gamma) = \left(\frac{\epsilon}{\Gamma}\right)'[\Phi(0,\Gamma)-\Phi(\epsilon,0)]+\frac{\epsilon}{\Gamma}[\Phi_\Gamma(0,\Gamma)-\Phi_\epsilon(\epsilon,0)\epsilon']+\Phi_\epsilon(0,\Gamma)P'(\Gamma)-\Phi_\Gamma(\epsilon,0)\epsilon'.$$

Then since

$$\lim_{\Gamma\to 0^-} \epsilon' = -\lim_{\Gamma\to 0^-}\left(\sqrt{\Gamma P(\Gamma)}\right)' = \sqrt{P'(0)}$$

and

$$\lim_{\Gamma\to 0^-}\left(\frac{\epsilon}{\Gamma}\right)' = -\lim_{\Gamma\to 0^-}\left(\frac{\sqrt{\Gamma P(\Gamma)}}{\Gamma}\right)' = \lim_{\Gamma\to 0^-}\frac{P(\Gamma) - \Gamma P'(\Gamma)}{2\Gamma\sqrt{\Gamma P(\Gamma)}} = \frac{P''(0)}{4\sqrt{P'(0)}}$$

it follows

$$\lim_{\Gamma\to 0^-} E'(\Gamma) = \sqrt{P'(0)}\Big(\Phi_\Gamma(0,0) - \Phi_\epsilon(0,0)\sqrt{P'(0)}\Big)$$
$$+ \Phi_\epsilon(0,0)P'(0) - \Phi_\Gamma(0,0)\sqrt{P'(0)} = 0.$$

Next,

$$E''(\Gamma) = \left(\frac{\epsilon}{\Gamma}\right)''\Big(\Phi(0,\Gamma) - \Phi(\epsilon,0)\Big) + 2\left(\frac{\epsilon}{\Gamma}\right)'\Big(\Phi_\Gamma(0,\Gamma) - \Phi_\epsilon(\epsilon,0)\epsilon'\Big)$$
$$+ \left(\frac{\epsilon}{\Gamma}\right)\Big(\Phi_{\Gamma\Gamma}(0,\Gamma) - \epsilon''\Phi_\epsilon(\epsilon,0) - \Phi_{\epsilon\epsilon}(\epsilon,0)(\epsilon')^2\Big) + P''(\Gamma)\Phi_\epsilon(0,\Gamma)$$
$$+ P'(\Gamma)\Phi_{\epsilon\Gamma}(0,\Gamma) - \epsilon''\Phi_\Gamma(\epsilon,0) - (\epsilon')^2\Phi_{\Gamma\epsilon}(\epsilon,0)$$

since

$$\lim_{\Gamma\to 0^-} \epsilon'' = \lim_{\Gamma\to 0^-}\left(\frac{P(\Gamma)+\Gamma P'(\Gamma)}{-2\sqrt{\Gamma P(\Gamma)}}\right) = \frac{P''(0)}{2\sqrt{P'(0)}},$$

and

$$\lim_{\Gamma\to 0^-}\left(\frac{\epsilon}{\Gamma}\right)'' = \frac{-(1/4)(P''(0))^2 + (2/3)P'(0)P'''(0)}{4\big(\sqrt{P'(0)}\big)^3},$$

therefore

$$\lim_{\Gamma\to 0^-} E''(\Gamma) = 2\left(\frac{P''(0)}{4\sqrt{P'(0)}}\right)\left[\Phi_\Gamma(0,0) - \Phi_\epsilon(0,0)\sqrt{P'(0)}\right]$$
$$+ \sqrt{P'(0)}\left[\Phi_{\Gamma\Gamma}(0,0) - \frac{P''(0)}{2\sqrt{P'(0)}}\Phi_\epsilon(0,0) - \Phi_{\epsilon\epsilon}(0,0)P'(0)\right]$$
$$+ P''(0)\Phi_\epsilon(0,0) + P'(0)\Phi_{\epsilon\Gamma}(0,0) - \frac{P''(0)}{2\sqrt{P'(0)}}\Phi_\Gamma(0,0) - P'(0)\Phi_{\Gamma\epsilon}(0,0) = 0.$$

Finally,

$$\begin{aligned}E'''(\Gamma) =&\Big(\frac{\epsilon}{\Gamma}\Big)'''\Big(\Phi(0,\Gamma)-\Phi(\epsilon,0)\Big)+3\Big(\frac{\epsilon}{\Gamma}\Big)''\Big(\Phi_\Gamma(0,\Gamma)-\Phi_\epsilon(\epsilon,0)\epsilon'\Big)\\&+3\Big(\frac{\epsilon}{\Gamma}\Big)'\Big(\Phi_{\Gamma\Gamma}(0,\Gamma)-\epsilon''\Phi_\epsilon(\epsilon,0)-(\epsilon')^2\Phi_{\epsilon\epsilon}(\epsilon,0)\Big)\\&\Big(\frac{\epsilon}{\Gamma}\Big)\Big[\Phi_{\Gamma\Gamma\Gamma}(0,\Gamma)-\epsilon'''\Phi_\epsilon(\epsilon,0)-3\epsilon'\epsilon''\Phi_{\epsilon\epsilon}(\epsilon,0)-(\epsilon')^3\Phi_{\epsilon\epsilon\epsilon}(\epsilon,0)\Big]\\&+P'''(\Gamma)\Phi_\epsilon(0,\Gamma)+2F''(\Gamma)\Phi_{\epsilon\Gamma}(0,\Gamma)+F'(\Gamma)\Phi_{\epsilon\Gamma\Gamma}(0,\Gamma)-\epsilon'''\Phi_\Gamma(\epsilon,0)\\&-3\epsilon'\epsilon''\Phi_{\Gamma\epsilon}(\epsilon,0)-(\epsilon')^3\Phi_{\Gamma\epsilon\epsilon}(\epsilon,0),\end{aligned}$$

$$\lim_{\Gamma\to0^-}\epsilon'''=\frac{4P'(0)P'''(0)-(3/2)(P''(0))^2}{8(P'(0))^{3/2}}$$

and

$$\lim_{\Gamma\to0^-}\Big(\frac{\epsilon}{\Gamma}\Big)'''=\frac{-(5/2)(P'(0))^2P'''(0)+(1/2)P'(0)P''(0)+(19/2)P'(0)(P''(0))^2}{8\big(\sqrt{P'(0)}\big)^5}$$

Therefore we get

$$\begin{aligned}\lim_{\Gamma\to0^-}E'''(\Gamma)=&-3\left[\frac{(1/4)(P''(0))^2-(2/3)P'(0)P'''(0)}{4\big(\sqrt{P'(0)}\big)^3}\right]\Big[\Phi_\Gamma(0,0)-\Phi_\epsilon(0,0)\sqrt{P'(0)}\Big]\\&+\frac{3P''(0)}{4\sqrt{P'(0)}}\left[\Phi_{\Gamma\Gamma}(0,0)-\frac{P''(0)}{2\sqrt{P'(0)}}\Phi_\epsilon(0,0)-\Phi_{\epsilon\epsilon}(0,0)P'(0)\right]\\&+\sqrt{P'(0)}\Bigg[\Phi_{\Gamma\Gamma\Gamma}(0,0)-\left(\frac{4P'(0)P'''(0)-(3/2)(P''(0))^2}{8(P'(0))^{3/2}}\right)\Phi_\epsilon(0,0)\\&-\frac{3\sqrt{P'(0)}P''(0)}{2\sqrt{P'(0)}}\Phi_{\epsilon\epsilon}(0,0)-\big(\sqrt{P'(0)}\big)^3\Phi_{\epsilon\epsilon\epsilon}(0,0)\Bigg]+P'''(0)\Phi_\epsilon(0,0)\\&+2P''(0)\Phi_{\epsilon\Gamma}(0,0)+P'(0)\Phi_{\epsilon\Gamma\Gamma}(0,0)-\left(\frac{4P'(0)P'''(0)-(3/2)(P''(0))^2}{8(P'(0))^{3/2}}\right)\Phi_\Gamma(0,0)\\&-\frac{3\sqrt{P'(0)}P''(0)}{2\sqrt{P'(0)}}\Phi_{\Gamma\epsilon}(0,0)-\big(\sqrt{P'(0)}\big)^3\Phi_{\Gamma\epsilon\epsilon}(0,0).\end{aligned}$$

However from (5.3) it follows

$$\Phi_{\Gamma\Gamma}(\Gamma,\epsilon) - \Phi_{\epsilon\epsilon}(\Gamma,\epsilon)P'(\Gamma) = 0$$

$$\Phi_{\Gamma\Gamma\epsilon}(\Gamma,\epsilon) - \Phi_{\epsilon\epsilon\epsilon}(\Gamma,\epsilon)P'(\Gamma) = 0$$

and

$$\Phi_{\Gamma\Gamma\Gamma}(\Gamma,\epsilon) - P''(\Gamma)\Phi_{\epsilon\epsilon}(\Gamma,\epsilon) - \Phi_{\epsilon\epsilon\Gamma}(\Gamma,\epsilon)P'(\Gamma) = 0.$$

Consequently

$$\lim_{\Gamma\to 0^-} E'''(\Gamma) = -\frac{1}{2}P''(0)[\sqrt{P'(0)}\Phi_{\epsilon\epsilon}(0,0) - \Phi_{\Gamma\epsilon}(0,0)]. \tag{5.10}$$

On the other hand, since Φ is strictly convex, we know that for all nonzero $(a,b) \in \mathbb{R}^2$

$$(a,b)\begin{pmatrix} \Phi_{\epsilon\epsilon} & \Phi_{\epsilon\Gamma} \\ \Phi_{\Gamma\epsilon} & \Phi_{\Gamma\Gamma} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} > 0,$$

or equivalently

$$\Phi_{\epsilon\epsilon}(0,0)a^2 + 2\Phi_{\epsilon\Gamma}(0,0)ab + \Phi_{\Gamma\Gamma}(0,0)b^2 > 0. \tag{5.11}$$

Now we demonstrate that the expression $\sqrt{P'(0)}\Phi_{\epsilon\epsilon}(0,0) - \Phi_{\Gamma\epsilon}(0,0)$ in (5.10) can be rewritten in a form of the left hand size of (5.11) with a and b appropriately chosen. To prove that we modify this expression by an additive, equal to zero term $\alpha\Phi_{\Gamma\Gamma}(0,0) - \alpha\Phi_{\epsilon\epsilon}(0,0)P'(0)$, where α to be determined.

Thus a and b have to be chosen so that

$$(\sqrt{P'(0)} - \alpha P'(0))\Phi_{\epsilon\epsilon} - \Phi_{\Gamma\epsilon} + \alpha\Phi_{\Gamma\Gamma} = \Phi_{\epsilon\epsilon}a^2 + 2\Phi_{\epsilon\Gamma}ab + \Phi_{\Gamma\Gamma}b^2 \tag{5.12}$$

holds, where the derivatives of Φ are at $(0,0)$.

Consequently we require that

$$\begin{aligned} a^2 &= \sqrt{P'(0)} - \alpha P'(0) \\ 2ab &= -1 \\ \alpha &= b^2 \end{aligned}$$

Now solving the system for a, b and α, we get

$$a = \pm\frac{\sqrt[4]{P'(0)}}{\sqrt{2}},$$

$$b = \mp\frac{1}{\sqrt{2}\sqrt[4]{P'(0)}},$$

and

$$\alpha = \frac{1}{2\sqrt{P'(0)}}.$$

In this way, we conclude from (5.10) that

$$\lim_{\Gamma\to 0^-} E'''(\Gamma) > 0.$$

Now we need the following *Lemmas*.

Lemma 1. *Let f and f' be continuous functions for $\Gamma < 0$. If $\lim_{\Gamma\to 0^-} f(\Gamma) = 0$ and $\lim_{\Gamma\to 0^-} f'(\Gamma) > 0$, then there exists $\Gamma_0 \in (-1, 0)$, such that $f(\Gamma) < 0$ for $\Gamma_0 < \Gamma < 0$.*

Proof.

$$\text{Let} \quad G(\Gamma) = \begin{cases} f(\Gamma), & \text{if } \Gamma < 0, \\ 0 & \text{if } \Gamma = 0, \end{cases}$$

then $\lim_{\Gamma\to 0^-} G(\Gamma) = \lim_{\Gamma\to 0^-} f(\Gamma) = 0 = G(0)$, that is G is a continuous function for $\Gamma \leq 0$. Besides this we have also $\lim_{\Gamma\to 0^-} G'(\Gamma) = \lim_{\Gamma\to 0^-} f'(\Gamma) > 0$, which implies that there exists $\Gamma_0 \in (-1, 0)$, such that $G'(\Gamma) > 0$ for $\Gamma_0 < \Gamma < 0$. Since $G(0) = 0$, we conclude that $G(\Gamma) < 0$ for $\Gamma \in (\Gamma_0, 0)$, and the same is true for $f(\Gamma)$.

□

Lemma 2. *Let f and f' be continuous functions for $\Gamma < 0$. If $\lim_{\Gamma\to 0^-} f(\Gamma) = 0$ and $\lim_{\Gamma\to 0^-} f'(\Gamma) < 0$ then, $f(\Gamma) > 0$ for $\Gamma_1 < \Gamma < 0$, where $\Gamma_1 \in (-1, 0)$.*

Proof. Apply *Lemma* 1, with $g(\Gamma) = -f(\Gamma)$. □

Now applying *Lemma* 1 with $f(\Gamma) = E''(\Gamma)$, we conclude that $E''(\Gamma) < 0$ for Γ close to zero. Next from *Lemma* 2, with $f(\Gamma) = E'(\Gamma)$, we infer that $E'(\Gamma) > 0$ near zero. Using *lemma* 1 again, now with $f(\Gamma) = E(\Gamma)$, we conclude that $E(\Gamma) < 0$ near zero. $\square$

5.0.1 Entropy condition for a solution of $IBVP_{V_0}$

It is difficult to describe explicitly all entropy functions Φ. Nevertheless, employing separation of variables, we can figure out one of them, which we shall call a standard entropy function.

For this purpose we set $\Phi(V, \Gamma) = a(V) + b(\Gamma)$, where a and b are functions to be determined, in (5.3). Then we obtain

$$a''(V)P'(\Gamma) = b''(\Gamma),$$

which is equivalent to

$$a''(V) = \frac{b''(\Gamma)}{P'(\Gamma)}.$$

Since V and Γ are independent variables, therefore there exists a constant denoted by c such that

$$a''(V) = \frac{b''(\Gamma)}{P'(\Gamma)} = c,$$

which implies

$$a(V) = c\frac{V^2}{2} + c_1 V + c_2$$

and

$$b(\Gamma) = c\int_0^\Gamma P(w)dw + c_3\Gamma + c_4.$$

Assuming $c_1 = c_2 = c_3 = c_4 = 0$, we get

$$\Phi(V, \Gamma) = c\frac{V^2}{2} + c\int_0^\Gamma P(w).$$

Now substituting Φ into (5.2), results in

$$\begin{aligned} \Psi_V &= -cP(\Gamma) \\ \Psi_\Gamma &= -cVP'(\Gamma) \end{aligned} \tag{5.13}$$

A solution of the system (5.13) is

$$\Psi(V,\Gamma) = -cVP(\Gamma).$$

Now strict convexity of Φ is equivalent to positive definiteness of its Hessian matrix. Here this matrix is

$$D^2\Phi(V,\Gamma) = \begin{pmatrix} 2c & 0 \\ 0 & cP'(\Gamma) \end{pmatrix}$$

so that $D^2\Phi > 0$ iff $c > 0$ and $P'(\Gamma) > 0$. Also without loss of generality we may put $c = 1$, thereby obtaining

$$\Phi(V,\Gamma) = \frac{V^2}{2} + \int_0^\Gamma P(w)dw \tag{5.14}$$

and

$$\Psi(V,\Gamma) = -P(\Gamma)V. \tag{5.15}$$

The function (5.14) is well known entropy function for a p-system, [1], which we call a standard entropy function.

For the solution $S(\Gamma_l)$, (4.38), the condition (5.7) can be simplified into (5.8). Here, $P(0) = 0$, $\Phi(-V_0, 0) = \frac{V_0^2}{2}$, $\Phi(\Gamma, 0) = \int_0^\Gamma P(w)dw$, $\Psi(-V_0, 0) = 0$ and $\Psi(0,\Gamma) = 0$, so that (5.8) becomes

$$2\int_0^{\Gamma_l} P(w)dw \leq \Gamma_l P(\Gamma_l). \tag{5.16}$$

This is the entropy condition for $S(\Gamma_l)$ corresponding to a standard entropy function, (5.14), and Ψ given by (5.15).

Remark 11. *The assertion of Proposition 5.2 does not say how far from* 0 *the inequality still holds or it already does not hold. It is rather difficult, except of* *a linear case, to answer this question without having more particular information about the entropy functions. That is why we concentrate ourselves on studying the inequality (5.16), for previously listed models of elastic materials.*

We notice the following facts which clarify an importance of genuine nonlinearity condition, (2.1.3), in studying the entropy condition (5.16).

Proposition 5.3. *If* $P(0) = 0$ *and* $P''(\Gamma) < 0$ *for all* $\Gamma \in (-1, 0)$*, then* $S(\Gamma_l)$ *satisfies (5.16) for all* $\Gamma_l \in (-1, 0]$.

Proof. Indeed, let $G(\Gamma_l) = 2\int_0^{\Gamma_l} P(w)dw - \Gamma_l P(\Gamma_l)$. Then $G' = P - \Gamma_l P'$ and $G'' = -\Gamma_l P''$. Therefore $G''(\Gamma_l) < 0$ and consequently G' is decreasing for $-1 < \Gamma_l < 0$. However $G'(0) = 0$. Hence $G'(\Gamma_l) > 0$ for $-1 < \Gamma_l < 0$. Now $G(\Gamma_l)$ is increasing there, and since $G(0) = 0$, therefore $G(\Gamma_l) < 0$ for $-1 < \Gamma_l < 0$. □

Proposition 5.4. *If* $P(0) = 0$ *and* $P''(\Gamma_l) > 0$ *for* $\overline{\Gamma} < \Gamma_l < 0$*, where* $\overline{\Gamma} \in (-1, 0)$*, then* $S(\Gamma_l)$ *does not satisfy (5.16). Therefore* $S(\Gamma_l)$ *does not satisfy the entropy condition for* $\overline{\Gamma} < \Gamma_l < 0$.

Proof. Let $G(\Gamma_l) = 2\int_0^{\Gamma_l} P(w)dw - \Gamma_l P(\Gamma_l)$. Then $G' = P - \Gamma_l P'$ and $G'' = -\Gamma_l P''$. Therefore $G''(\Gamma_l) > 0$ and consequently G' is increasing for $\overline{\Gamma} < \Gamma_l < 0$. However $G'(0) = 0$. Hence $G'(\Gamma_l) < 0$ for $\overline{\Gamma} < \Gamma_l < 0$. Now $G(\Gamma_l)$ is decreasing there, and since $G(0) = 0$, therefore $G(\Gamma_l) > 0$ for $\overline{\Gamma} < \Gamma_l < 0$. If entropy condition does not hold for a particular entropy function then the solution in question is not an entropy solution. □

CHAPTER 6
RESULTS ON HYPERBOLICITY

Throughout this and the remaining chapters the condition of no interpenetration of matter, $\Gamma > -1$, is assumed.

6.1 St.Venant-Kirchhoff model

From (3.12),

$$P(\Gamma) = \Big(\frac{\lambda + 2\mu}{2\rho_0}\Big)(1+\Gamma)(2+\Gamma)\Gamma.$$

We want to know when $P' > 0$, what is equivalent to

$$3\Gamma^2 + 6\Gamma + 2 > 0.$$

Hence $P'(\Gamma) > 0$ iff $\Gamma < -1 - \frac{1}{\sqrt{3}}$ or $\Gamma > -1 + \frac{1}{\sqrt{3}}$. Since we require $\Gamma > -1$, therefore the corresponding p-system is hyperbolic iff $\Gamma > -1 + \frac{1}{\sqrt{3}}$.

6.2 Modified Kirchhoff model

Here it is convenient to use instead of Γ a variable $s = \Gamma + 1$, restricted by $s > 0$, since Γ is subject to $\Gamma > -1$. Then we define

$$\widetilde{P}(s) = P(s-1). \tag{6.1}$$

Setting $s = \Gamma + 1$ in (3.13) we obtain,

$$\widetilde{P}(s) = \mu s^3 - \mu s + \frac{\lambda \ln(s)}{s}.$$

Introducing a parameter $\alpha = \frac{\lambda}{\mu}$ we can rewrite $\widetilde{P}(s)$ into

$$\widetilde{P}(s) = \mu s^3 - s\mu + \frac{\alpha\mu \ln(s)}{s}.$$

Hence

$$\widetilde{P}'(s) = \frac{\mu(3s^4 - s^2 - \alpha\ln(s) + \alpha)}{s^2},$$

so that the sign of $\widetilde{P}'(s)$ is determined by the sign of $T(s)$, where

$$T(s) = 3s^4 - s^2 - \alpha\ln(s) + \alpha,$$

T has only one real positive critical point.

$$s_p = \frac{\sqrt{3 + 3\sqrt{1 + 12\alpha}}}{6}.$$

Indeed,

$$T'(s) = \frac{12s^4 - 2s^2 - \alpha}{s}$$

and its zeros are

$$\pm\frac{\sqrt{3 + 3\sqrt{1 + 12\alpha}}}{6},\ \pm\frac{\sqrt{3 - 3\sqrt{1 + 12\alpha}}}{6}.$$

However $\lim\limits_{s\to 0^+} T(s) = \infty$ and $\lim\limits_{s\to\infty} T(s) = \infty$. Therefore T has at s_p an absolute minimum.

Thus $\widetilde{P}'(s) > 0$ for all $s > 0$ iff $T(s_p) > 0$. To work out that condition we substitute $s = s_p$ in T, and we obtain $T(s_p) = L(\alpha)$, where

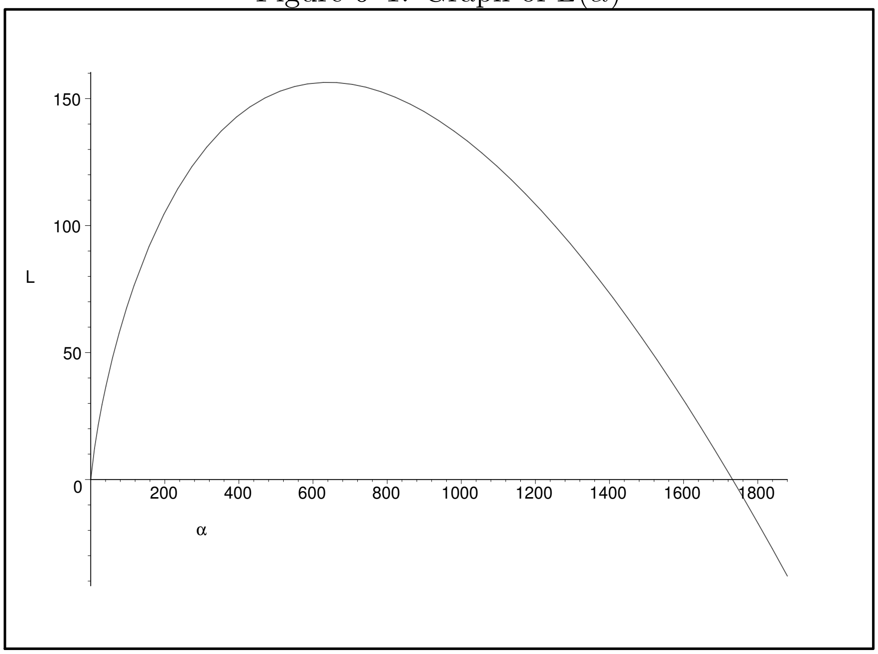
$$L(\alpha) = -\frac{1}{24} - \frac{\sqrt{1 + 12\alpha}}{24} + \frac{5}{4}\alpha + \alpha\ln(2) + \alpha\ln(3) - \frac{1}{2}\alpha\ln(3 + 3\sqrt{1 + 12\alpha}).$$

Now

$$L'(\alpha) = 1 - \ln\left(\frac{1}{6}\sqrt{3 + 3\sqrt{1 + 12\alpha}}\right).$$

Consequently L is an increasing function of α up to its unique critical point, $\alpha_c = 12e^4 - 2e^2$, where $L(\alpha_c) \approx 156.405393$, and then it is decreasing to $-\infty$ as α goes to infinity. Since $L(0) = -1/2$, so there are two positive zeros of L: $\alpha_1 < \alpha_2$. Therefore $\widetilde{P}'(s) > 0$ for all $s > 0$, iff $\alpha_1 < \alpha < \alpha_2$. An approximate inequality for α is $0.04465672959 < \alpha < 17732.05696$ (see Figure 6–1).

Figure 6–1: Graph of $L(\alpha)$



6.3 Ogden model

From (3.14),

$$P(\Gamma) = \lambda\Gamma + \mu\frac{\Gamma^2 + 2\Gamma}{\Gamma + 1},$$

and

$$P'(\Gamma) = \lambda + \mu\frac{(\Gamma + 1)^2 + 1}{(\Gamma + 1)^2}$$

Therefore $P'(\Gamma) > 0$ for all $\Gamma > -1$, and the corresponding p-system is hyperbolic for all $\Gamma > -1$.

6.4 Blatz-Ko-Ogden model

Setting $s = 1 + \Gamma$ in (3.15), results in

$\widetilde{P}(s) = \mu s\big[f\mu + (1-f)s^{2\beta-2} - fs^{-2\beta-2} - \frac{(1-f)}{s^4}\big]$

and

$$\widetilde{P}'(s) = \frac{\mu}{s^4}(2fs^{-2\beta+2}\beta + fs^4 - s^{2\beta+2} + s^{2\beta+2}f + fs^{-2\beta+2} + 3 - 3f + 2s^{2\beta+2}\beta - 2s^{2\beta+2}f\beta).$$

Thus $\widetilde{P}'(s) > 0$ iff

$$2fs^{-2\beta+2}\beta + fs^4 + fs^{-2\beta+2} + 3(1-f) + s^{2\beta+2}(1-f)(2\beta-1) > 0. \tag{6.2}$$

It is obvious that (6.2) holds for all $s > 0$ provided $\beta \geq \frac{1}{2}$. Next we ask about validity of (6.2) for all $s > 0$ if $0 < \beta < 1/2$.

We notice that $s_0 = \big(\frac{3}{1-2\beta}\big)^{\frac{1}{2\beta+2}}$ is an unique positive zero of a decreasing function $g(s) = 3 - s^{2\beta+2}(1-2\beta)$, so that $g(s) > 0$ for $0 < s < s_0$, $g(s_0) = 0$ and $g(s) < 0$ for $s > s_0$. Consequently

- if $0 < s \leq s_0$ then

$$3(1-f) - s^{2\beta+2}(1-f) + s^{2\beta+2}(1-f)2\beta = (1-f)g(s) \geq 0.$$

  Hence (6.2) is true.

- If $0 < \beta < 1/2$ and $s > s_0$, we multiply the equation (6.2) by $s^{2\beta}$ to obtain

$$2fs^2\beta + fs^{4+2\beta} + fs^2 + 3(1-f)s^{2\beta} + s^{4\beta+2}(1-f)(2\beta-1) > 0,$$

  which is equivalent to

$$[s^2(2\beta+1) + s^{4+2\beta} - s^{2\beta}(3 - (1-2\beta)s^{2\beta+2})]f > -3s^{2\beta} + (1-2\beta)s^{4\beta+2}, \tag{6.3}$$

and next to

$$f > r(s),$$

where

$$r(s) = \frac{s^{2\beta}[(1-2\beta)s^{2\beta+2}-3]}{s^2[(1+2\beta)+s^{2\beta+2}]+s^{2\beta}[(1-2\beta)s^{2\beta+2}-3]}.$$

We observe that $\lim_{s\to\infty} r(s) = 0$, $\lim_{s\to s_0} r(s) = 0$ and for all $s > s_0$

$$0 < r(s) < 1.$$

Therefore there exists a maximum value of $r(s)$, $\max_{s>s_0} r(s)$ and

$$0 < \max_{s>s_0} r(s) < 1.$$

Consequently if $0 < \beta < \frac{1}{2}$ then the hyperbolicity condition for all $s > 0$ is equivalent to

$$f > \max_{s>s_0} r(s).$$

Now the goal is to get an upper bound for $\max_{s>s_0} r(s)$. For that the following *lemma* is useful.

Lemma 3. *Let a, b and $\alpha > 0$. If $b \geq a$ and $b > \alpha$ then*

$$\frac{a-\alpha}{b-\alpha} \leq \frac{a}{b}$$

Proof.

$$\frac{a-\alpha}{b-\alpha} - \frac{a}{b} = \frac{\alpha(a-b)}{b(b-\alpha)} \leq 0. \qquad \square$$

First we notice that for $s > s_0$

$$\frac{-3s^{2\beta}+(1-2\beta)s^{4\beta+2}}{s^2(2\beta+1)+s^{4+2\beta}-s^{2\beta}(3-(1-2\beta)s^{2\beta+2})} < \frac{-3s^{2\beta}+(1-2\beta)s^{4\beta+2}}{s^{4+2\beta}-s^{2\beta}(3-(1-2\beta)s^{2\beta+2})}.$$

Consequently

$$r(s) < \frac{(1-2\beta)s^{4\beta+2}-3s^{2\beta}}{s^{4+2\beta}+(1-2\beta)s^{4\beta+2}-3s^{2\beta}}.$$

Next set $a = (1-2\beta)s^{4\beta+2}$, $b = s^{4+2\beta} + (1-2\beta)s^{4\beta+2}$ y $\alpha = 3s^{2\beta}$. Clearly $b \geq a$ holds. We demonstrate that $b > \alpha$. Indeed, since $0 < \beta < \frac{1}{2}$ and $s > s_0$ therefore

$$s^4 + (1-2\beta)s^{2\beta+2} > (1-2\beta)s_0^{2\beta+2},$$

which is equivalent to $s^4 + (1-2\beta)s^{2\beta+2} > 3$.

Consequently $s^{4+2\beta} + (1-2\beta)s^{4\beta+2} \geq 3s^{2\beta}$, which means that $b > \alpha$.

Therefore by *Lemma* 3 we obtain

$$r(s) < \frac{(1-2\beta)s^{4\beta+2}}{s^{4+2\beta} + (1-2\beta)s^{4\beta+2}}.$$

However

$$\frac{(1-2\beta)s^{4\beta+2}}{s^{4+2\beta} + (1-2\beta)s^{4\beta+2}} < \frac{1-2\beta}{s_0^{2-2\beta} + 1 - 2\beta}.$$

Consequently

$$\max_{s>s_0} r(s) < \frac{1-2\beta}{s_0^{2-2\beta} + 1 - 2\beta} < 1.$$

So if $0 < \beta < 1/2$ the hyperbolicity condition holds all $s > 0$ provided $f \geq \frac{1-2\beta}{s_0^{2-2\beta}+1-2\beta}$

.

CHAPTER 7
RESULTS ON GENUINE NONLINEARITY

We are interested if $P''(0) < 0$. If that is the case then $P''(\Gamma) < 0$ near $s = 0$ as well. Having in mind the entropy inequality for weak solutions, we want to know how $P''(\Gamma)$ behaves for $\Gamma \in (-1, 0)$.

7.1 St.Venant-Kirchhoff model

From (3.12),

$$P(\Gamma) = \Big(\frac{\lambda + 2\mu}{\rho_0}\Big)(\Gamma^3 + 3\Gamma^2 + 2\Gamma).$$

Then

$$P''(\Gamma) = \frac{6(\lambda + 2\mu)(\Gamma + 1)}{\rho_0},$$

so that $P'' > 0$ for all $\Gamma > -1$. Consequently the condition of genuine nonlinearity cannot be satisfied for any $\Gamma > -1$.

7.2 Modified Kirchhoff model

Setting $s = \Gamma + 1$ in (3.13), we obtain

$$\widetilde{P}(s) = \mu s^3 - \mu s + \frac{\alpha\mu \ln s}{s}.$$

Hence

$$\widetilde{P}''(s) = \frac{\mu(6s^4 - 3\alpha + 2\alpha \ln s)}{s^3},$$

so that $\widetilde{P}''(1) < 0$ iff $\alpha > 2$; genuine nonlinearity holds at $s = 1$ ($\Gamma = 0$) iff $\alpha > 2$.

More generally $\widetilde{P}''(s) < 0$ iff

$$6s^4 - 3\alpha + 2\alpha \ln s < 0.$$

Now we study that inequality on $(0, \infty)$. For this purpose we define $G(s) = 6s^4 - 3\alpha + 2\alpha \ln s$. Then since $G'(s) = \frac{2(12s^4+\alpha)}{s} > 0$, G is a strictly increasing function on $(0, \infty)$. Besides this, $\lim_{s\to\infty} G(s) = \infty$ and $\lim_{s\to 0^+} G(s) = -\infty$. Therefore G has an unique zero s_0 in $(0, \infty)$. We can express that zero by means of the *LambertW* function, where *LambertW* is the inverse of the function

$$f : [-1, \infty) \to [-e^{-1}, \infty),$$

given by

$$f(w) = we^w.$$

Consequently

$$LambertW : [-e^{-1}, \infty) \to [-1, \infty).$$

Now we find s_0:

$$6s_0^4 - 3\alpha + 2\alpha \ln s_0 = 0,$$

$$\ln s_0^4 = \frac{6\alpha - 12s_0^4}{\alpha},$$

and

$$s_0^4 e^{\frac{12s_0^4}{\alpha}} = e^6.$$

Setting $w = \frac{12s_0^4}{\alpha}$, that is $s_0 = \left(\frac{\alpha w}{12}\right)^{1/4}$, the last equation takes the following form

$$we^w = \frac{12e^6}{\alpha},$$

which is equivalent to

$$w = LambertW\left(\frac{12e^6}{\alpha}\right).$$

Hence

$$s_0 = \left[\frac{\alpha}{12} LambertW\left(\frac{12e^6}{\alpha}\right)\right]^{1/4}.$$

Thus for $0 < s < s_0$, $\widetilde{P}''(s) < 0$, for $s = s_0$ $\widetilde{P}''(s) = 0$ and for $s > s_0$, $\widetilde{P}''(s) > 0$.

In addition we observe that $s_0 > 1$, if and only if $\alpha > 2$. Indeed $s_0 > 1$ is equivalent to

$$LambertW\left(\frac{12e^6}{\alpha}\right) > \frac{12}{\alpha} \Leftrightarrow \frac{12e^6}{\alpha} > f\left(\frac{12}{\alpha}\right) \Leftrightarrow \frac{12e^6}{\alpha} > \frac{12}{\alpha}e^{\frac{12}{\alpha}} \Leftrightarrow \alpha > 2.$$

Therefore we conclude that $\widetilde{P}''(s) < 0$ holds for all $s \in (0, 1]$ iff $s_0 > 1$, what is equivalent to $\alpha > 2$.

7.3 Ogden model

From (3.14),

$$P(\Gamma) = \frac{1}{\rho_0}\left(\lambda\Gamma + \frac{\mu(\Gamma^2 + 2\Gamma)}{\Gamma + 1}\right).$$

Then

$$P''(\Gamma) = -\frac{2\mu}{(1+\Gamma)^3}.$$

Thus $P''(\Gamma) < 0$ for all $\Gamma > -1$.

7.4 Blatz-Ko-Ogden model

Setting $s = \Gamma + 1$ in (3.15), we get

$$\widetilde{P}(s) = \mu s\left[f\mu + (1-f)s^{2\beta-2} - fs^{-2\beta-2} - \frac{(1-f)}{s^4}\right].$$

and

$$\widetilde{P}''(s) = -2\mu\frac{(1-f)[s^{2\beta+2}(1-\beta)(2\beta-1) + 6] + fs^{-2\beta+2}(2\beta+1)(\beta+1)}{s^5}$$

Hence $\widetilde{P}''(1) = -2\mu[4f(\beta^2-1)-(\beta+1)(2\beta-5)]$. Now $\widetilde{P}''(1) < 0$, is equivalent to

$$4f(1-\beta) < 5-2\beta. \tag{7.1}$$

To analyze (7.1), we consider the following cases.

- If $\beta = 1$, clearly (7.1) holds.
- If $1 < \beta < 5/2$, then $5-2\beta$ is positive and $4f(1-\beta)$ is negative therefore (7.1) holds.
- If $\beta \geq 5/2$, the inequality (7.1) holds provided that

  $$f > \frac{2\beta-5}{4(\beta-1)}.$$

  Notice that in this case

  $$0 \leq \frac{2\beta-5}{4(\beta-1)} < 1.$$
- If $0 < \beta < 1$, then $4f(1-\beta) < 4(1-\beta)$ since $0 < f < 1$, and additionally we have also that $4(1-\beta) < 5-2\beta$. Thereby (7.1) holds.

Therefore we conclude that if $\beta \in (0, \frac{5}{2}]$ then $\widetilde{P}''(1) < 0$ without any other than $0 < f < 1$ restriction for f.

Now, we want to know when $\widetilde{P}''(s) < 0$, which is equivalent to

$$s^{2\beta+2}(1-f)(2\beta-1)(1-\beta) + 6(1-f) + fs^{-2\beta+2}(2\beta+1)(\beta+1) > 0. \tag{7.2}$$

It is clear that if $\frac{1}{2} \leq \beta \leq 1$ then (7.2) holds.

Next we consider the case of $\beta \in (0, \frac{1}{2}) \cup (1, \infty)$.

For this purpose we multiply (7.2) by $s^{2\beta}$ and rewrite it into

$$fs^2(2\beta+1)(\beta+1) - (1-f)s^{2\beta}[s^{2\beta+2}(2\beta-1)(\beta-1) - 6] > 0. \tag{7.3}$$

Let $T(s) = s^{2\beta+2}(2\beta-1)(\beta-1) - 6$. Then $s_0 = \left(\frac{6}{(2\beta-1)(\beta-1)}\right)^{\frac{1}{2\beta+2}}$ is its unique positive zero. Note that T is an increasing function, so that $T(s) < 0$ for $0 < s < s_0$ and $T(s) > 0$ for $s > s_0$. We consider two cases now.

- $s \leq s_0$. Then (7.3) holds since $T(s) \leq 0$.
- $s > s_0$. Then $T(s) > 0$, so that (7.3) can be rewritten in the following form

$$f > H(s),$$

where

$$H(s) = \frac{s^{4\beta+2}(2\beta-1)(\beta-1) - 6s^{2\beta}}{s^{4\beta+2}(2\beta-1)(\beta-1) - 6s^{2\beta} + s^2(2\beta+1)(\beta+1)}.$$

We verify that H is a increasing function for all $s > s_0$. Indeed

$$H'(s) = \frac{4s^{2\beta+1}(\beta-1)(2\beta+1)(\beta+1)[s^{2\beta+2}\beta(2\beta-1) - 3]}{[s^{4\beta+2}(2\beta-1)(\beta-1) - 6s^{2\beta} + s^2(2\beta+1)(\beta+1)]^2},$$

H' is greater than zero when the numerator is greater than zero. It is clearly so for $0 < \beta < \frac{1}{2}$.

To ensure $H'(s) > 0$ for $\beta > 1$, we have to solve the inequality

$$2s^{2\beta+2}\beta^2 - s^{2\beta+2}\beta - 3 > 0.$$

The result is $s > s_1$, where $s_1 = \left(\frac{3}{\beta(2\beta-1)}\right)^{\frac{1}{2\beta+2}}$. However one can verify that $s_0 > s_1$. consequently for $s > s_0$, $H(s)$ is increasing and moreover $\lim_{s\to\infty} H(s) = 1$. Thereby we arrive at the conclusion that 1 is the least upper bound for H.

Thus for $\beta \in (0, \frac{1}{2}) \cup (1, \infty)$ and $s > s_0$, there exists s_2 such that $\widetilde{P}''(s) < 0$ up to s_2 and then it changes its sign.

Finally we point out the following fact. If $\beta \in (0, \frac{5}{2}]$ then $\widetilde{P}''(s) < 0$ for all $s \in (0, 1]$. Indeed, from our discussion, it is true for $\beta \in [\frac{1}{2}, 1]$. For $\beta \in (0, \frac{1}{2}) \cup (1, \frac{5}{2})$, it is so because $\beta \leq \frac{5}{2}$ is equivalent to $s_0 > 1$.

Next we show some graphs for H, they show what we confirmed theoretically.

Figure 7–1: Graph of $H(\beta)$ with $\beta = 2.3$

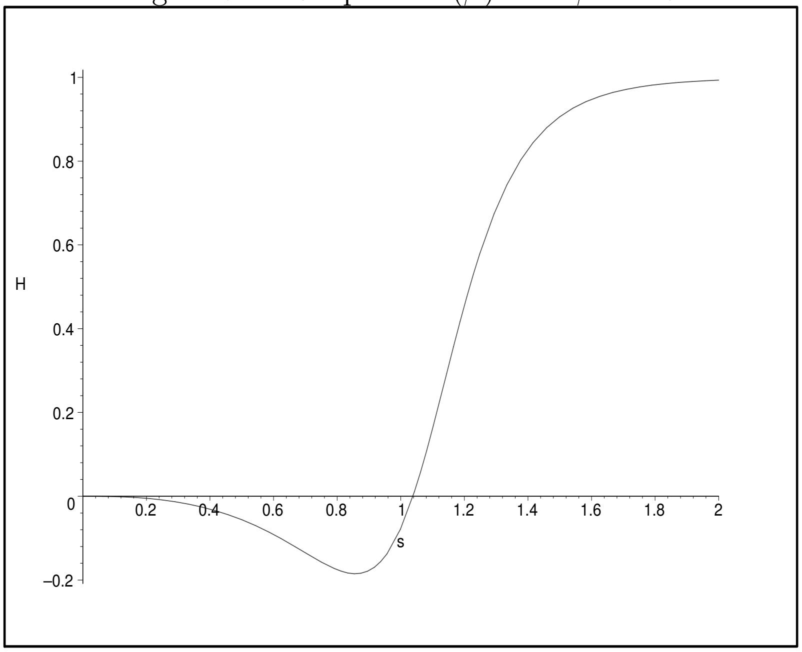
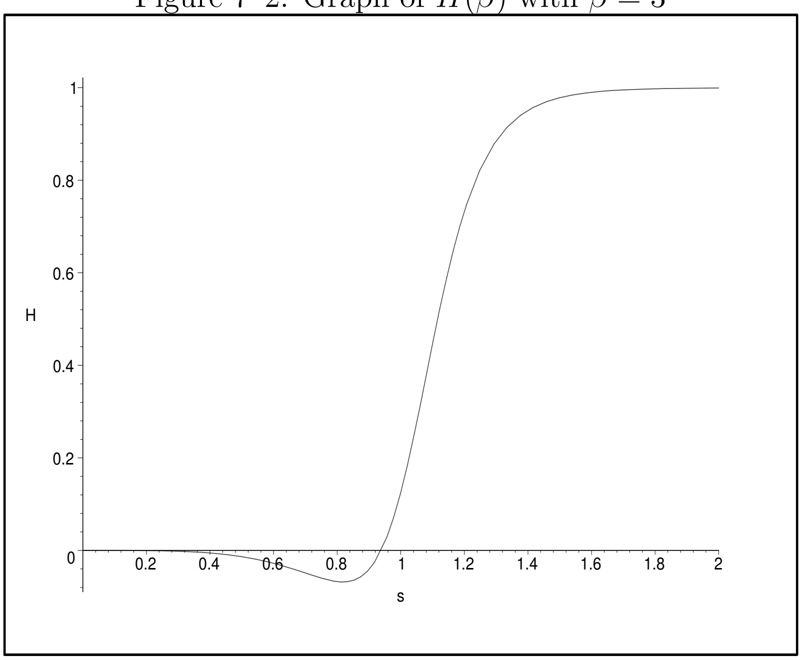


Figure 7–2: Graph of $H(\beta)$ with $\beta = 3$



CHAPTER 8
RESULTS ON ENTROPY CONDITION WITH A STANDARD ENTROPY FUNCTION

Here we shall see if the inequality (5.16) holds for our models of study.

8.1 St.Venant-Kirchhoff model

Proposition 8.1. *For the St.Venant-Kirchhoff model, $S(\Gamma_l)$ does not satisfy the entropy condition for $-1 < \Gamma_l < 0$.*

Proof. From (3.12),

$$P(\Gamma) = c^2(1+\Gamma)(2+\Gamma)\Gamma,$$

where $c^2 = \frac{\lambda+2\mu}{\rho_0}$,

so that

$$\int_0^\Gamma P(w)dw = \frac{c^2}{2}\left(\frac{\Gamma^4}{4} + \Gamma^3 + \Gamma^2\right),$$

and

$$\Gamma P(\Gamma) = c^2(\Gamma^4 + 3\Gamma^2 + 2\Gamma^2).$$

Thereby the inequality (5.16) is equivalent to

$$\frac{V_0^2}{2} \geq \frac{c^2}{2}\left(\frac{\Gamma^4}{4} + \Gamma^3 + \Gamma^2\right),$$

and since $\Gamma P(\Gamma) = V_0^2$, to

$$\Gamma^3(\Gamma + 2) \geq 0,$$

which does not hold for $-1 < \Gamma < 0$. ∎

8.2 Modified Kirchhoff model

Setting $s = \Gamma + 1$ in (3.13), we obtain

$$\widetilde{P}(s) = \mu s^3 - \mu s + \frac{\alpha\mu \ln s}{s}.$$

Then

$$\int_1^s \widetilde{P}(w)dw = \frac{1}{4}\mu[s^4 - 2s^2 + 2\alpha(\ln s)^2 + 1],$$

therefore (5.16) is equivalent to the following inequality

$$s(s+1)(s-1)^3 + 2\alpha[(s - 1 - s\ln s)\ln s] \geq 0, \tag{8.1}$$

It is clear that (8.1) holds for $s = 1$. Now we study it for $0 < s < 1$.

First we note that the function multiplying α is positive. It is so because $\ln s < 0$ for $0 < s < 1$ and because the function $r(s) = s - 1 - s\ln s$ is negative as well, since $r(s)$ is increasing and $r(1) = 0$. Consequently for $0 < s < 1$ (8.1) is equivalent to

$$\alpha \geq K(s),$$

where

$$K(s) = \frac{s(s+1)(1-s)^3}{2[(s - 1 - s\ln s)\ln s]}. \tag{8.2}$$

Moreover since $\ln s < s - 1$ therefore

$$0 < \frac{1-s}{-\ln s} < 1. \tag{8.3}$$

Also, if $g(s) = 2(1 - s + s\ln s) - (1-s)^2$ then $g'(s) = 2(\ln s + 1 - s)$ which is negative. Thus $g(s)$ is a strictly decreasing function and $g(1) = 0$, what implies $g(s) > 0$ for all $0 < s < 1$. Now because of (8.3) and $g(s) > 0$ we obtain

$$K(s) < s(s+1) < 2,$$

for $0 < s < 1$.

Therefore we conclude that for $\alpha \geq 2$ (5.16) holds, for all $0 < s \leq 1$. Next, we ask what happens if $\alpha < 2$. To answer this question we observe that $K(s)$ can be written as a product of three positive functions, namely:

$$f_1 = s(s+1),$$
$$f_2 = \frac{1-s}{-2\ln s},$$
$$f_3 = \frac{(1-s)^2}{(1-s+s\ln s)}.$$

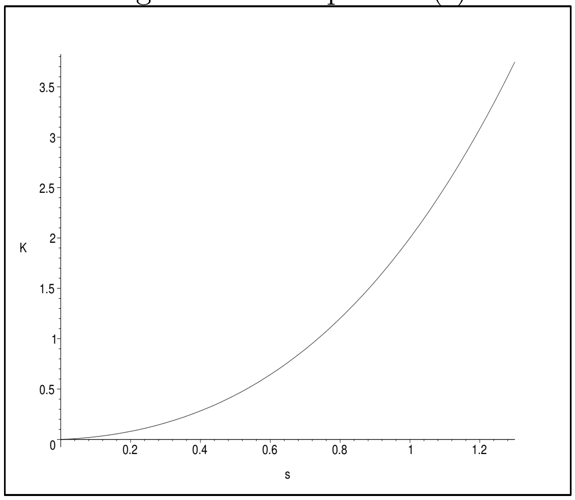
Moreover the derivatives of these functions are positive for all $0 < s < 1$,

$$f_1' = 2s + 1,$$
$$f_2' = \frac{s\ln s - s + 1}{2s(\ln s)^2},$$
$$f_3' = \frac{(s-1)[(s+1)\ln s + 2(1-s)]}{(s\ln s - s + 1)^2}.$$

Therefore we conclude by the product rule that K is a strictly increasing function and additionally $\lim_{s\to 1} K(s) = 2$ and $\lim_{s\to 0^+} K(s) = 0$ (see Figure 8–1). Hence if $0 < \alpha < 2$, then (5.16) holds for all $s \in (0, s_e]$ and does not for $s \in (s_e, 1)$, where s_e is an unique solution in the interval $(0, 1)$ of the equation

$$\frac{s(s+1)(1-s)^3}{2(s-1-s\ln s)\ln s} = \alpha.$$

Figure 8–1: Graph of K(s)



8.3 Ogden model

From (3.14),

$$P(\Gamma) = \frac{1}{\rho_0}\left(\lambda\Gamma + \frac{\mu(\Gamma^2 + 2\Gamma)}{\Gamma + 1}\right).$$

Then

$$\int_0^\Gamma P(w)dw = -\frac{\mu}{\rho_0}\ln(\Gamma + 1) + \frac{(\mu + \lambda)\Gamma^2}{2\rho_0} + \frac{\mu\Gamma}{\rho_0},$$

so that the inequality (5.16) is equivalent to prove

$$2(\Gamma + 1)\ln(\Gamma + 1) \geq \Gamma(\Gamma + 2). \tag{8.4}$$

Setting $s = \Gamma + 1$ in (8.4) one arrives at

$$2s\ln s - s^2 + 1 \geq 0. \tag{8.5}$$

To prove (8.5) we define $t(s) = 2s\ln s - s^2 + 1$. Then $t'(s) = 2\ln s + 2 - 2s < 0$, because $\ln s \leq s - 1$ for $0 < s < 1$ consequently $t(s)$ is strictly decreasing function since $t(1) = 0$ then $t(s) > 0$ for $0 < s < 1$. That prove (8.5) for $0 < s \leq 1$ and (8.4) for $-1 < \Gamma \leq 0$. Therefore we conclude that for Ogden model (5.16) holds for all $-1 < \Gamma_l \leq 0$.

8.4 Blatz-Ko-Ogden model

We know from Chapter 7 that if $\beta \in (0, \frac{5}{2}]$ then $\widetilde{P}''(s) < 0$ for all $s \in (0, 1]$ and consequently by *Proposition* 5.3 the inequality (5.16) holds for all $s \in (0, 1]$. Additionally from Chapter 6 we know that $\widetilde{P}'(1) = 2 + 2\beta > 0$ and $\widetilde{P}(1) = 0$, therefore by *Proposition* 5.2 we conclude that if $\beta \in (0, \frac{5}{2}]$, $S(\Gamma_l)$ satisfies the entropy condition for all Γ_l sufficiently close to zero.

Setting $s = \Gamma + 1$ in (3.14), we obtain

$$\widetilde{P}(s) = \mu s\left[f\mu + (1-f)s^{2\beta-2} - fs^{-2\beta-2} - \frac{(1-f)}{s^4}\right].$$

Then

$$\int_1^s \widetilde{P}(w)dw = \frac{\mu}{2\beta}\frac{[s^4 f\beta + s^{-2\beta+2}f + (1-f)(\beta + s^{2+2\beta}) - s^2(1+\beta)]}{s^2},$$

so that (5.16) is equivalent to

$$\begin{aligned} &[s^{4\beta}(\beta(1-s)+s) + s^{2\beta-1}\beta(1-s^3) + (1-s)\beta(1-s^{2\beta-2}) - s]f \\ &- [s^{2\beta-2}(s^{2\beta+3}(1-\beta) + \beta s^{2+2\beta} + \beta(2s-1) - s^3(1+\beta))] \geq 0. \end{aligned} \tag{8.6}$$

Next we rewrite (8.6) in the following form

$$B(s)f \geq s^{2\beta-2}A(s), \tag{8.7}$$

where

$$B(s) = -s^{4\beta+1}\beta - \beta s + 2s^{2\beta-1}\beta - \beta s^{2+2\beta} + s^{4\beta}\beta + \beta - s^{2\beta-2}\beta + s^{4\beta+1} - s$$

and

$$A(s) = -\beta s^{2\beta+3} + 2\beta s + \beta s^{2+2\beta} - \beta + s^{2\beta+3} - s^3\beta - s^3.$$

One of the difficulties that arise with these functions $A(s)$ and $B(s)$ is that they are not analytic at 0 for all $\beta > 0$. Next we know that $A(1) = A'(1) = A''(1) =$

$B(1) = B'(1) = B''(1) = 0$, that is 1 is a zero of multiplicity 3. Therefore there exists analytic functions at 1, $g_1(s)$ and $g_2(s)$ such that $A(s) = (1-s)^3 g_1(s)$ and $B(s) = (1-s)^3 g_2(s)$, but the problem here it is that we can not find explicitly these functions $g_1(s)$ and $g_2(s)$. For these reasons we discuss validity of (8.7) for the particular case of $\beta = \frac{n}{2}$, where n is an integer number greater than 5. Now it is convenient to define $a(s) = 2A(s)$ and $b(s) = 2B(s)$. With these changes $a(s)$ and $b(s)$ become polynomial functions. We factorize them now.
Setting $\beta = n/2$ in $a(s)$, we obtain

$$a(s) = -n + 2ns - (2+n)s^3 + ns^{n+2} + (2-n)s^{n+3}. \tag{8.8}$$

We verify that $a(1) = a'(1) = a''(1) = 0$, so that we can rewrite $a(s)$ in a form of

$$a(s) = (1-s)^3 \sum_{k=0}^{n} a_k s^k, \tag{8.9}$$

where $a_0, \dots, a_n$ are to be determined. It is straight forward to obtain that

$$\begin{aligned}(1-s)^3 \sum_{k=0}^{n} a_k s^k &= \sum_{k=0}^{n} a_k s^k - 3\sum_{k=0}^{n} a_k s^{k+1} + 3\sum_{k=0}^{n} a_k s^{k+2} - \sum_{k=0}^{n} a_k s^{k+3} \\ &= a_0 + (-3a_0 + a_1)s + (3a_0 - 3a_1 + a_2)s^2 \\ &+ \sum_{k=3}^{n}(a_k - 3a_{k-1} + 3a_{k-2} - a_{k-3})s^k \\ &+ (-a_{n-2} + 3a_{n-1} - 3a_n)s^{n+1} + (-a_{n-1} + 3a_n)s^{n+2} - a_n s^{n+3}.\end{aligned}$$

Therefore from (8.8) and (8.9) we obtain

$a_0 = -n$,

$a_1 = -n$,

$a_2 = 0$,

$a_3 = n-2$,

$a_4 = 2(n-3)$,

$a_5 = 3(n-4)$

and proceeding inductively we find that for $k = 1, \ldots, n$

$a_k = (k-2)(n+1-k)$.

Consequently $a(s) = (1-s)^3[-n - ns + \sum_{k=3}^{n}(k-2)(n+1-k)s^k]$.

Next setting $\beta = n/2$ in $b(s)$, we obtain:

$$b(s) = -ns^{2n+1} - ns + 2ns^{n-1} - ns^{n+2} + ns^{2n} + n - ns^{n-2} + 2s^{2n+1} - 2s. \quad (8.10)$$

Then as in case of $a(s)$, we can find real coefficients b_k, for $k = 0, 1, \ldots, 2n-2$, such that

$$b(s) = (1-s)^3 \sum_{k=0}^{2n-2} b_k s^k. \quad (8.11)$$

Indeed,

$$\begin{aligned}(1-s)^3 \sum_{k=0}^{2n-2} b_k s^k &= \sum_{k=0}^{2n-2} b_k s^k - 3\sum_{k=0}^{2n-2} b_k s^{k+1} + 3\sum_{k=0}^{2n-2} b_k s^{k+2} - \sum_{k=0}^{2n-2} b_k s^{k+3} \\ &= \sum_{k=0}^{2n-2} b_k s^k - 3\sum_{k=1}^{2n-1} b_{k-1} s^k + 3\sum_{k=2}^{2n} b_{k-2} s^k - \sum_{k=3}^{2n+1} b_{k-3} s^k \\ &= b_0 + (b_1 - 3b_0)s + (b_2 - 3b_1 + 3b_0)s^2 \\ &+ \sum_{k=3}^{2n-2}(b_k - 3b_{k-1} + 3b_{k-2} - b_{k-3})s^k + (-3b_{2n-2} + 3b_{2n-3} - b_{2n-4})s^{2n-1} \\ &+ (3b_{2n-2} - b_{2n-3})s^{2n} - b_{2n-2}s^{2n+1}.\end{aligned}$$

Then from (8.10) and (8.11) we get:

$b_0 = n$,

$b_1 = 2(n-1)$,

$b_2 = 3(n-1)$,

and proceeding inductively we find that for $k = 0, \ldots, n-3$

$b_k = (k+1)(n-k)$.

Further,

$b_{n-2} = n-2,$

$b_{n-1} = 0,$

$b_n = 0,$

$b_{n+1} = n-2,$

$b_{n+2} = 2(n-3),$

$b_{n+3} = 3(n-4),$

and by induction we find that for $l = 0, \dots, n-1$

$b_{n+l} = l(n-l-1).$

Consequently,

$$b(s) = (1-s)^3 \left[\sum_{k=0}^{n-3} (k+1)(n-k)s^k + (n-2)s^{n-2} + \sum_{k=n+1}^{2n-2} (k-n)(2n-k-1)s^k \right].$$

Next we notice some relevant facts about $a(s)$ and $b(s)$ in the following *Proposition*.

Proposition 8.2.

1. *There exists $s_1 \in (0,1)$ such that $a(s) < 0$ for $s \in (0, s_1)$, $a(s_1) = 0$ and $a(s) > 0$ for $s \in (s_1, 1)$.*
2. *$b(s) > 0$ for $s \in (0,1)$.*
3. *let $M(s) = \frac{s^{2\beta-2}a(s)}{b(s)}$, therefore $M(s) < 1$, for $s > 0$.*

Proof.

1. Let
$$\widetilde{a}(s) = -n - ns + \sum_{k=3}^{n} (k-2)(n+1-k)s^k.$$
$\widetilde{a}(s)$ has one variation in sign; $n+1-k > 0$ since $k \leq n$. Therefore by Descartes' Rule of signs, $\widetilde{a}(s)$ has at most one positive zero. The existence of such zero in $(0,1)$ is guaranteed by the Intermediate Value Theorem. Indeed:
$\widetilde{a}(0) = -n$

and

$$\begin{aligned}\widetilde{a}(1) &= -2n + \sum_{k=3}^{n}(k-2)(n+1-k) \\ &= -2n + n\sum_{k=3}^{n}(k-2) - \sum_{k=3}^{n}(k-2)(k-1) \\ &= -2n + \frac{n(n-2)(n-1)}{2} - \frac{n(n-2)(n-1)}{3} \\ &= \frac{n(n+2)(n-5)}{6}.\end{aligned}$$

Therefore $\widetilde{a}(0) < 0$ and $\widetilde{a}(1) > 0$, because $n > 5$. Denoting this zero by s_1 we see that $\widetilde{a}(s) < 0$ for $s \in (0, s_1)$, $\widetilde{a}(s_1) = 0$ and $\widetilde{a}(s) > 0$ for $s \in (s_1, 1)$. And the same is true for $a(s)$.

2. Let

$$\widetilde{b}(s) = \sum_{k=0}^{n-3}(k+1)(n-k)s^k + (n-2)s^{n-2} + \sum_{k=n+1}^{2n-2}(k-n)(2n-k-1)s^k.$$

Note that for $\widetilde{b}(s)$ all its coefficients are strictly positive. Indeed $n-k > 0$, for $k \leq n-3$. $2n-k-1 > 0$, for $k \leq 2n-2$, and finally $n-2 > 0$, for $n > 5$. Therefore $\widetilde{b}(s)$ does not have positive zeros. Consequently $b(s)$ does not have zeros in $(0,1)$ and it is positive there.

3.

$$\frac{s^{n-2}(-n-ns+\sum_{k=3}^{n}(k-2)(n-(k-1))s^k))}{\sum_{k=0}^{n-3}(k+1)(n-k)s^k+(n-2)s^{n-2}+\sum_{k=n+1}^{2n-2}(k-n)(2n-k-1)s^k}$$

$$\leq \frac{\sum_{k=3}^{n}(k-2)(n-(k-1))s^{n+k-2}}{\sum_{k=0}^{n-3}(k+1)(n-k)s^k+(n-2)s^{n-2}+\sum_{k=n+1}^{2n-2}(k-n)(2n-k-1)s^k}$$

$$= \frac{\sum_{k=1}^{n-2}k(n-(k+1))s^{n+k}}{\sum_{k=0}^{n-3}(k+1)(n-k)s^k+(n-2)s^{n-2}+\sum_{k=n+1}^{2n-2}(k-n)(2n-k-1)s^k}$$

$$= \frac{\sum_{k=n+1}^{2n-2}(k-n)(2n-k-1)s^k}{\sum_{k=0}^{n-3}(k+1)(n-k)s^k+(n-2)s^{n-2}+\sum_{k=n+1}^{2n-2}(k-n)(2n-k-1)s^k}$$

$$< 1.$$

□

Remark 12. *An experimentation with plots indicates that for a given value of* $\beta > \frac{5}{2}$ *there exist* $f_\beta \in (0,1)$ *such that,* $S(\Gamma_l)$ *satisfies the entropy condition (5.16), for all* $s \in (0,1]$, *provided* $f \geq f_\beta$, *where* $f_\beta = \max_{s\in(0,1]} M(s)$. *If however* $f < f_\beta$, *then there exists* $s_\beta \in (0,1]$ *such that (5.16) holds for all* $s \in (0, s_\beta)$ *and does not for* $s \in (s_\beta, 1)$, *while at* $s = 1$ *it holds again. The Proposition 8.2 confirms partially these observations.*

Next we show some graphs of $M(s)$ for different values of β, that confirm our assertions.

Figure 8–2: $\beta = 2$

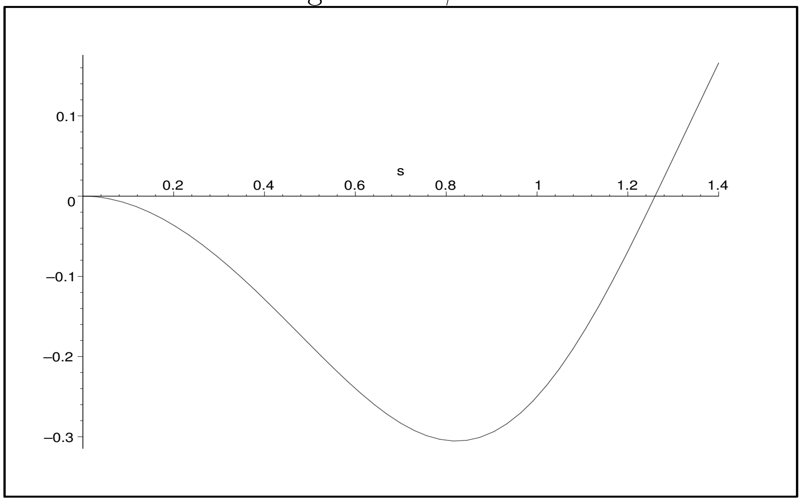


Figure 8–3: $\beta = 2.5$

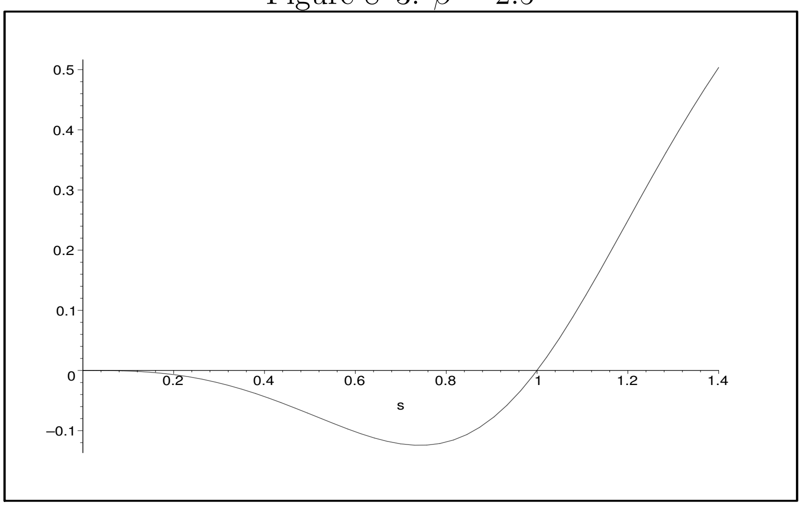
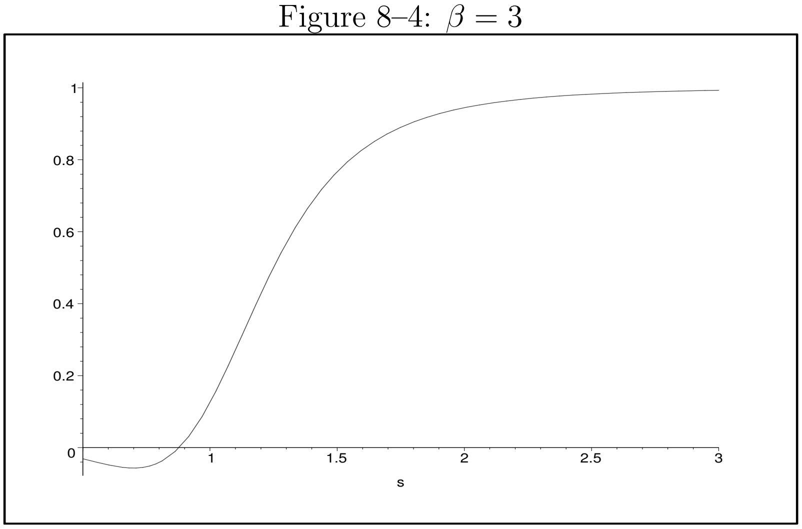


Figure 8–4: $\beta = 3$



CHAPTER 9
NUMERICAL COMPARISON OF THE COMPRESSION SHOCKS FOR VARIOUS MODELS

In this Chapter we obtain numerical values of Γ_l, for the compression shock, corresponding to given values of V_0; more specifically we use

$$\widetilde{V}_0 = \frac{\rho_0 V_0^2}{\mu}.$$

We do this for the following models: Modified Kirchhoff, Ogden and Blatz-Ko-Ogden.

Here Γ_l is determined by the first equation in (4.38), which after substituting $\lambda = 2\mu\beta$ can be rewritten in the following form:

$$Q(\Gamma) = \widetilde{V}_0.$$

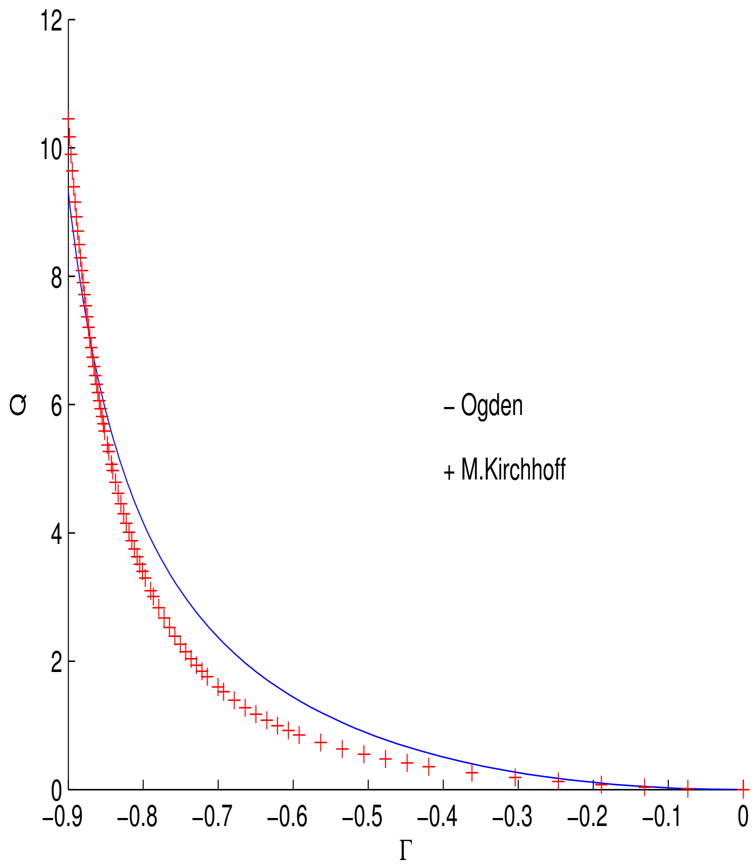
$$Q(\Gamma) = \Gamma\big((1+\Gamma)^3 - (1+\Gamma) + 2\beta\frac{\ln(1+\Gamma)}{(1+\Gamma)}\big) \quad \text{(Modified Kirchhoff)}$$

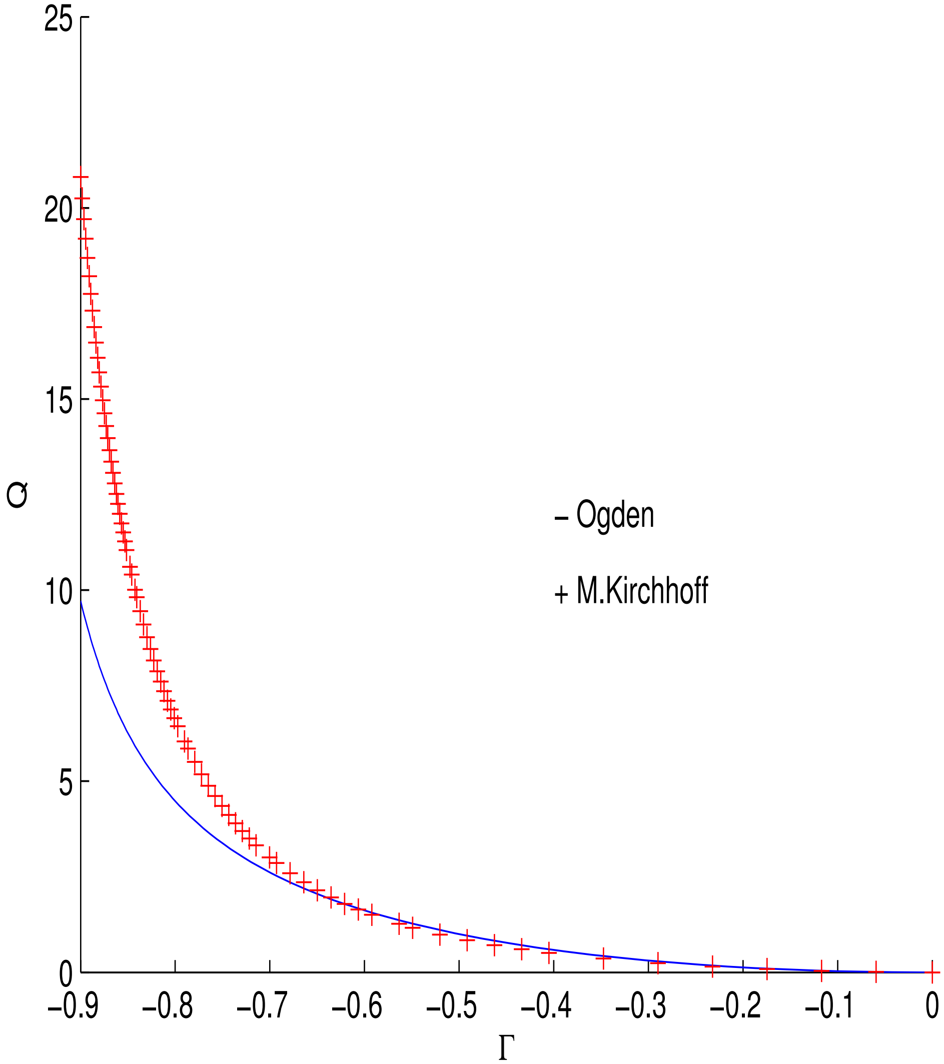
$$Q(\Gamma) = \Gamma\big(2\beta\Gamma + \frac{(2+\Gamma)\Gamma}{\Gamma+1}\big) \quad \text{(Ogden model)}$$

$$Q(\Gamma) = \Gamma(1+\Gamma)\left\{f\left[1-(1+\Gamma)^{-2\beta-2}\right]+\frac{(1-f)}{(1+\Gamma)^4}\left[(1+\Gamma)^{2\beta+2}-1\right]\right\} \quad \text{(Blatz-Ko-Ogden model)}$$

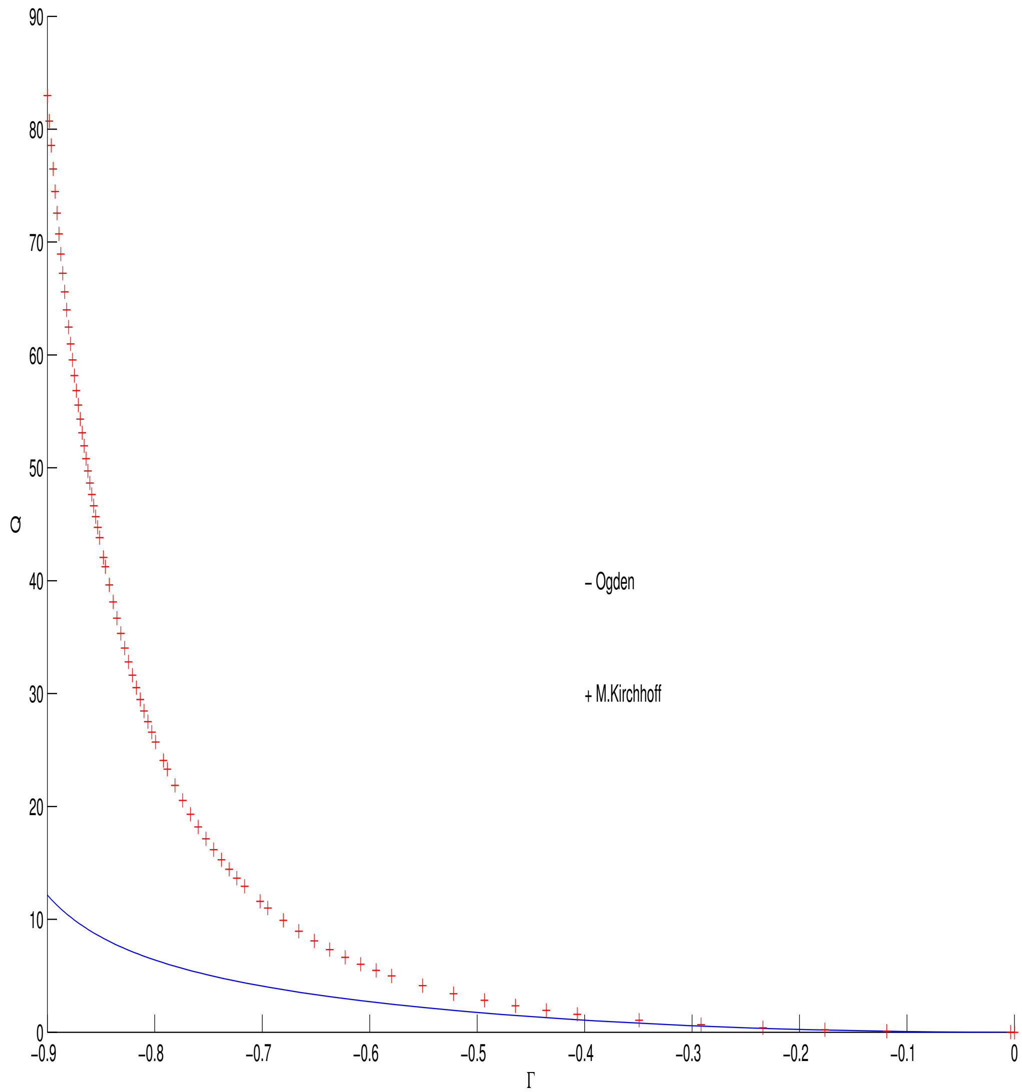
The reason of the change of λ into $\lambda = 2\mu\beta$, is to have $\widetilde{V}_0$ dimensionless what is also true for the function $Q(\Gamma)$ and β.

Next we show graphs of the function $Q(\Gamma)$ in the ΓQ plane for the models under consideration, for the values of β and f specified. These graphs demonstrate that given a value of $\widetilde{V}_0$, the horizontal line $Q = \widetilde{V}_0$ intersects the graph of $Q(\Gamma)$ in an unique point, whose first coordinate is Γ_l. We show the graphs of $Q(\Gamma)$ for the models of Modified Kirchhoff and Ogden in the same plane. The graph of $Q(\Gamma)$ for the model of Blatz-Ko-Ogden is shown separately due to behavior of the models near $\Gamma = -1$.

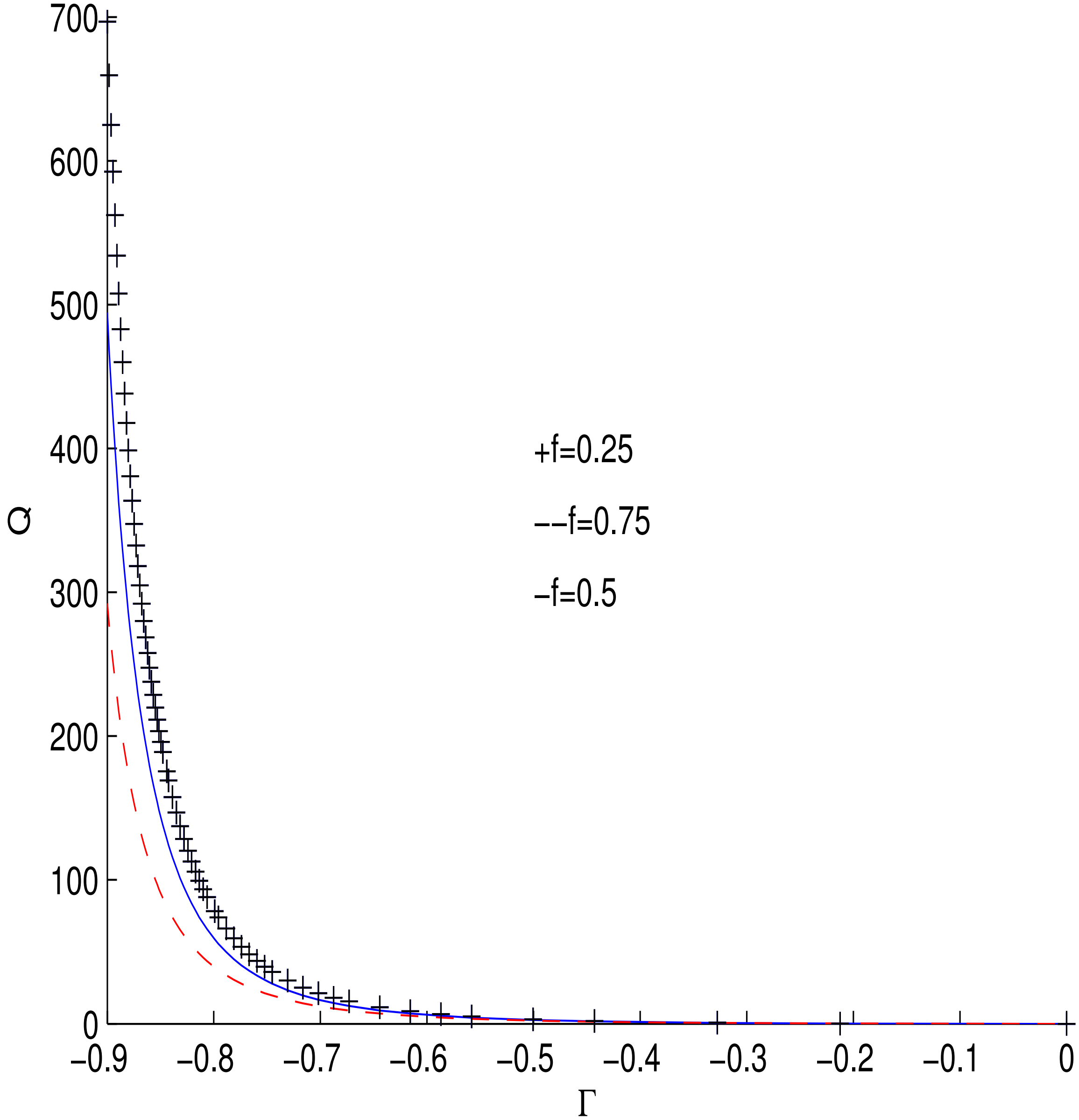


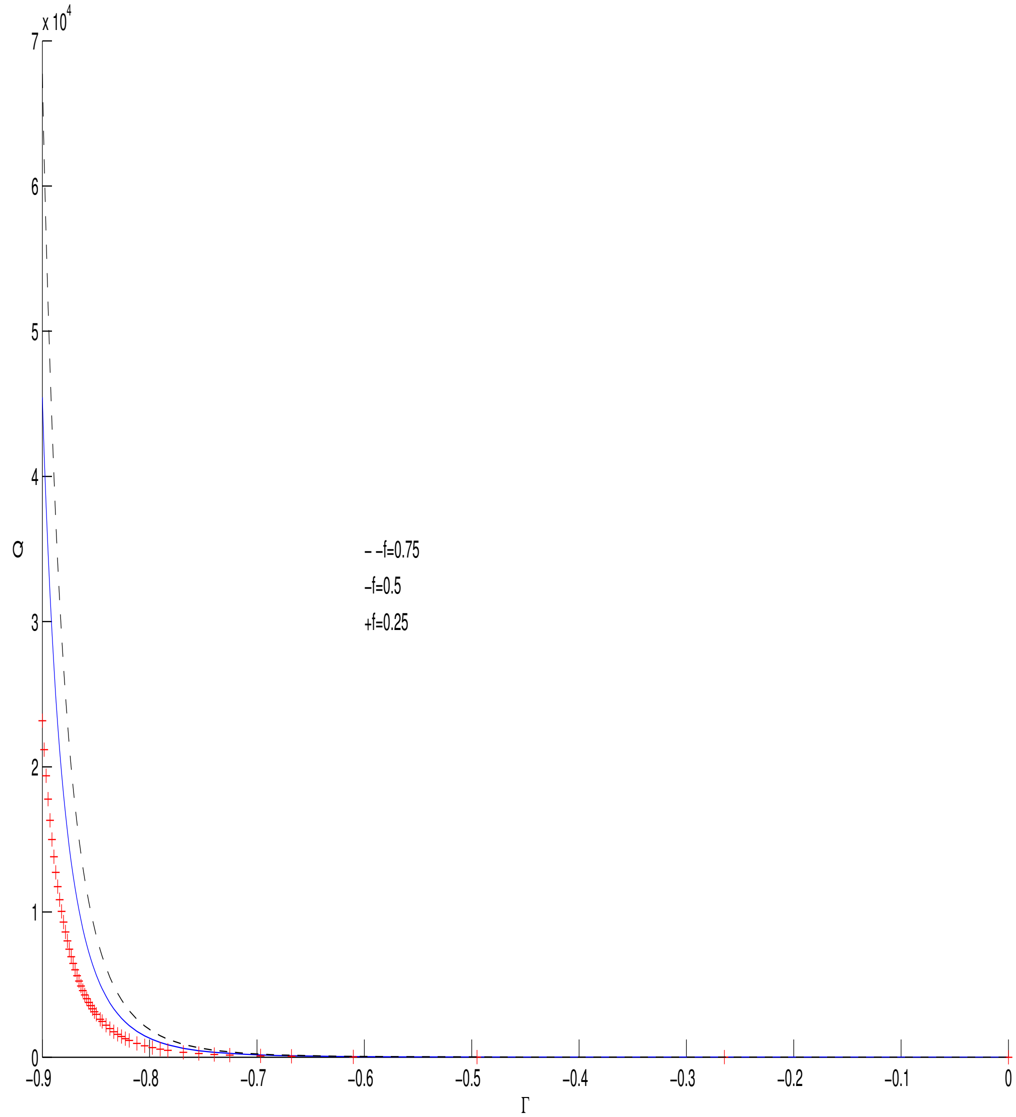


25
20
15
10
5
0
Q
−0.9
−0.8
−0.7
−0.6
−0.5
−0.4
−0.3
−0.2
−0.1
0
Γ
– Ogden
+ M.Kirchhoff


σ
Γ
– Ogden
+ M.Kirchhoff

We show some graphs of $Q(\Gamma)$ for Blatz-Ko-Ogden model with $\beta = 1/2$ and $\beta = 2$ respectively.




x 10^4
7
6
5
4
3
2
1
0
σ
– –f=0.75
–f=0.5
+f=0.25
−0.9
−0.8
−0.7
−0.6
−0.5
−0.4
−0.3
−0.2
−0.1
0
Γ

The numerical values of Γ_l which correspond to a given values of $\widetilde{V}_0$ for fixed values of β and f are shown in the following tables.

Table 9–1: $\beta = 0.25$

$\widetilde{V}_0$ \ Γ	Ogden	M.Kirchhoff	Blatz-Ko ($f = 0.25$)	Blat-Ko ($f = 0.5$)
0.1	−0.1912	−0.217420	−0.373581	−0.447296
0.25	−0.2929	−0.351888	−0.386761	−0.457802
0.5	−0.3978	−0.486632	−0.407276	−0.474264
2	−0.6667	−0.733399	−0.495098	−0.547908
4	−0.7938	−0.818724	−0.559164	−0.604841
10	−0.9063	−0.897073	−0.646584	−0.684415
40	−0.9753	−0.960995	−0.760038	−0.787456

Table 9–2: $\beta = 0.5$

$\widetilde{V}_0$ \ Γ	Ogden	M.Kirchhoff	Blatz-Ko ($f = 0.25$)	Blat-Ko ($f = 0.5$)
0.1	−0.1764	−0.187793	−0.396762	−0.467705
0.25	−0.2722	−0.294512	−0.406156	−0.475495
0.5	−0.3729	−0.401976	−0.42134	−0.488051
2	−0.6446	−0.638674	−0.495152	−0.550033
4	−0.7808	−0.739561	−0.556282	−0.603611
10	−0.9027	−0.842347	−0.643929	−0.682669
40	−0.9678	−0.9357075	−0.759011	−0.786753

Table 9–3: $\beta = 2$

$\widetilde{V}_0$ \ Γ	Ogden	M.Kirchhoff	Blatz-Ko ($f = 0.25$)	Blat-Ko ($f = 0.5$)
0.1	−0.1276	−0.123866	−0.516574	−0.571569
0.25	−0.2	−0.189781	−0.518832	−0.573691
0.5	−0.2798	−0.257764	−0.522632	−0.577243
2	−0.5298	−0.440641	−0.546019	−0.598641
4	−0.6951	−0.546173	−0.576544	−0.625725
10	−0.8757	−0.681674	−0.645033	−0.685736
40	−0.9731	−0.843239	−0.758247	−0.786419

Table 9–4: $\beta = 5$

$\widetilde{V}_0$ \ Γ	Ogden	M.Kirchhoff	Blatz-Ko ($f = 0.25$)	Blat-Ko ($f = 0.5$)
0.1	−0.0909	−0.145165	−0.646008	−0.68264
0.25	−0.1433	−0.223122	−0.646471	−0.683109
0.5	−0.2020	−0.302771	−0.647248	−0.683894
2	−0.3975	−0.506407	−0.652028	−0.688701
4	−0.5501	−0.614488	−0.658692	−0.695328
10	−0.7941	−0.743021	−0.680005	−0.715904
40	−0.9684	−0.881932	−0.759992	−0.788159

CHAPTER 10
CONCLUSIONS

1. A definition of a weak solution of an initial and boundary value problem for a p-system, in the first quadrant of the Xt-plane, is provided. There are two unknown functions $V(X,t)$ and $\Gamma(X,t)$. Consequently, there are two initial conditions (at $t=0$) and only one boundary condition (at $X=0$). There are considered four types of boundary conditions: the first, (4.1), for $V(0,t)$, the second, (4.8), for $\Gamma(0,t)$ and the other two are mixed boundary conditions involving $V(0,t)$ and $\Gamma(0,t)$, (4.9) and (4.10) respectively. The first two types of boundary conditions are particular cases of the other two. All of that is consistent with what is known in case of classical solutions of linear systems, [19].
2. A particular weak solution of a p-system, called a compression shock is constructed. It satisfies the initial and boundary conditions given by (4.27), which is a particular case of (4.1). This solution, denoted by $S(V_0)$, is determined by one parameter $V_0 > 0$, which can be interpreted as an impact velocity. $S(V_0)$ is constant by parts, having jump discontinuities of V and Γ along the line $X = \sigma t$; $(V,\Gamma) = (-V_0, 0)$, for $X > \sigma t$ and $(V,\Gamma) = (0, \Gamma_l)$, for $X < \sigma t$, where the constants $\sigma > 0$ and $\Gamma_l < 0$ are solutions of the Rankine-Hugoniot conditions. It is the hyperbolicity condition and furthermore $P(0) = 0$ and $\lim_{\Gamma \to -1^+} P(\Gamma) = -\infty$ which guarantee the existence and uniqueness of such σ and Γ_l for the models under study. Because of that the notation $S(\Gamma_l)$ is used for $S(V_0)$.

3. We confirm, *Proposition* 5.2, a known in the literature fact, [2], that for each entropy/entropy-flux pair (Φ, Ψ), where Φ is strictly convex, under the hypothesis of hyperbolicity and genuine nonlinearity at $\Gamma = 0$, the solutions like $S(\Gamma_l)$ satisfy the entropy condition for all Γ_l sufficiently close to zero. We do that for this particular solution, $S(\Gamma_l)$, only. The proof is based on Taylor's expansion formula.
4. For the St.Venant-Kirchhoff model according to *Proposition* 8.1, $S(\Gamma_l)$ does not satisfy the entropy condition. Consequently we can consider this model as inadequate to describe the compression shock. For the Kirchhoff modified, Ogden and Blatz-Ko-Ogden models we can verify that they satisfy, under certain restrictions on their parameters, the hypothesis of the *Proposition* 5.2. These restrictions appear in details in Chapters 6 and 7. Therefore in those models, under the corresponding restrictions $S(\Gamma_l)$ satisfies the entropy condition, for Γ_l sufficiently close to zero.
5. The *Proposition* 5.2 does not provide an exact information about the interval for Γ_l in which the entropy condition for $S(\Gamma_l)$ holds. That is why we concentrate on the entropy condition with a well known in literature [1], entropy/entropy-flux pair (Φ, Ψ), which we call a standard entropy/entropy-flux pair. We provide the restrictions for the parameters μ,λ, f and for Γ_l, under which $S(\Gamma_l)$ fulfills the entropy condition with this standard entropy function. This discussion is complete, except of the Blatz-Ko-Ogden model for $\beta > \frac{5}{2}$. In this case we clarify the validity of the entropy condition only for $\beta = \frac{n}{2}$, where n is an integer number greater than 5.
6. An open question remains about the validity of the entropy condition with a general entropy function.

CHAPTER 11
FUTURE WORKS

1. Clarify the validity of the entropy condition for the Blatz-Ko-Ogden model in case of $\beta > \frac{5}{2}$ and such that 2β is not an integer number.
2. If $V_0 < 0$ then the system of Rankine-Hugoniot conditions, (4.37), has a solution (σ, Γ_l) where $\sigma > 0$ and $\Gamma_l > 0$. The corresponding $S(V_0)$ can be called an extension shock. The question is if $S(V_0)$ satisfies the entropy condition.
3. Generalize this work by considering a deformation $\phi(X, t)$ of a slightly more general form than (3.1),

$$\begin{cases} \phi^1(\mathbf{X}, t) = X^1 + U_1(X^1, t) \\ \phi^2(\mathbf{X}, t) = X^2 + U_2(X^1, t) \\ \phi^3(\mathbf{X}, t) = X^3 + U_3(X^1, t). \end{cases}$$

   In this case the displacements of particles are in all three directions of X_1, X_2 and X_3 axes, but they depend only on X^1.
4. Extend the discussion of the Chapter 8 for a general entropy function.

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