ON THE NUMBER OF $\tau_{(n)}$-FACTORS

By

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In 2006, Anderson and Frazier defined the theory of τ-factorization as a generalization of the theory of comaximal factorizations given by McAdam and Swan. The theory of τ-factorization was built on integral domains. The idea can be associated with a restriction to the multiplicative operation. That is, they considered a symmetric relation τ on the nonzero nonunit elements of an integral domain and allowed two or more elements to be multiplied if and only if they were pairwise related.

Formally, a nonzero nonunit element a of an integral domain D (denoted by $D^{\#}$) has a τ-factorization if $a = \lambda a_1 * \cdots * a_k$, where λ is an unit element and for any $i \neq j$, $a_i \, \tau \, a_j$. In order to expand our vocabulary with respect to this concept, the a_i's will be called τ-factors and a a τ-product of the a_i's.

This work studied a specific relation on the set of integers defined by Frazier and Anderson, and further studied by Hamon, Ortiz, Lanterman, Florescu, Serna and Barrios. Note that the notation and definition that will be used follows from

Hamon's work. They defined the relation $\tau_{(n)}$ (also denoted as τ_n) on $\mathbb{Z}^{\#}$ by $a\,\tau_{(n)}\,b$ if and only if $a - b \in (n)$, the principal ideal generated by n. The main goal in this research was to find formulas that count the number of $\tau_{(n)}$-factors of an integer distinct to 0, 1 and -1. These formulas would be very useful tool to identify which elements cannot be $\tau_{(n)}$-factored properly.

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SOBRE EL NÚMERO DE $\tau_{(n)}$-FACTORES

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En 2006, Anderson y Frazier definen la teoría de τ-factorización como una generalización de la teoría de factorizaciones comaximales hecha por McAdam y Swan. La teoría de τ-factorización se construyó sobre dominios integrales. La idea puede ser asociada con una restricción de la multiplicación. Es decir, se considera una relación simétrica τ sobre los elementos no unidades y distintos de cero de un dominio integral, y que dos o más elementos pueden multiplicarse si y sólo si están relacionados entre si.

Formalmente, un elemento distinto de cero y no unidad a de un dominio integral D (denotado por $D^{\#}$) tiene una τ-factorización si $a = \lambda a_1 * \cdots * a_k$, donde λ es una unidad, y para cualquier $i \neq j$, $a_i \, \tau \, a_j$. Con el fin de ampliar nuestro vocabulario con respecto a este concepto, los a_i's serán llamados τ-factores, y a un τ-producto de los a_i's.

En este trabajo se estudió una relación específica sobre el conjunto de los enteros definida por Frazier y Anderson, y estudiada posteriormente por Hamon, Ortiz,

Lanterman, Florescu, Serna y Barrios. Debe tenerse en cuenta que la notación y definición que se utiliza es la misma del trabajo de Hamon. Ellos definen la relación $\tau_{(n)}$ (también conocida como τ_n) sobre $\mathbb{Z}^{\#}$ como $a\,\tau_{(n)}\,b$ si y sólo si $a-b \in (n)$, el ideal generado por n. El principal objetivo de esta investigación era encontrar fórmulas que cuenten el número de $\tau_{(n)}$-factores de un número entero distinto de 0, 1 y -1. Estas fórmulas serán una herramienta muy útil para identificar los elementos que no pueden ser $\tau_{(n)}$-factorizados propiamente.

To my parents Ricardo and Edita.

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LIST OF ABBREVIATIONS AND SYMBOLS

$\mathbb{Z}$	The set of integers.
$\nu(a)$	The number of positive factors of an integer a.
$\phi(a)$	The number of positive integer less than a that are relatively prime to a.
D	Integral domain.
D^*	Set of nonzero elements of D.
$\mathrm{U}(D)$	Group of units of D.
$D^\#$	Set of nonzero nonunit elements of D.
τ	A symmetric relation on $D^\#$.
$\cdot$	Usual product.
$*$	τ-product.
$\cdots$	It is used to indicate a product of an indexed list of integers a_1, a_2, $\ldots$, a_n.
$*\cdots*$	It is used to indicate a τ-product of an indexed list of integers a_1, a_2, $\ldots$, a_n.
$\|A\|$	Cardinality of the set A.
$\equiv_n$	Conguence relation modulo n.
$\tau_{(n)}$	Relation defined by $a\,\tau_{(n)}\,b$ if and only if $a-b \in (n)$, the ideal generated by $n \in \mathbb{Z}$.
$\tau'_{(n)}$	Relation defined by $a\,\tau'_{(n)}\,b$ if and only if $a\,\tau_{(n)}\,b$ or $a\,\tau_{(n)}\,-b$.
$\nu_{\tau_{(n)}}(a)$	The number of positive $\tau_{(n)}$-factors of an integer $a \in \mathbb{Z}^\#$.

CHAPTER 1
INTRODUCTION

Number theory is one of the oldest branches of mathematics, taking its impulse trying to find the solutions of Diophantine's equations. For this, it was necessary to study the properties of the integers and the developement of algorithms to answer some basic notions such as: the greatest common divisor of two integers, the least common multiple of two integers, how many distinct divisors does an integer have, among others. Number theory has spread to the point that nowadays it splits in 3 main areas: analytic number theory, algebraic number theory and computational number theory.

The first signs of modern algebra arose when Galois was trying to find all the roots of a polynomial. Several authors attempted to give the structural properties of the integers to sets, generalizing the notions of number theory. One of these structures is called an integral domain, which is a commutative ring with identity and without zero divisors. The set of integers (denoted by $\mathbb{Z}$) is an example of an integral domain with other interesting properties such as unique factorization into primes, division algorithm, and others.

Let D be an integral domain. For simplicity, denote the set of nonzero elements of D by D^*, the set of elements with multiplicative inverse, called *units*, by $\mathrm{U}(D)$ and the set of nonzero nonunits elements by $D^\# = D - \big(U(D) \cup \{0\}\big)$. Anderson and

Frazier in [3], motivated by the work of McAdam and Swan on comaximal factorizations [4], defined the theory of τ-factorizations or generalized factorizations. They observed that the idea of two elements being comaximal can be represented by a symmetric relation. Formally, they considered a symmetric relation τ on the nonzero nonunit elements and called a *τ-factorization* of an element $a \in D^{\#}$ an expression of the form $a = \lambda \cdot a_1 * a_2 * \cdots * a_n$, where $\lambda \in \mathrm{U}(D)$ and $a_i \,\tau\, a_j$ for each $i \neq j$ in the set $\{1, 2, \ldots, n\}$. Note that a and $\lambda(\lambda^{-1}a)$ are both τ-factorizations, more known as the trivial ones. For simplicity, a is called a *τ-product* of a_i's and each a_i is called a *τ-factor* of a, usually denoted by $a_i \,|_\tau\, a$. In the above factorization, "$\cdot$" denotes the usual product and "$*$" denotes a τ-product; that is, a product of the factors of x which are related under τ. With this new definition, they generalized the notion of factorization theory, by choosing $\tau = S \times S$, where S represents a particular set of elements of interest. For example, S can represent the set of elements such as primes (irreducibles, primals, rigids and primary elements) and the τ-factorizations are the factorizations into primes (resp. irreducibles, primals, rigids and primary elements). As a nonstandard example of τ-factorizations, Frazier considered the relation defined on $\mathbb{Z}^{\#}$ by $a \,\tau_n\, a$ if and only if $a \equiv b (\mathrm{mod}\ n)$ where $n \geq 2$ is an integer. They considered cases where $n \in \{2, 3, 4, 5, 6\}$ and showed the form that the elements of $\mathbb{Z}^{\#}$ must have in order to be a τ_n-atom (that is, have no nontrivial τ_n-factorizations).

In 2007, Hamon [1], redefined the relation $\tau_{(n)}$ by $a \,\tau_{(n)}\, b$ if and only if $a - b \in (n)$, where (n) is the principal ideal generated by n. In case of $n > 1$, both definitions are equivalent. But in this case, Hamon's definition extends Frazier's definition to any $n \in \mathbb{Z}$. Since $(-n) = (n)$, then it is sufficient to analyze when $n \geq 0$. So, when $n = 0$, the relation $\tau_{(0)}$ is equivalent to "$=$". Since 1 divides the difference of any two integers, then the $\tau_{(1)}$-factorizations coincide with the usual factorizations. Hamon found some properties that the $\tau_{(n)}$-factorizations satisfy. Her main result was the fact that $\mathbb{Z}$ is $\tau_{(n)}$-atomic (that is, any nonzero nonunit integer can be written as a finite $\tau_{(n)}$-product of $\tau_{(n)}$-atoms) for $n \in \{0, 1, 2, 3, 4, 5, 6, 8, 10, 12\}$. Hamon's proof used Dirichlet's infinite sequence of primes to rule out most of the cases. Then she checked case by case giving such a list of integers. Unfortunately, she made a mistake when $n = 12$. In [5], Juett showed that $\mathbb{Z}$ is not τ_{12}-atomic.

In 2011, Figueroa and Ortiz [2], considered the set $\mathbb{Z}$ with the relation $\tau_{(n)}$ and extended the multiplicative function ν, which counts the number of positive factors of an integer. Moreover, if $a = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ is the canonical factorization of a, then $\nu(a) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_m + 1)$. They established formulas to count the number of $\tau_{(n)}$-factors of an integer when $|n| \leq 4$.

In 2012, J. Lanterman [6], exposed new properties about the form of $\tau_{(n)}$-atoms for $n \in \{7, 11\}$. Also, he defined the relation $\mu_{(n)}$ over $\mathbb{Z}$ given by $a\,\mu_{(n)}\,b$ if and only if $a\,\tau_{(n)}\,\pm b$, and proved that the number of equivalence classes generated by $\mu_{(n),}$ with $n > 2$ a prime number is $\lceil \frac{n}{2} \rceil$.

In 2014, C. Serna [7], studying the associate preserving closure of a relation τ, denoted by τ', notes that the relation $\mu_{(n)}$ defined by Lanterman is the associate preserving closure of $\tau_{(n)}$. It must be noted that $\mu_{(n)}$ in Serna's work was denoted as $\tau'_{(n)}$, and this provided a characterization of its connection to the relation $\tau_{(n)}$. For example, for all $a \in \mathbb{Z}^{\#}$ the $\tau_{(n)}$-factors of a are the same $\tau'_{(n)}$-factors of a, and a is a $\tau_{(n)}$-atom if and only if a is a $\tau'_{(n)}$-atom. The main importance of this last relation with respect to this work is the reduction of the number of equivalence classes, and the fact that each $\tau'_{(n)}$-factor (resp. $\tau'_{(n)}$-atom) is a $\tau_{(n)}$-factor (resp. $\tau_{(n.)}$-atom).

This work establishes formulas to count the number of $\tau_{(n)}$-factors of an element in $\mathbb{Z}^{\#}$ for $n \geq 5$. These formulas are a tool to see the structure of $\tau_{(n)}$-atoms, since the number of $\tau_{(n)}$-factors of a $\tau_{(n)}$-atom is exactly 1. The cases given in [3], [1] and [6] on the structure of some $\tau_{(n)}$-atoms were verified using this technique.

1.1 Summary of chapters

In this work the reader can find formulas to calculate the number of $\tau_{(n)}$-factors of an integer distinct from 0, 1 and -1. Also, there is a comparison and verification of the form of the $\tau_{(n)}$-atom presented by Anderson and Frazier [3], Hamon [1] and Lanterman [6].

The second chapter presents basic properties of number theory and formalize the notion of the theory of τ-factorization on generalized factorizations. Basically, it contains most of the basic necessary results in order to understand the known results, and the techniques needed in this research. Also, it includes the formulas, found by Figueroa and Ortiz [2], to calculate the number of $\tau_{(n)}$-factor for $\phi(n) \leq 2$ (except for the case when $n = 6$).

The third chapter contains the research's main results. It gives the formula to count the number of $\tau_{(6)}$-factors. This completes the formulas to count the number of $\tau_{(n)}$-factors when $\phi(n) = 2$. Then it presents the formulas to count the number of $\tau_{(n)}$-factors when $\phi(n) = 4$. Setting the formulas equal to 1, this work determined the form of the $\tau_{(n)}$-atoms when $0 \leq n \leq 6$. Such characterization coincides with the work done by Hamon [1] and Lanterman [6].

The fourth chapter presents the formulas to count the number of $\tau_{(n)}$-factors for integers having a particular form. And finally, the fifth chapter presents the conclusions and future work.

CHAPTER 2
PRELIMINARIES

This chapter is a brief review of some concepts of number theory needed in this research and the main concepts of the theory of τ-factorization.

2.1 Some notions of number theory

In this section there are some properties of the integers, including division and counting properties. Some of these definitions and properties were taken from [8].

Definition 2.1 (Definition 2.1.1, [8])**.** *Let a and b be integers where $a \neq 0$. It is said that a divides b (denoted as $a|b$) if there is an integer c such that $b = ac$. If no such c exists, then a does not divide b (denoted by $a \nmid b$). If a divides b, it is said that a is a divisor, or a factor of b, and b is divisible by a.*

A positive integer $p > 1$ is called a *prime* if the only positive divisors of p are 1 and p. If a positive integer greater than 1 is not a prime, then it is said to be a composite integer. Note that the definition implies that every positive integer greater than 1 can be decomposed as a product of prime numbers. One could think about primes as atoms in chemistry and the fact that everything is composed by atoms. A fundamental tool in number theory is the division theorem which states when an integer a is divisible by an integer b. This happens only if the remainder is zero. Otherwise the remainder must be positive integer less than b.

Proposition 2.1 (The Division Theorem, [8]). *Given two integers a and b with $b \neq 0$, there exist unique integers q and r such that*

$$a = bq + r \text{ and } 0 \leq r <\mid b \mid .$$

In the proposition above q is called the *quotient* of a divided by b, and r is called the *remainder*.

Definition 2.2 (Definition 2.1.8, [8]). *If a is a real number, the greatest integer function or floor function $\lfloor a \rfloor$ is defined to be the largest integer less than or equal to a. Similarly, the smallest integer greater than or equal to a is denoted by $\lceil a \rceil$, called the ceiling function.*

By definition $\lfloor 3.45 \rfloor = 3$, $\lfloor 4 \rfloor = 4$, $\lceil -4.5 \rceil = -4$ and $\lceil 2 \rceil = 2$. Observe that the quotient in Proposition 2.1 when a is divided by b is given by $q = \lfloor \frac{a}{b} \rfloor$. Some properties of the floor and the ceiling functions are given in the following proposition.

Proposition 2.2. *Let n be an odd positive integer. Then*

1. $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$
2. $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$
3. $n = \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor$

The last part of the proposition holds for any integer. Notice that for all $n \in \mathbb{N}$, $\lceil \frac{n}{2} \rceil = \lfloor \frac{n+1}{2} \rfloor$.

An important result in the theory of numbers is The Fundamental Theorem of Arithmetic which states that every positive integer greater than 1 can be written as $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ with every integer $\alpha_i > 0$ and where the p_i are distinct primes.

This expression is called the *canonical factorization* of a. Negative integers's factorization can be obtain by multiplying in the above expression by -1. A simple way to count the divisors of an integer is given by the following proposition.

Proposition 2.3 (Proposition 2.3.2, [8]). *Let a be a positive integer with canonical factorization $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$. The number of positive divisors $\nu(a)$ of a (including 1 and a) is*

$$\nu(a) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_n + 1).$$

Example. *Note that since $12 = 2^2 \cdot 3$, then by the proposition above 12 has 6 positive divisors, 1,2,3,4,6 and 12.*

Definition 2.3 (Definition 3.1.1, [8]). *If a, b and n are integers, it is said that a is congruent to b modulo n (denoted by $a \equiv b(mod\ n)$) if $n|(a-b)$. If $n \nmid (a-b)$, we write $a \not\equiv b(mod\ n)$ and say that a is not congruent, or incongruent, to b modulo n.*

Since $2 = 7 - 5$, then the definition implies that $5 \equiv 7(\text{mod } 2)$. Note that $a \equiv b(\text{mod } n)$ if and only if a and b have the same remainder when divided by n.

Proposition 2.4 (Proposition 3.1.3, [8]). *If a, b, c and n are integers, then*

1. *$a \equiv a(mod\ n)$ for every a.*
2. *$a \equiv b(mod\ n)$ if and only if $b \equiv a(mod\ n)$.*
3. *$a \equiv b(mod\ n)$ and $b \equiv c(mod\ n)$ imply that $a \equiv c(mod\ n)$.*

The Proposition 2.4 implies that for every integer n, the congruence modulo n is an equivalence relation on $\mathbb{Z}$. For simplicity $a \equiv b(\text{mod } n)$ can be written as $a \equiv_n b$ and "$\equiv_n$" is an equivalence relation on $\mathbb{Z}$. The equivalence class of an integer a with respect to the congruence modulo n is denoted and defined by

$[a]_n = \{b \in \mathbb{Z} \mid b \equiv a(\text{mod } n)\}$. The set of all the equivalence classes modulo n is called a *quotient set*, and is denoted by $\mathbb{Z}/\equiv_n$.

Observe that $\mathbb{Z}/\equiv_n$ with the following well-defined operations

$$\begin{aligned} + : \Big(\mathbb{Z}/\equiv_n\Big) \times \Big(\mathbb{Z}/\equiv_n\Big) &\longrightarrow \mathbb{Z}/\equiv_n \\ \big([a]_n, [b]_n\big) &\longmapsto [a+b]_n \end{aligned} \tag{2.1}$$

$$\begin{aligned} \bullet : \Big(\mathbb{Z}/\equiv_n\Big) \times \Big(\mathbb{Z}/\equiv_n\Big) &\longrightarrow \mathbb{Z}/\equiv_n \\ \big([a]_n, [b]_n\big) &\longmapsto [a \cdot b]_n \end{aligned} \tag{2.2}$$

is a finite commutative ring with identity of order $\Big|\mathbb{Z}/\equiv_n\Big| = n$.

Let a, $b \in \mathbb{Z}$ not both be zero. The *greast common divisor of a and b*, denoted by $gcd(a, b)$, is defined as the largest positive integer dividing both a and b. Two integers a and b are said to be *relatively prime*, if $gcd(a, b) = 1$.

Proposition 2.5 (Propositions 3.1.7 and 3.1.10, [8]). *Let $a, b, c \in \mathbb{Z}$ and $n, m, d \in \mathbb{N}$. Then the following statement holds:*

1. *If $a \equiv b(mod\ m)$ and $d|m$, then $a \equiv b(mod\ d)$.*
2. *If $a \equiv b(mod\ m)$, then $ca \equiv cb(mod\ m)$.*
3. *$ac \equiv bc(mod\ m)$ implies that $a \equiv b(mod\ \frac{m}{gcd(c,m)})$.*
4. *Suppose $gcd(m, n) = 1$. Then $a \equiv b(mod\ m)$ and $a \equiv b(mod\ n)$ if and only if $a \equiv b(mod\ mn)$.*

Definition 2.4 (Definition 3.2.1, [8]). *A number b is called the inverse of a modulo n, if $ab \equiv 1(mod\ n)$. It is said that a is invertible modulo n, if it has an inverse.*

For example, $2 \cdot 2 \equiv 1 (\text{mod } 3)$, then the inverse of 2 modulo 3 is 2.

Proposition 2.6 (Proposition 3.2.3, [8])**.** *A nonzero integer a is invertible modulo n if and only if $gcd(a, n) = 1$. If a has an inverse, then it is unique modulo n.*

The importance of Proposition 2.6 is that the representatives of the equivalence classes that are units of $\mathbb{Z}/\equiv_n$, are relatively prime integers with respect to n.

Definition 2.5 (Definition 4.2.1, [8])**.** *The number of positive integers less than n which are relatively prime to n is denoted by $\phi(n)$. The funtion ϕ is called Euler's totient function or Euler's ϕ-function and $\phi(n)$ is called the Euler number of n.*

For example, $\phi(8) = 4$. The numbers less than 8 that are relatively prime to 8 are 1, 3, 5 and 7. An important fact is the multiplicative property of the ϕ-function. That is, if $(m, n) = 1$, then $\phi(mn) = \phi(m)\phi(n)$. A consequence of this fact is given in the following proposition.

Proposition 2.7 (Corollary 4.2.4, [8])**.** *If $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ is the canonical factorization of m, then*

$$\phi(m) = \prod_{i=1}^{n} p_i^{\alpha_i - 1}(p_i - 1).$$

By the proposition above, $\phi(18) = \phi(2 \cdot 3^2) = (2-1)(9-3) = 6$. Observe that $\phi(m)$ is always even, except when $m = 2$. By Definition 2.5 and Proposition 2.6, $\left| U\Big(\mathbb{Z}/\equiv_n\Big) \right| = \phi(n)$.

2.2 Some notions of theory of τ-factorization

This section gives some properties of the theory of τ-factorizations. This notion was defined by Anderson and Frazier in 2006. It is considered that the reader is familiar with some basics definitions of modern algebra. Let D be an integral domain, defined $D^* = D - \{0\}$, $\mathrm{U}(D)$ the group of units of D and $D^\# = D^* - \mathrm{U}(D)$, the nonzero nonunits elements of D. We say that two elements are *associated* if and only if $(a) = (b)$ (equivalently, there is $\lambda \in \mathrm{U}(D)$ such that $a = \lambda b$). Otherwise a and b are called non-associated. Consider τ a symmetric relation on $D^\#$, that is, a subset of order pairs (a, b) of elements in $D^\#$. Since it is symmetric, for each $(a, b) \in \tau$, (b, a) is also an element of τ. For simplicity, one could write $a\tau b$, if $(a, b) \in \tau$.

Definition 2.6 (Page 5, [3]). *Let $a \in D^\#$ and τ a symmetric relation on $D^\#$. The expression $a = \lambda \cdot a_1 * a_2 * \cdots * a_n$ (where $a_i \in D^\#$ and $\lambda \in U(D)$) is called a τ-factorization of a, if $a_i \, \tau \, a_j$ for each $i \neq j$.*

For simplicity, a is called a τ-*product* of the a_i's and each a_i is called a τ-*factor* of a (usually denoted by $a_i \,|_\tau\, a$). In the above factorization, "$\cdot$" denotes the usual product and "$*$" denotes a τ-product; that is, a product of two factors of a which are related under τ. We will use $a_1 \cdots a_n$ (resp. $a_1 * \cdots * a_n$) to represent the usual product (resp. the τ-product) of the indexed list of integers a_1, a_2, $\ldots$, a_n. Also, it is said that a τ-*divided* b if there is a τ-factorization of b that contains a as a τ-factor. If $\lambda = 1$ in the product of above, then the expresion $a = a_1 * a_2 * \cdots * a_n$ is called a reduced τ-factorization.

Consider $a \in D^\#$. Any expression of the form $a = a$ or $a = \lambda(\lambda^{-1}a)$, where $\lambda \in \mathrm{U}(D)$, is called a trivial τ-factorization of a. Therefore, a is a τ-factor of a. Any non-associated τ-factor of a is called a proper τ-factor of a.

Definition 2.7 (Page 7, [3]). *Let D be an integral domain. An element $a \in D^{\#}$ is called a τ-atom or a τ-irreducible element if the only τ-factorizations are trivial. If each element of $D^{\#}$ has a τ-factorization into τ-atom, then $D^{\#}$ is said to be τ-atomic.*

Let τ be an equivalence relation on $D^{\#}$ and $a \in D^{\#}$. Then the equivalence class of a with respect to τ is denoted by $[a]_\tau$. Therefore, $b \in [a]_\tau$ if and only if $a\,\tau\,b$.

2.2.1 The definition of $\tau_{(n)}$

For $n \in \mathbb{Z}$ define the relation $\tau_{(n)}$ on $\mathbb{Z}^{\#}$ by $a\tau_{(n)}b$ if and only if $a - b \in (n)$, the principal ideal generated by n. The relation $\tau_{(n)}$ is an equivalence relation. The proof follows from the fact that (n) is an ideal. It is reflexive because $a - a = 0 \in (n)$. For the symmetric property, notice that $-(a - b) \in (n)$ if and only if $a - b \in (n)$. Finally the closure of the additive operation warrants the transitive property, because $(a - b) + (b - c) = a - c$.

Since $\tau_{(n)}$ is an equivalence relation, then for each $a \in \mathbb{Z}^{\#}$, there is an equivalence class $[a]_{\tau_{(n)}}$. The reader must observe that even though the difference of $0 - n \in (n)$, this does not imply that $0 \in [n]_{\tau_{(n)}}$, because $0 \notin \mathbb{Z}^{\#}$. Similarly, $-1 \notin [n+1]_{\tau_{(n)}}$ and $1 \notin [n+1]_{\tau_{(n)}}$. Observe that if $n > 1$, $\tau_{(n)}$ and $\equiv_n$ coincide on $\mathbb{Z}^{\#}$. Therefore, for $n > 1$, $\mathbb{Z}^{\#}/\tau_{(n)} = \{[n]_{\tau_{(n)}}, [n+1]_{\tau_{(n)}}, [2]_{\tau_{(n)}}, \ldots, [n-1]_{\tau_{(n)}}\}$, $\left|\mathbb{Z}^{\#}/\tau_{(n)}\right| = n$ and $\left|U\left(\mathbb{Z}^{\#}/\tau_{(n)}\right)\right| = \phi(n)$.

n	$\tau_{(n)}$-atom
0	$\pm p_1^{\alpha_1} \cdots p_t^{\alpha_t}$, p_i primes with $gcd(\alpha_1 \dots \alpha_t) = 1$
1	$\pm p$, p prime
2	$\pm p$, $\pm 2k$ with p prime and k odd.
3	$\pm p$, $\pm 3k$ with p prime and $3 \not| k$.
4	$\pm p$, $\pm 2k$ with p prime and k odd.
5	$\pm p$, $\pm 5k$, $\pm p_1 p_2 \cdots p_t$ with p prime, $5 \not| k$, $p_1 \equiv_5 \pm 2$ and $p_j \equiv_5 \pm 1$ for $j \neq 1$.
6	$\pm p$, $\pm 2k$, $\pm 3m$ with p prime, k odd and $3 \not| m$.

Table 2–1: The $\tau_{(n)}$-atoms when $0 \leq n \leq 6$, [1].

In [1], Sections 2.2 - 2.7 were devoted to the study of the $\tau_{(n)}$-relation when $n \in \{0, 1, 2, 3, 4, 5, 6\}$. Table 2–1 summarizes some of the facts obtained in these sections with respect to the classification of $\tau_{(n)}$-atoms.

Hamon did not prove that the $\tau_{(n)}$-atoms presented in the Table 2–1 are the only $\tau_{(n)}$-atoms, but she describes the cases when $\mathbb{Z}$ is not τ-atomic. She concludes (incorrectly) that $\mathbb{Z}$ is a $\tau_{(n)}$-atomic domain for $n \leq 12$, except when $n \in \{7, 9, 11\}$. Later in [5], Juett proved that $\mathbb{Z}$ is not $\tau_{(12)}$-atomic. To see why it fails, he considers the $\tau_{(12)}$-factorization $432 = 12 * 36$, and proves that this is the only $\tau_{(12)}$-factorization of 432 (up to choice of sign and the order of factors). Note that 36 is not a $\tau_{(12)}$-atom, because $6 * 6$ is a $\tau_{(12)}$-factorization of 36. On the other hand, 6 is not $\tau_{(12)}$-related to 12. Therefore, 432 cannot be factored in terms of $\tau_{(12)}$-atoms. Hamon did not consider the integer congruent modulo 12 to give any problems. Because most of the issues of such type usually only arose in the classes represented by an integer relative prime to n.

In [6], Lanterman tried to figure out the form of the $\tau_{(n)}$-atoms. He proved that Hamon had found all the $\tau_{(n)}$-atoms for $0 \leq n \leq 6$, but notes that for higher values of n, finding the form of the $\tau_{(n)}$-atoms becomes more difficult by the negative factors. For example, $6 = 2 \cdot 3$ is not a $\tau_{(5)}$-factorization of 6, because 2 and 3 are not related under $\tau_{(5)}$. So 6 can be considered a $\tau_{(5)}$-atom. On the other hand, $6 = (-1) \cdot (-2) * 3$ is a $\tau_{(5)}$-factorization of 6. Following this observation he defined the relation $\mu_{(n)}$ by $a\mu_{(n)}b$ if and only if $a\tau_{(n)} \pm b$. He proved that the relation $\mu_{(n)}$ reduces the number of equivalence classes to $\lceil n/2 \rceil$, when n is a prime. With this fact he found the form of the $\tau_{(7)}$-atoms and $\tau_{(11)}$-atoms.

2.2.2 The definition of $\tau'_{(n)}$

In [7], Serna defined a *unitary relation* as a symmetric relation τ on $D^{\#}$ that satisfies the property "$a\tau b$ implies $(\lambda a)\tau(\lambda b)$, for all $\lambda \in \mathrm{U}(D)$". He also defined

$$\begin{aligned} \tau' &= \left\{(a,b) : \exists \lambda, \mu \in \mathrm{U}(D) \text{ such that } (\lambda a)\tau(\mu b)\right\} \\ &= \left\{(\lambda a, \mu b) : \lambda, \mu \in \mathrm{U}(D) \text{ and } a\tau b\right\} \end{aligned} \tag{2.3}$$

Anderson and Frazier, [3], defined associated-preserving relation as a symmetric relation τ on $D^{\#}$ such that if b' is an associate of b and $a\tau b$, then $a\tau b'$. Serna proved that τ' is the smallest symmetric relation on $D^{\#}$ containing τ which is associated-preserving. The relation between the $|_\tau$ and τ-atom, and the $|_{\tau'}$ and τ'-atom is given in the following proposition.

Proposition 2.8 (Theorem 17, [7])**.** *Let D be an integral domain and τ an unitary equivalence relation on $D^{\#}$. If $a, b \in D^{\#}$, then*

1. *$a \mid_\tau b$ if and only if $a \mid_{\tau'} b$*
2. *a is a τ-atom if and only if a is a τ'-atom.*

By Proposition 2.5, $\tau_{(n)}$ is an unitary equivalence relation on $\mathbb{Z}^{\#}$; and the definition of $\mu_{(n)}$ satisfies Equation 2.3. Thus, the relation $\mu_{(n)}$ defined by Lanterman, is the associated-preserving closure of the relation $\tau_{(n)}$. Hence $\tau'_{(n)}$ coincides with $\mu_{(n)}$. Observe that for $n > 1$, $b \in [n+k]_{\tau_{(n)}}$ if and only if $-b \in [n-k]_{\tau_{(n)}}$ for all $b \in \mathbb{Z}$

and $k \in \{0, 1, 2, \ldots, n-1\}$. Hence, define the relation $\tau'_{(n)}$ as in [7] (and [6]),

$$\begin{aligned} a\tau'_{(n)}b &\Longleftrightarrow a-b \in (n) \text{ or } a+b \in (n) \\ &\Longleftrightarrow a\tau_{(n)}b \text{ or } a\tau_{(n)}(-b) \end{aligned}$$

It is clear that $\tau'_{(n)}$ is an equivalence relation. Thus, define the equivalence classes of $\tau'_{(n)}$ as $[n+k]_{\tau'_{(n)}} = [n+k]_{\tau_{(n)}} \cup [n-k]_{\tau_{(n)}}$ where $k \in \{0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The fact that $k \leq \lfloor \frac{n}{2} \rfloor$ is because $[n + \lfloor \frac{n}{2} \rfloor]_{\tau_{(n)}} = [n - \lfloor \frac{n}{2} \rfloor]_{\tau_{(n)}}$. Note that this last part proves that $\left|\mathbb{Z}^{\#}/\tau'_{(n)}\right| = \lfloor \frac{n}{2} \rfloor + 1$. It is clear that $\left|U\left(\mathbb{Z}^{\#}/\tau'_{(n)}\right)\right| = \left\lceil \frac{\phi(n)}{2} \right\rceil$. Note that $\phi(n)$ is even for all $n > 2$ and $\phi(2) = 1$, so the ceiling function only is to guarantee the cardinality of $U\left(\mathbb{Z}^{\#}/\tau'_{(n)}\right)$ when $n = 2$. By Proposition 2.6, $[a]_{\tau'_n} \in U\left(\mathbb{Z}^{\#}/\tau'_{(n)}\right)$ if and only if $gcd(a, n) = 1$.

Proposition 2.8 implies that to find the $\tau_{(n)}$-factors (resp. $\tau_{(n)}$-atoms) is equivalent to find the $\tau'_{(n)}$-factors (resp. $\tau'_{(n)}$-atoms).

2.3 On $\tau_{(n)}$-number theory

Several questions arise as in classical number theory. For example:

1. Finding which elements work as primes.
2. The maximum common $\tau_{(n)}$-factor ($\tau_{(n)}$-mcd).
3. The minimum common $\tau_{(n)}$-product.
4. The number of $\tau_{(n)}$-factors.
5. The Euler ϕ-function with respect to $\tau_{(n)}$-product.

The first question has been attempted by all mathematicians that have studied this theory. This work also is looking for the form of the $\tau_{(n)}$-atoms. For the remaining of problems, Ortiz's research group has formalized the ideas and are still working on them. In particular, Barrios in [9], found formulas for the $\tau_{(n)}$-mcd for $n \in \{5, 6, 8, 10, 12\}$. She also provided an algorithm that may compute the $\tau_{(n)}$-mcd

n	Canonical factorization of $a \in \mathbb{Z}^{\#}$	$\nu_{\tau_{(n)}}(a)$
0	$\pm p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}$	$\nu\big(gcd(\alpha_1, \alpha_2, \ldots, \alpha_l)\big)$
1	$\pm p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}$	$\nu(\|a\|) - 1$
2	$\pm 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}$	$(\alpha - 1)\nu(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}) + 1$, if a is even
		$\nu(\|a\|) - 1$, if a is odd
3	$\pm 3^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}$	$(\alpha - 1)\nu(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}) + 1$, if $\alpha \geq 1$
		$\nu(\|a\|) - 1$, if $\alpha = 0$
4	$\pm 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}$	1, if $\alpha = 1$
		$(\alpha - 2)\nu(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}) + 1$, if $\alpha > 1$
		$\nu(\|a\|) - 1$, if $\alpha = 0$

Table 2–2: The $\nu_{\tau_{(n)}}(a)$ when $0 \leq n \leq 4$, [2].

when $n \in \{7, 9, 14, 18\}$. The third and fifth problem only have been worked out for the cases when $n \in \{0, 1, 2, 3, 4\}$. The fourth problem was developed by Ortiz and Figueroa in 2011 [2] for the cases $n \in \{0, 1, 2, 3, 4\}$. This work is the first attempt at concluding their work. Table 2–2 summarizes what Ortiz and Figueroa found [2]. Note that Figueroa found the formula for $\nu_{\tau_{(n)}}$ for $n \in \{0, 1, 2, 3, 4\}$. $\nu_{\tau_{(n)}}(a)$ count the number of $\tau_{(n)}$-factors of a up to associates.

CHAPTER 3
COUNTING THE NUMBER OF $\tau_{(n)}$-FACTORS

In the previous chapter, we present an introduction and some preliminary results with respect to the number of $\tau_{(n)}$-factors when $n \in \{0, 1, 2, 3, 4\}$, done by Ortiz and Figueroa [2]. In this chapter we adress the same problem, but for $n \in \{5, 6, 8, 10, 12\}$.

We present first the case when $n = 6$, because they share some properties with the cases studied by Ortiz and Figueroa [2]. Later, we adress the case when $n \in \{6, 8, 10, 12\}$. They have something in common and the technique used in most of the cases is very similar. Of course, with the exception of some special and very particular type of elements.

By Proposition 2.8, it is known that counting the number of $\tau_{(n)}$-factors of $a \in \mathbb{Z}^{\#}$ is equivalent to counting the number of $\tau'_{(n)}$-factors of a. Unless specified, the results will be given in terms of $\tau'_{(n)}$. In order to address the questions about $\tau_{(n)}$-atoms, just look at the form of the nonzero nonunit integers with $\nu_{\tau'_{(n)}}(a) = 1$.

3.1 The number of $\tau_{(6)}$-factors

By definition $\phi(n)$ is an even integer, except when $n = 2$. So, this work starts by completing the cases when $\phi(n) = 2$. Observe that $\phi(n) = 2$ only for $n \in \{3, 4, 6\}$. Since, Figueroa in [2], found the formulas for $\nu_{\tau_{(3)}}(a)$ and $\nu_{\tau_{(4)}}(a)$, this section focuses in the formula of $\nu_{\tau_{(6)}}(a)$. Note that $\mathbb{Z}^{\#}/\tau_{(6)} = \{[6]_{\tau_{(6)}}, [7]_{\tau_{(6)}}, [2]_{\tau_{(6)}}, [3]_{\tau_{(6)}}, [4]_{\tau_{(6)}}, [5]_{\tau_{(6)}}\}$, but $\mathbb{Z}^{\#}/\tau'_{(6)} = \{[6]_{\tau'_{(6)}}, [7]_{\tau'_{(6)}}, [2]_{\tau'_{(6)}}, [3]_{\tau'_{(6)}}\}$. So $\tau'_{(6)}$ generates less equivalence clases

	$[6]_{\tau'_{(6)}}$	$[7]_{\tau'_{(6)}}$	$[2]_{\tau'_{(6)}}$	$[3]_{\tau'_{(6)}}$
$[6]_{\tau'_{(6)}}$	$[6]_{\tau'_{(6)}}$	$[6]_{\tau'_{(6)}}$	$[6]_{\tau'_{(6)}}$	$[6]_{\tau'_{(6)}}$
$[7]_{\tau'_{(6)}}$	$[6]_{\tau'_{(6)}}$	$[7]_{\tau'_{(6)}}$	$[2]_{\tau'_{(6)}}$	$[3]_{\tau'_{(6)}}$
$[2]_{\tau'_{(6)}}$	$[6]_{\tau'_{(6)}}$	$[2]_{\tau'_{(6)}}$	$[2]_{\tau'_{(6)}}$	$[6]_{\tau'_{(6)}}$
$[3]_{\tau'_{(6)}}$	$[6]_{\tau'_{(6)}}$	$[3]_{\tau'_{(6)}}$	$[6]_{\tau'_{(6)}}$	$[3]_{\tau'_{(6)}}$

Table 3–1: Cayley's multiplicative table for $\mathbb{Z}^{\#}/\tau'_{(6)}$.

than $\tau_{(6)}$ on $\mathbb{Z}^{\#}$. More over, the reader must notice that if $a\tau'_{(6)}b$ and $a\tau'_{(6)}c$, then $a\tau'_{(6)}bc$. This property is very nice and it is called the multiplicative property of a relation. In this work $\tau'_{(6)}$ is the only relation that satisfies such property.

Table 3–1 gives an idea of how does the equivalence classes (with respect to $\tau'_{(6)}$) interact to each other. Before we present the formulas of the number of non-associated $\tau'_{(6)}$-factors, lets formalize the structure of the integers in every equivalence class of $\tau'_{(6)}$ in the following lemma.

Lemma 3.1. *Let $a = \pm 2^{\alpha}3^{\beta}p_1^{\alpha_1}\cdots p_l^{\alpha_l} \in \mathbb{Z}^{\#}$, where $p_i \in [7]_{\tau'_{(6)}}$ are non-associated positive primes and $\alpha, \beta, \alpha_i, \in \mathbb{N}\cup\{0\}$. Then*

$$a \in [6]_{\tau'_{(6)}} \textit{ if and only if } \alpha \neq 0 \textit{ and } \beta \neq 0$$

$$a \in [7]_{\tau'_{(6)}} \textit{ if and only if } \alpha = \beta = 0$$

$$a \in [2]_{\tau'_{(6)}} \textit{ if and only if } \alpha \neq 0 \textit{ and } \beta = 0$$

$$a \in [3]_{\tau'_{(6)}} \textit{ if and only if } \alpha = 0 \textit{ and } \beta \neq 0$$

Proof. It is clear that in each item the converse is true by Table 3–1. Consider the set of equivalence classes with respect to $\tau'_{(6)}$, $\mathbb{Z}^{\#}/\tau'_{(6)} = \{[6]_{\tau'_{(6)}}, [7]_{\tau'_{(6)}}, [2]_{\tau'_{(6)}}, [3]_{\tau'_{(6)}}\}$ where $[6]_{\tau'_{(6)}}$ is the "0" and $[7]_{\tau'_{(6)}}$ is the identity. By Table 3–1 and simple calculations show that $[2]^m_{\tau'_{(6)}} = [2]_{\tau'_{(6)}}$ and $[3]^m_{\tau'_{(6)}} = [3]_{\tau'_{(6)}}$ for all $m \in \mathbb{N}$.

Notice that by Table 3–1, $a \in [6]_{\tau'_{(6)}}$ if and only if $6|a$. That says, $a \in [6]_{\tau'_{(6)}}$ if and only if $\alpha \neq 0$ and $\beta \neq 0$.

Suppose that $a \in [7]_{\tau'_{(6)}}$, and either $\alpha \neq 0$ or $\beta \neq 0$. First let us consider $\alpha \neq 0$. Then $a \in [6]_{\tau'_{(6)}}$ if $\beta \neq 0$, or $a \in [2]_{\tau'_{(6)}}$ if $\beta = 0$. Similarly, if $\alpha = 0$ and $\beta \neq 0$, then $a \in [3]_{\tau'_{(6)}}$. This is a contradiction with the fact that $a \in [7]_{\tau'_{(6)}}$. In consequence, if $a \in [7]_{\tau'_{(6)}}$, then $\alpha = \beta = 0$.

Now assume that $a \in [2]_{\tau'_{(6)}}$, and $\alpha = 0$ or $\beta \neq 0$. Consider $\alpha = 0$. Note that if $\beta = 0$, then $a \in [7]_{\tau'_{(6)}}$, and if $\beta \neq 0$, then $a \in [3]_{\tau'_{(6)}}$. Similarly, if $\alpha \neq 0$ and $\beta \neq 0$, then $a \in [6]_{\tau'_{(6)}}$. This is a contradiction with the fact that $a \in [2]_{\tau'_{(6)}}$. Therefore, if $a \in [2]_{\tau'_{(6)}}$, then $\alpha \neq 0$ and $\beta = 0$.

Similarly, if $a \in [3]_{\tau'_{(6)}}$, then $\alpha = 0$ and $\beta \neq 0$. $\square$

It is now possible calculate the number of $\tau'_{(6)}$-factors for an integer $a \in \mathbb{Z}^{\#}$.

Proposition 3.1. *Let $a = \pm 2^{\alpha}3^{\beta}p_1^{\alpha_1}\cdots p_l^{\alpha_l} \in \mathbb{Z}^{\#}$, where $p_i \in [7]_{\tau'_{(6)}}$ are non-associated positive primes. Then*

$$\nu_{\tau'_{(6)}}(a) = \begin{cases} |(\alpha-1)(\beta-1)|\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l}) + 1, & \text{if } \alpha \neq 0 \text{ or } \beta \neq 0 \\ \nu\big(|a|\big) - 1, & \text{if } \alpha = \beta = 0 \end{cases}$$

Proof. Suppose that a is as described in the statement. In order to prove the formula, we split the problem into the four cases of Lemma 3.1. First, lets calculate the number of $\tau'_{(6)}$-factors of $a' = 2^{\alpha}3^{\beta}$, where $\alpha \neq 0$ and $\beta \neq 0$. Consider the positive integer $b = 2^i 3^j$, where $0 \leq i \leq \alpha$ and $0 \leq j \leq \beta$. Let $i = 0$. Then $b = 3^j \in [3]_{\tau'_{(6)}}$, if $1 \leq j \leq \beta$. If $\pm b * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(6)}$-factorization of a', then every $c_s \in [3]_{\tau'_{(6)}}$. But $c_1 \cdot c_2 \cdots c_k = 2^{\alpha}3^{\beta-j} \notin [3]_{\tau'_{(6)}}$, a contradiction. Note that if $j = 0$, then $b = 1$, and 1 can not be a $\tau'_{(6)}$-factor of a'. Therefore, if $i = 0$, then b is not a $\tau'_{(6)}$-factor of a'. Now consider $1 \leq i \leq \alpha - 1$ with i fixed. Observe that $\pm b * 2^{\alpha-i}3^{\beta-j}$ is a $\tau'_{(6)}$-factorization of a' if $1 \leq j \leq \beta - 1$. Because b, $2^{\alpha-i}3^{\beta-j} \in [6]_{\tau'_{(6)}}$. It is clear that if $j \in \{0, \beta\}$, the b is not a $\tau'_{(6)}$-factor of a'. Therefore, if $1 \leq i \leq \alpha - 1$ and i is fixed, the number of possible $\tau'_{(6)}$-factors of a' is $\beta - 1$. Finally consider $i = \alpha$, then b is not a $\tau'_{(6)}$-factor of a' when $1 \leq j \leq \beta - 1$. Otherwise if $\pm b * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(6)}$-factorization of a', then each c_s must be in $[6]_{\tau'_{(6)}}$; but $c_1 \cdot c_2 \cdots c_k = 3^{\beta-j} \notin [6]_{\tau'_{(6)}}$. Similarly, if $j = 0$, then b is not a $\tau'_{(6)}$-factor of a'. On the other hand, if $j = \beta$, then $b = a'$ is a $\tau'_{(6)}$-factor of a'. Therefore, $a' = 2^\alpha 3^\beta$ has $(\alpha-1)(\beta-1)+1$ $\tau'_{(6)}$-factors.

Consider $a \in [6]_{\tau'_{(6)}}$. By Lemma 3.1, $a = \pm 2^\alpha 3^\beta p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, where $\alpha \neq 0$ and $\beta \neq 0$. Let $\left(2^i 3^j\right) * \left(2^{\alpha-i}3^{\beta-j}\right)$ a $\tau'_{(6)}$-factorization not trivial of $2^\alpha 3^\beta$, and c is a positive factor of $P = p_1^{\alpha_1} \cdots p_l^{\alpha_l} \in [7]_{\tau'_{(6)}}$. Since $c \in [7]_{\tau'_{(6)}}$ and $2^i 3^j$, $2^{\alpha-i}3^{\beta-j} \in [6]_{\tau'_{(6)}}$, then

$$\pm\Big(\left(2^i 3^j\right)c\Big) * \Big(\left(2^{\alpha-i}3^{\beta-j}\right)\frac{P}{c}\Big)$$

is a $\tau'_{(6)}$-factorization of a. Hence, $\left(2^i 3^j\right)c$ is a proper $\tau'_{(6)}$-factor of a. Since c is any positive factor of P, for each proper $\tau'_{(6)}$-factor of $2^\alpha 3^\beta$, a has $\nu(P)$ $\tau'_{(6)}$-factors. Thus $\nu_{\tau'_{(6)}}(a) = (\alpha - 1)(\beta - 1)\nu(P) + 1$, by adding a itself.

Let $a \in [7]_{\tau'_{(6)}}$. By Lemma 3.1 $a = \pm p_1^{\alpha_1} \cdots p_l^{\alpha_l}$. Consider b_1 be any integer greater than 1 such that $a = \pm b_1 b_2$. Since $b_1 \in [7]_{\tau'_{(6)}}$, then either $b_2 \in [7]_{\tau'_{(6)}}$ or $b_2 = 1$. Therefore any factor of a (except 1) is a $\tau'_{(6)}$-factor. Hence $\nu_{\tau'_{(6)}}(a) = \nu(|a|) - 1$.

Suppose $a \in [2]_{\tau'_{(6)}}$. By Lemma 3.1, $a = \pm 2^\alpha p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, where $\alpha \neq 0$. By a similar argument as in the first case (when $a \in [6]_{\tau'_{(6)}}$), it is only sufficient to find $\nu_{\tau'_6}(2^\alpha)$ ($\alpha \neq 0$). Because $\nu_{\tau'_6}(a) = \left(\nu_{\tau'_6}(2^\alpha) - 1\right)\nu(P) + 1$, where $P = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$. In order to obtain $\nu_{\tau'_6}(2^\alpha)$, consider 2^i a factor of 2^α. If $1 \leq i \leq \alpha - 1$, then (by Lemma 3.1)

$2^i * 2^{\alpha-i}$ is a $\tau'_{(6)}$-factorization of 2^α. Thus, 2^i is a $\tau'_{(6)}$-factor of 2^α for $1 \le i \le \alpha - 1$. Note that $1 = 2^0$ is not a $\tau'_{(6)}$-factor of 2^α, but 2^α is. Therefore, $\nu_{\tau'_6}(2^\alpha) = \alpha$ and $\nu_{\tau'_6}(a) = \big(\alpha - 1\big)\nu(P) + 1$.

Finally, suppose that $a \in [3]_{\tau'_{(6)}}$. By Lemma 3.1, $a = \pm 3^\beta p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, where $\beta \neq 0$. By an analogous argument to the previous case, $\nu_{\tau'_6}(3^\beta) = \beta$. Therefore, $\nu_{\tau'_6}(a) = (\beta - 1)\nu(P) + 1$ where $P = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$.
This concludes the proof. $\square$

As an example of Proposition 3.1, let's calculate the number of $\tau'_{(6)}$-factors of $2^3 \cdot 7^2 \cdot 11$. Note that $\nu_{\tau'_{(6)}}(2^3 \cdot 7^2 \cdot 11) = |(3-1)(0-1)|\nu(7^2 \cdot 11) + 1 = 13$. The reader can easily verify that the following are all $\tau'_{(6)}$-factors of $2^3 \cdot 7^2 \cdot 11$.

- 2
- $2 \cdot 7$
- $2 \cdot 7^2$
- $2 \cdot 11$
- $2 \cdot 7 \cdot 11$
- $2 \cdot 7^2 \cdot 11$
- 2^2
- $2^2 \cdot 7$
- $2^2 \cdot 7^2$
- $2^2 \cdot 11$
- $2^2 \cdot 7 \cdot 11$
- $2^2 \cdot 7^2 \cdot 11$
- $2^3 \cdot 7^2 \cdot 11$

Observe that $\nu_{\tau'_{(6)}}(90) = \nu_{\tau_{(6)}}(2 \cdot 3^2 \cdot 5) = |(1-1)(2-1)|\nu(5) + 1 = 1$. Thus, in particular, Proposition 3.1 can be used to calculate the form of the $\tau_{(6)}$-atom. The following corollary shows this form.

Corollary 3.1. *The set of $\tau_{(6)}$-atoms is the set of elements of the form q, $\pm 2p_1 \cdots p_s$ and $\pm 3q_1 \cdots q_s$ where q, p_i and q_i are primes with $p_i \neq 2$ and $q_i \neq 3$.*

Proof. By Proposition 3.1 each of these elements is a $\tau_{(6)}$-atom. Let $a \in \mathbb{Z}^\#$, with canonical factorization $\pm 2^\alpha 3^\beta p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ where each $p_i \in [7]_{\tau'_{(6)}}$ are positive primes.

Then a is a $\tau_{(6)}$-atom, if $\nu_{\tau'_6}(a) = 1$. First, assume that $\alpha \neq 0$ or $\beta \neq 0$. Then $|(\alpha-1)(\beta-1)|\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l}) + 1 = 1$ from where $(\alpha-1)(\beta-1) = 0$, because $\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l}q_1^{\beta_1}\cdots q_m^{\beta_m}) \neq 0$. Thus, $\tau'_{(6)}$-atom when a has the following form

- $a = \pm 2^{\alpha}\cdot 3p_1\cdots p_s$, where $p_i > 3$ are positive prime integers, $s \geq 0$ and $\alpha \geq 0$.
- $a = \pm 2\cdot 3^{\beta}p_1\cdots p_s$, where $p_i > 3$ are positive prime integers, $s \geq 0$ and $\beta \geq 0$.

Now, if $\alpha = \beta = 0$, then $\nu(|a|) - 1 = 1$. This happens only when a is a prime integer non-asociated to 2 or 3.

This concludes the proof. $\square$

The results in Corollary 3.1 are exactly the results that Hamon found in [1].

3.2 The number of $\tau_{(n)}$-factors, when $\phi(n) = 4$

There are only four integers with $\phi(n) = 4$. To find them, consider $n \in \mathbb{N}$ with canonical factorization $n = p_1^{\alpha_1}\cdots p_l^{\alpha_l}$. By Proposition 2.7, if $l = 1$, then $\phi(n) = p_1^{\alpha_1 - 1}(p_1 - 1) = 4$ only when $n = 2^3$ or $n = 5$. Now assume $l = 2$. Then for every $i \in \{1,2\}$, $p_i^{\alpha_i - 1}(p_i - 1) \in \{1,2,4\}$. Observe that $p_i^{\alpha_i-1}(p_i - 1) = 1$ only if $p_i = 2$ and $\alpha_i = 1$; and $p_i^{\alpha_i - 1}(p_i - 1) = 2$ only if $p_i = 2$ with $\alpha_i = 2$, or $p_i = 3$ with $\alpha_i = 1$. Since $p_i \neq p_j$ for $i \neq j$, the only integers n with $\phi(n) = 4$ (when $l = 2$) are $n = 10$ and $n = 12$. Observe that as $p_i \neq p_j$ for $i \neq j$ and the only positive factors of 4 are 1,2 and 4, then $l > 2$ is impossible. Otherwise, $p_i^{\alpha_i - 1}(p_i - 1) = 1$ for at least two distinct prime (except when 4 is factored as $1\cdot 2\cdot 2$, in which only is obtained $n = 12$), but the only form that this happens is that $p_i = 2$ and $\alpha = 1$. Therefore, $\phi(n) = 4$ if and only if $n \in \{5, 8, 10, 12\}$.

The following lemma turns out to be a useful tool when trying to calculate the number of $\tau'_{(n)}$-factors when $n \in \{5, 8, 10, 12\}$. Since 1 is a factor of any integer and

$1 = p_1^0 p_2^0 \cdots p_l^0$, then for the lemma it will be considered that 1 is the product of zero primes.

Lemma 3.2. *Let $a \in \mathbb{Z}^+ - \{1\}$, then the number of positive factors of a, which are formed by a product of an odd number of not necessarily distinct primes is $\left\lfloor \frac{\nu(a)}{2} \right\rfloor$. In addition, the number of positive factors of a, which are formed by a product of an even number of not necessarily distinct primes is $\left\lceil \frac{\nu(a)}{2} \right\rceil$.*

Proof. The proof is by induction on the number of distinct primes in the canonical factorization of a. First, let $a = p_1^{\alpha_1}$. It is sufficient to know how many non-negative odd numbers are less than or equal to α_1. The number of positive factors of a, which are the product of an odd number of not necessarily distinct primes, is given by

$$\left\lceil \frac{\alpha_1}{2} \right\rceil = \left\lfloor \frac{\alpha_1 + 1}{2} \right\rfloor = \left\lfloor \frac{\nu(p_1^{\alpha_1})}{2} \right\rfloor.$$

Similarly, the number of positive factors of a, which are the product of an even number of not necessarily distinct primes (incluiding 1), is given by

$$\left\lfloor \frac{\alpha_1}{2} \right\rfloor + 1 = \left\lceil \frac{\alpha_1 - 1}{2} \right\rceil + 1 = \left\lceil \frac{\alpha_1 - 1}{2} + 1 \right\rceil = \left\lceil \frac{\nu(p_1^{\alpha_1})}{2} \right\rceil.$$

Suppose that the number of positive factors of $p_1^{\alpha_1} \cdots p_{l-1}^{\alpha_{l-1}}$, where each p_j is a distinct prime, formed by a product of an odd number of not necessarily distinct primes is $\left\lfloor \frac{\nu(p_1^{\alpha_1} \cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rfloor$; and the number of positive factors formed by a product of an even number of not necessarily distinct primes is $\left\lceil \frac{\nu(p_1^{\alpha_1} \cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rceil$.

Consider $a = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, where the p_j's are distinct primes. Need to find the number of factors of the form $p_1^{i_1} \cdots p_l^{i_l}$, with $0 \leq i_j \leq \alpha_j$ where $\sum_{j=1}^{l} i_j$ is odd. If i_l is even (odd), then $\sum_{j=1}^{l-1} i_j$ is odd (respectively even). By the induction hypothesis, the number of factors of the form $p_1^{i_1} \cdots p_{l-1}^{i_{l-1}}$, where $\sum_{j=1}^{l-1} i_j$ is odd, is $\left\lfloor \frac{\nu(p_1^{\alpha_1} \cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rfloor$, and the number of factors of the form $p_1^{i_1} \cdots p_{l-1}^{i_{l-1}}$, where $\sum_{j=1}^{l-1} i_j$ is even, is $\left\lceil \frac{\nu(p_1^{\alpha_1} \cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rceil$.

Then the number of factors of the form $p_1^{i_1}\cdots p_l^{i_l}$ with $0 \le i_j \le \alpha_j$, where $\sum_{j=1}^{l} i_j$ is odd, is given by

$$\left\lceil \frac{\alpha_l+1}{2} \right\rceil \left\lfloor \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rfloor + \left\lfloor \frac{\alpha_l+1}{2} \right\rfloor \left\lceil \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rceil \tag{3.1}$$

First assume that α_l to be an odd integer. Then

$$\begin{aligned}
&\left\lceil \frac{\alpha_l+1}{2} \right\rceil \left\lfloor \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rfloor + \left\lfloor \frac{\alpha_l+1}{2} \right\rfloor \left\lceil \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rceil \\
&\qquad = \frac{\alpha_l+1}{2} \left\lfloor \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rfloor + \frac{\alpha_l+1}{2} \left\lceil \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rceil \\
&\qquad = \frac{\alpha_l+1}{2} \left(\left\lfloor \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rfloor + \left\lceil \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rceil \right) \\
&\qquad = \frac{\alpha_l+1}{2} \nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}}) \\
&\qquad = \frac{\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l})}{2} \\
&\qquad = \left\lfloor \frac{\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l})}{2} \right\rfloor
\end{aligned}$$

Now assume that α_l is even. Then by definition of the floor and ceiling functions, Equation 3.1 can be re-written as follows:

$$\begin{aligned}
&\left\lceil \frac{\alpha_l+1}{2} \right\rceil \left\lfloor \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rfloor + \left\lfloor \frac{\alpha_l+1}{2} \right\rfloor \left\lceil \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rceil \\
&\qquad = \left\lfloor \frac{\alpha_l+2}{2} \right\rfloor \left\lfloor \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rfloor + \left\lceil \frac{\alpha_l}{2} \right\rceil \left\lceil \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rceil \qquad (3.2) \\
&\qquad = \frac{\alpha_l+2}{2} \left\lfloor \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rfloor + \frac{\alpha_l}{2} \left\lceil \frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2} \right\rceil
\end{aligned}$$

In order to finish the proof, split this in two cases; when $\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})$ is an even integer or an odd integer. First, suppose that $\nu(p_1^{\alpha_1}\cdots p_{n-1}^{\alpha_{n-1}})$ is an even integer.

So, re-writting Equation 3.2,

$$\begin{aligned}
\left\lceil\frac{\alpha_l+1}{2}\right\rceil\left\lfloor\frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2}\right\rfloor+\left\lfloor\frac{\alpha_l+1}{2}\right\rfloor\left\lceil\frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2}\right\rceil& \\
=\frac{\alpha_l+2}{2}\frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2}+\frac{\alpha_l}{2}\frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2}& \\
=\frac{2\alpha_l+2}{2}\frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2}& \\
=\frac{\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l})}{2}& \\
=\left\lfloor\frac{\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l})}{2}\right\rfloor&
\end{aligned}$$

Now if $\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})$ is odd, then

$$\begin{aligned}
\left\lceil\frac{\alpha_l+1}{2}\right\rceil\left\lfloor\frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2}\right\rfloor+\left\lfloor\frac{\alpha_l+1}{2}\right\rfloor\left\lceil\frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2}\right\rceil& \\
=\frac{\alpha_l+2}{2}\left\lceil\frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})-1}{2}\right\rceil+\frac{\alpha_l}{2}\left\lfloor\frac{\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})+1}{2}\right\rfloor& \\
=\frac{\big(\alpha_l+2\big)\big(\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})-1\big)+\alpha_l\big(\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})+1\big)}{4}& \\
=\frac{2\alpha_l\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})+2\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})-2}{4}& \\
=\frac{\alpha_l\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})+\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})-1}{2}& \\
=\frac{\big(\alpha_l+1\big)\nu(p_1^{\alpha_1}\cdots p_{l-1}^{\alpha_{l-1}})}{2}-\frac{1}{2}& \\
=\frac{\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l})}{2}-\frac{1}{2}& \\
=\left\lfloor\frac{\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l})}{2}\right\rfloor&
\end{aligned}$$

Therefore, we have proved that the number of positive factors of $p_1^{\alpha_1}\cdots p_l^{\alpha_l}$, which are formed by a product of an odd number of not necessarily distinct primes is $\left\lfloor\frac{\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l})}{2}\right\rfloor$.

The Proposition 2.2 implies that $\nu(a)=\left\lfloor\frac{\nu(a)}{2}\right\rfloor+\left\lceil\frac{\nu(a)}{2}\right\rceil$ for all integer a. Thus, the number of positive factors (including 1) of $p_1^{\alpha_1}\cdots p_l^{\alpha_l}$, which are formed by a product of an even number of not necessarily distinct primes is $\left\lceil \frac{\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l})}{2} \right\rceil$. This concludes the proof. $\square$

The following subsections will study the cases for n, when $\phi(n) = 4$. Note that in these cases $\left| U\left(\mathbb{Z}^{\#}/\tau'_{(n)}\right) \right| = 2$.

3.2.1 The number of $\tau_{(5)}$-factors

This subsection presents the formula to count the number of $\tau_{(n)}$-factors of an element $a \in \mathbb{Z}^{\#}$. Note that $\mathbb{Z}^{\#}/\tau_{(n)} = \{[5]_{\tau_{(5)}}, [6]_{\tau_{(5)}}, [2]_{\tau_{(5)}}, [3]_{\tau_{(5)}}, [4]_{\tau_{(5)}}\}$ and $U(\mathbb{Z}^{\#}/\tau_{(n)}) = \{[6]_{\tau_{(5)}}, [2]_{\tau_{(5)}}, [3]_{\tau_{(5)}}, [4]_{\tau_{(5)}}\}$. The main drawback Figueroa found when studying the cases where $n = 5$ is due to the number of classes in $\mathbb{Z}^{\#}/\tau_{(5)}$, because the canonical factorization of an element $a \in \mathbb{Z}^{\#}$ is given by $\pm 5^{\alpha} PQRS$, where P, Q, R and S are the product of non-associated positive primes in the class $[6]_{\tau_{(5)}}$, $[2]_{\tau_{(5)}}$, $[3]_{\tau_{(5)}}$ and $[4]_{\tau_{(5)}}$, respectively. So, in order to count the number of $\tau_{(5)}$-factors of a, it is necessary to know how 5^{α} and the primes in P, Q, R, and S behave with each other. An advantage of using the relation $\tau'_{(n)}$ is that the cardinality of $\mathbb{Z}^{\#}/\tau'_{(n)}$ is 3, and is given by $\{[5]_{\tau'_{(5)}}, [6]_{\tau'_{(5)}}, [2]_{\tau'_{(5)}}\}$. Hence, the canonical factorization of an element $a \in \mathbb{Z}^{\#}$ is given by $\pm 5^{\alpha} PQ$, where P and Q are the product of non-associated positive primes in the class $[6]_{\tau'_{(5)}}$ and $[2]_{\tau'_{(5)}}$, respectively. Therefore $\tau'_{(5)}$ makes easier to count the number of $\tau_{(5)}$-factors of an element a.

As in the case $n = 6$, Table 3–2 gives an idea of the structure of the integers in every equivalence class of $\tau'_{(5)}$. The following lemma shows this structure.

	$[5]_{\tau'_{(5)}}$	$[6]_{\tau'_{(5)}}$	$[2]_{\tau'_{(5)}}$
$[5]_{\tau'_{(5)}}$	$[5]_{\tau'_{(5)}}$	$[5]_{\tau'_{(5)}}$	$[5]_{\tau'_{(5)}}$
$[6]_{\tau'_{(5)}}$	$[5]_{\tau'_{(5)}}$	$[6]_{\tau'_{(5)}}$	$[2]_{\tau'_{(5)}}$
$[2]_{\tau'_{(5)}}$	$[5]_{\tau'_{(5)}}$	$[2]_{\tau'_{(5)}}$	$[6]_{\tau'_{(5)}}$

Table 3–2: Cayley's multiplicative table for $\mathbb{Z}^{\#}/\tau'_{(5)}$

Lemma 3.3. *Let $a = \pm 5^{\alpha} p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m} \in \mathbb{Z}^{\#}$, where $\alpha, \alpha_i, \beta_j \in \mathbb{N} \cup \{0\}$, each $p_i \in [6]_{\tau'_{(5)}}$ and each $q_j \in [2]_{\tau'_{(5)}}$ are non-associated positive primes. Then*

$$a \in [5]_{\tau'_{(5)}} \text{ if and only if } \alpha \neq 0$$

$$a \in [6]_{\tau'_{(5)}} \text{ if and only if } \alpha = 0 \text{ and } \sum_{j=1}^{m} \beta_j \text{ is even}$$

$$a \in [2]_{\tau'_{(5)}} \text{ if and only if } \alpha = 0 \text{ and } \sum_{j=1}^{m} \beta_j \text{ is odd}$$

Proof. It is clear that in each item the converse is true. For the other part, note that $\mathbb{Z}^{\#}/\tau'_{(5)} = \{[5]_{\tau'_{(5)}}, [6]_{\tau'_{(5)}}, [2]_{\tau'_{(5)}}\}$, where $[5]_{\tau'_{(5)}}$ is the "0" and $[6]_{\tau'_{(5)}}$ is the identity. By Table 3–2 and simple modular calculations,

$$[2]^m_{\tau'_{(5)}} = \begin{cases} [6]_{\tau'_{(5)}} & \text{if } m \text{ is even} \\ [2]_{\tau'_{(5)}} & \text{if } m \text{ is odd} \end{cases}$$

Assume that $a \in [5]_{\tau'_{(5)}}$ and $\alpha = 0$. Then $a = p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$ where p_i's are primes distinct to 5. But $p_1^{\alpha_1} \cdots p_l^{\alpha_l} \in [6]_{\tau'_{(5)}}$, and either $q_1^{\beta_1} \cdots q_m^{\beta_m} \in [6]_{\tau'_{(5)}}$ or $q_1^{\beta_1} \cdots q_m^{\beta_m} \in [2]_{\tau'_{(5)}}$ (depending on the parity of $\sum_{j=1}^{m} \beta_j$). So either, $a \in [6]_{\tau'_{(5)}}$ or $a \in [2]_{\tau'_{(5)}}$, which is a contradiction. As a consequence, if $a \in [5]_{\tau'_{(5)}}$, then $\alpha \neq 0$.

Now, for the last two parts, it is clear that $\alpha = 0$ and $p_1^{\alpha_1} \cdots p_l^{\alpha_l} \in [6]_{\tau'_{(5)}}$. So, consider that $a \in [6]_{\tau'_{(5)}}$, $\alpha = 0$ and $\sum_{j=1}^{m} \beta_j$ is odd. Then $a = p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$. Since $\sum_{j=1}^{m} \beta_j$ is odd, $q_1^{\beta_1} \cdots q_m^{\beta_m} \in [2]_{\tau'_{(5)}}$. Therefore $a \in [2]_{\tau'_{(5)}}$, a contradiction. As a consequence, if $a \in [6]_{\tau'_{(5)}}$, then $\alpha = 0$ and $\sum_{j=1}^{m} \beta_j$ is even. Similarly, if $a \in [2]_{\tau'_{(5)}}$, then $\alpha = 0$ and $\sum_{j=1}^{m} \beta_j$ is odd. □

Now it is possible to count the number of $\tau'_{(5)}$-factors of an element $a \in \mathbb{Z}^{\#}$.

Proposition 3.2. *Let $a = \pm 5^{\alpha} p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m} \in \mathbb{Z}^{\#}$, where $p_i \in [6]_{\tau'_{(5)}}$, $q_j \in [2]_{\tau'_{(5)}}$ are non-associated positive primes. Then*

$$\nu_{\tau'_{(5)}}(a) = \begin{cases} (\alpha - 1)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}) + 1, & \text{if } \alpha \neq 0 \\ \nu\big(|a|\big) - 1, & \text{if } \alpha = 0 \text{ and } \displaystyle\sum_{j=1}^{m} \beta_j \text{ is even} \\ \dfrac{1}{2}\nu\big(|a|\big) - \nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) + 1, & \text{if } \alpha = 0 \text{ and } \displaystyle\sum_{j=1}^{m} \beta_j \text{ is odd} \end{cases}$$

Proof. Consider $a = \pm 5^{\alpha} p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$, $P = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, $Q = q_1^{\beta_1} \cdots q_m^{\beta_m}$, $A = PQ$ and $a' = 5^{\alpha}$.

Suppose $a \in [5]_{\tau'_{(5)}}$. By Lemma 3.3, $\alpha \neq 0$. If $\alpha = 1$, then a' is a $\tau'_{(5)}$-atom. Let $\alpha \geq 2$ and consider 5^i where $1 \leq i \leq \alpha - 1$. Since $5^i * 5^{\alpha - i}$ is a $\tau'_{(5)}$-factorization of a', then for $1 \leq i \leq \alpha - 1$ 5^i is a $\tau'_{(5)}$-factor of a'. Therefore, $\nu_{\tau'_{(5)}}(5^{\alpha}) = \alpha$, because 5^{α} is a trivial $\tau'_{(5)}$-factor of itself.

Now take 5^i a proper $\tau'_{(5)}$-factor of 5^{α} and c a positive factor of A. If $5^i * c_1 * \cdots * c_k$ is a $\tau'_{(5)}$-factorization of 5^{α}, then

$$\pm\big(5^i \cdot c\big) * c_1 * \cdots * c_{j-1} * \left(c_j \cdot \frac{A}{c}\right) * c_{j+1} * \cdots * c_k$$

is a $\tau'_{(5)}$-factorization of a. Hence, each $5^i \cdot c$ is a proper $\tau'_{(5)}$-factors of a. Since c is any positive factor of A, for each proper $\tau'_{(5)}$-factor of a', a has $\nu(A)$ $\tau'_{(5)}$-factors. Thus $\nu_{\tau'_{(5)}}(a) = \big(\nu_{\tau'_{(5)}}(5^{\alpha}) - 1\big)\nu(A) + 1 = (\alpha - 1)\nu(A) + 1$.

Now assume that $\alpha = 0$. There are two cases. First suppose that $a \in [6]_{\tau'_{(5)}}$. By Lemma 3.3, $\sum_{j=1}^{m} \beta_j$ is an even integer. Consider $b_1 \in \mathbb{Z}^+ - \{1\}$ a factor of a. Say $a = \pm b_1 b_2$ where $b_2 \in \mathbb{Z}$. In order to obtain a $a \in [6]_{\tau'_{(5)}}$, there are two possible cases.

1. If $b_1 \in [6]_{\tau'_{(5)}}$, then $b_2 \in [6]_{\tau'_{(5)}}$ or $b_2 = 1$. Then either $a = \pm b_1 * b_2$ or $a = \pm b_2 * b_1$ is a $\tau'_{(5)}$-factorization of a. Hence b_1 is a $\tau'_{(5)}$-factor of a.
2. If $b_1 \in [2]_{\tau'_{(5)}}$, then $b_2 \in [2]_{\tau'_{(5)}}$. Therefore $\pm b_1 * b_2$ is a $\tau'_{(5)}$-factorization of a and b_1 is a $\tau'_{(5)}$-factor of a.

Since $b_1 \neq 1$ was any arbitrary factor of a, $\nu_{\tau'_{(5)}}(a) = \nu(|a|) - 1$.

For the second case, suppose that $a \in [2]_{\tau'_{(5)}}$. By Lemma 3.3 $\sum_{j=1}^{m} \beta_j$ is an odd integer. Let $b = q_1^{i_1} \cdots q_m^{i_m}$ where $0 \leq i_j \leq \beta_j$. If $\sum_{j=1}^{m} i_j$ is odd, then $b \in [2]_{\tau'_{(5)}}$. So $\pm b * \underbrace{q_1 * \cdots * q_1}_{\beta_1 - i_1} * \underbrace{q_2 * \cdots * q_2}_{\beta_2 - i_2} * * * \underbrace{q_m * \cdots * q_m}_{\beta_m - i_m}$ is a $\tau'_{(5)}$-factorization of Q. Hence b is a $\tau'_{(5)}$-factor of Q. If $\sum_{j=1}^{m} i_j$ is even, then b is not a $\tau'_{(5)}$-factor of Q. Otherwise, suppose $\pm b * b_1 * \cdots * b_t$ a $\tau'_{(5)}$-factorization of Q. Since $b \in [6]_{\tau'_{(5)}}$, then each $b_i \in [6]_{\tau'_{(5)}}$. Thus, $Q = b \cdot b_1 \cdots b_t \in [6]_{\tau'_{(5)}}$, contradicting the fact that $Q \in [2]_{\tau'_{(5)}}$. By Lemma 3.2, the number of $\tau'_{(5)}$-factors of Q, including Q, is $\left\lfloor \frac{\nu(Q)}{2} \right\rfloor = \frac{\nu(Q)}{2}$.

If b is a proper $\tau'_{(5)}$-factor of Q and c is a positive factor of P, then

$$\pm bc * \left(q_1 \cdot \frac{P}{c}\right) * \underbrace{q_1 * \cdots * q_1}_{\beta_1 - i_1 - 1} * \underbrace{q_2 * \cdots * q_2}_{\beta_2 - i_2} * * * \underbrace{q_m * \cdots * q_m}_{\beta_m - i_m}$$

is a $\tau'_{(5)}$-factorization of a. Hence bc is a proper $\tau'_{(5)}$-factor of a. Since c is any positive factor of P, for each proper $\tau'_{(5)}$-factor of Q, a has $\nu(P)$ $\tau'_{(5)}$-factors. Thus

$$\nu_{\tau'_{(5)}}(a) = \Big(\frac{\nu(Q)}{2} - 1\Big)\nu(P) + 1 = \frac{1}{2}\nu\big(|a|\big) - \nu(P) + 1$$

This concludes the proof. $\square$

As an example of Proposition 3.2, let's calculate the number of $\tau'_{(5)}$-factors of $2^3 \cdot 7^2 \cdot 11$. Since $2 \in [2]_{\tau'_{(5)}}$, $7 \in [2]_{\tau'_{(5)}}$, $11 \in [6]_{\tau'_{(5)}}$ and $\sum \beta_j = 5$ is odd, then $\nu_{\tau'_{(5)}}(2^3 \cdot 7^2 \cdot 11) = \frac{1}{2}\nu(2^3 \cdot 7^2 \cdot 11) - \nu(11) + 1 = \frac{1}{2}(4 \cdot 3 \cdot 2) - 2 + 1 = 11$. All the $\tau'_{(5)}$-factors of $2^3 \cdot 7^2 \cdot 11$ are:

- 2
- $2 \cdot 7^2$
- $2 \cdot 11$
- $2 \cdot 7^2 \cdot 11$
- $2^2 \cdot 7$
- $2^2 \cdot 7 \cdot 11$
- 2^3
- $2^3 \cdot 11$
- 7
- $7 \cdot 11$
- $2^3 \cdot 7^2 \cdot 11$

It clear that any prime integer is a $\tau'_{(5)}$-atom. Note that the only not trivial factorization of 10 is $2 \cdot 5$, but 2 and 5 are not related under $\tau'_{(5)}$. So $2 \cdot 5$ is not a $\tau'_{(5)}$-factorization of 10. This implies that 10 is a $\tau'_{(5)}$-atom. A similar reasoning proves that $22 = 2 \cdot 11$ is a $\tau'_{(5)}$-atom. These are examples of three types of $\tau'_{(5)}$-atom that are characterized in the following corollary.

Corollary 3.2. *The set of $\tau'_{(5)}$-atoms is the set of elements of the form q, $\pm 5p_1 \cdots p_s$ and $\pm pq_1 \cdots q_s$; where p, q, p_i and q_i are primes with $p \in [2]_{\tau'_{(5)}}$, $p_i \neq 5$ and $q_i \in [6]_{\tau'_{(5)}}$.*

Proof. By the previous proposition, each of these elements is a $\tau'_{(5)}$-atom. To show that any $\tau'_{(5)}$-atom is of these form, suppose $a \in \mathbb{Z}^{\#}$ with canonical factorization $\pm 5^{\alpha} p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$, where $p_i \in [6]_{\tau'_{(5)}}$ and $q_j \in [2]_{\tau'_{(5)}}$ are positive distinct primes. Suppose that a is a $\tau'_{(5)}$-atom, that is $\nu_{\tau'_{(5)}}(a) = 1$.

- Suppose $\alpha \neq 0$. In order for $(\alpha - 1)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}) + 1 = 1$, then α must be one. Thus, $\tau_{(5)}$-atoms, when $\alpha \neq 0$ have the form $\pm 5p_1 \cdots p_s$ where p_i are positive primes distinct to five and $s \geq 0$.

- If $\alpha = 0$ and $\sum_{j=1}^{m} \beta_j$ is even, $\nu(|a|)-1 = 1$ if and only if $\nu(|a|) = 2$. The $\tau'_{(5)}$-atoms, in this case, are the non-associated primes to 5.
- If $\alpha = 0$ and $\sum_{j=1}^{m} \beta_j$ is odd, then $\frac{\nu\left(|a|\right)}{2} - \nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l}) + 1 = 1$. Thus $\frac{\nu(q_1^{\beta_1}\cdots q_m^{\beta_m})}{2} = 1$, because $\nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l}) \neq 0$. Therefore the $\tau'_{(5)}$-atoms are of the form $\pm pp_1\cdots p_s$ where p and p_i are positive primes with $p \in [2]_{\tau'_{(5)}}$ and $p_i \in [6]_{\tau'_{(5)}}$.

$\square$

3.2.2 The number of $\tau_{(8)}$-factors

Note that $\mathbb{Z}^{\#}/\tau'_{(8)} = \{[8]_{\tau'_{(8)}}, [9]_{\tau'_{(8)}}, [2]_{\tau'_{(8)}}, [3]_{\tau'_{(8)}}, [4]_{\tau'_{(8)}}\}$, and it seens a lot more complicated, but such structure is very similar to $\mathbb{Z}^{\#}/\tau'_{(5)}$. In others words we have five clases, but $[8]_{\tau'_{(8)}}$, $[9]_{\tau'_{(8)}}$ and $[3]_{\tau'_{(8)}}$ behaves exactly as $[5]_{\tau'_{(5)}}$, $[6]_{\tau'_{(5)}}$ and $[2]_{\tau'_{(5)}}$, respectively. The classes $[2]_{\tau'_{(8)}}$ and $[4]_{\tau'_{(8)}}$ are special cases that are studied independently. Table 3–3 and the following lemma provide the structure of the integers in every equivalence class of $\tau'_{(8)}$.

Lemma 3.4. *Let $a = \pm 2^{\alpha} p_1^{\alpha_1}\cdots p_l^{\alpha_l} q_1^{\beta_1}\cdots q_m^{\beta_m} \in \mathbb{Z}^{\#}$, where $\alpha, \beta, \alpha_i, \in \mathbb{N}\cup\{0\}$, each $p_i \in [9]_{\tau'_{(8)}}$ and each $q_j \in [3]_{\tau'_{(8)}}$ are non-associated positive primes. Then*

1. *$a \in [8]_{\tau'_{(8)}}$ if and only if $\alpha \geq 3$,*
2. *$a \in [9]_{\tau'_{(8)}}$ if and only if $\alpha = 0$ and $\sum_{j=1}^{m} \beta_j$ is even,*
3. *$a \in [2]_{\tau'_{(8)}}$ if and only if $\alpha = 1$,*
4. *$a \in [3]_{\tau'_{(8)}}$ if and only if $\alpha = 0$ and $\sum_{j=1}^{m} \beta_j$ is odd,*
5. *$a \in [4]_{\tau'_{(8)}}$ if and only if $\alpha = 2$.*

Proof. Note that $\mathbb{Z}^{\#}/\tau'_{(8)} = \{[8]_{\tau'_{(8)}}, [9]_{\tau'_{(8)}}, [2]_{\tau'_{(8)}}, [3]_{\tau'_{(8)}}, [4]_{\tau'_{(8)}}\}$ where $[8]_{\tau'_{(8)}}$ is the "0" and $[9]_{\tau'_{(8)}}$ is the identity. By Table 3–3

- $[2]^m_{\tau'_{(8)}} = \begin{cases} [2]_{\tau'_{(8)}} & \text{if } m = 1 \\ [4]_{\tau'_{(8)}} & \text{if } m = 2 \\ [8]_{\tau'_{(8)}} & \text{if } m \geq 3 \end{cases}$

	$[8]_{\tau'_{(8)}}$	$[9]_{\tau'_{(8)}}$	$[2]_{\tau'_{(8)}}$	$[3]_{\tau'_{(8)}}$	$[4]_{\tau'_{(8)}}$
$[8]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$
$[9]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$	$[9]_{\tau'_{(8)}}$	$[2]_{\tau'_{(8)}}$	$[3]_{\tau'_{(8)}}$	$[4]_{\tau'_{(8)}}$
$[2]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$	$[2]_{\tau'_{(8)}}$	$[4]_{\tau'_{(8)}}$	$[2]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$
$[3]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$	$[3]_{\tau'_{(8)}}$	$[2]_{\tau'_{(8)}}$	$[9]_{\tau'_{(8)}}$	$[4]_{\tau'_{(8)}}$
$[4]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$	$[4]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$	$[4]_{\tau'_{(8)}}$	$[8]_{\tau'_{(8)}}$

Table 3–3: Cayley's multiplicative table for $\mathbb{Z}^{\#}/\tau'_{(8)}$

- $[3]^m_{\tau'_{(8)}} = \begin{cases} [9]_{\tau'_{(8)}} \text{ if } m \text{ is even} \\ [3]_{\tau'_{(8)}} \text{ if } m \text{ is odd} \end{cases}$

Hence the converse of each statement holds.

Reciprocally, suppose that $a \in [8]_{\tau'_{(8)}}$ and $\alpha \leq 2$. If $\alpha = 0$ (respectively, $\alpha = 1$ or $\alpha = 2$), then $a \in [9]_{\tau'_{(8)}}$ or $a \in [3]_{\tau'_{(8)}}$ (respectively, $a \in [2]_{\tau'_{(8)}}$ or $a \in [4]_{\tau'_{(8)}}$). This contradicts that $a \in [8]_{\tau'_{(8)}}$. Therefore, if $a \in [8]_{\tau'_{(8)}}$, then $\alpha \geq 3$.

The above reasoning also proves that if $a \in [2]_{\tau'_{(8)}}$ ($a \in [4]_{\tau'_{(8)}}$), then $\alpha = 1$ (respectively $\alpha = 2$). Now, since $[2]_{\tau'_{(8)}} \cdot [3]_{\tau'_{(8)}} = [2]_{\tau'_{(8)}}$ and $[4]_{\tau'_{(8)}} \cdot [3]_{\tau'_{(8)}} = [4]_{\tau'_4}$, then it is necessary that $\alpha = 0$ in order for $a \in [9]_{\tau'_{(8)}}$ or $a \in [3]_{\tau'_{(8)}}$. Consider that $[9]_{\tau'_{(8)}}$, $\alpha = 0$ and $\sum_{j=1}^m \beta_j$ is odd. Then $q_1^{\beta_1} \cdots q_m^{\beta_m} \in [3]_{\tau'_{(8)}}$, which implies that $a \in [3]_{\tau'_{(8)}}$. This is a contradiction. As a consequence, if $a \in [9]_{\tau'_{(8)}}$, then $\alpha = 0$ and $\sum_{j=1}^m \beta_j$ is even.

Similarly, if $a \in [3]_{\tau'_{(8)}}$, then $\alpha = 0$ and $\sum_{j=1}^m \beta_j$ is odd. □

Lemma 3.4 serves as a guide to study the number of non-asociated $\tau'_{(8)}$-factors of an element $a \in \mathbb{Z}^{\#}$. We split the problem into five cases given by the lemma, and obtain the following proposition.

Proposition 3.3. *Let* $a = \pm 2^{\alpha} p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m} \in \mathbb{Z}^{\#}$, *where each* $p_i \in [9]_{\tau'_{(8)}}$ *and each* $q_j \in [3]_{\tau'_{(8)}}$ *are non-associated positive primes, then*

$$\nu_{\tau'_{(8)}}(a) = \begin{cases} \Big(\nu_{\tau'_{(8)}}(2^{\alpha}) - 1\Big)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}) + 1, & \text{if } \alpha \neq 0 \\ \nu(|a|) - 1, & \text{if } \alpha = 0 \text{ and } \sum_{j=1}^{m} \beta_j \text{ is even} \\ \frac{1}{2}\nu\big(|a|\big) - \nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) + 1, & \text{if } \alpha = 0 \text{ and } \sum_{j=1}^{m} \beta_j \text{ is odd} \end{cases}$$

where

$$\nu_{\tau'_{(8)}}(2^{\alpha}) = \begin{cases} \alpha & \text{if } \alpha \text{ is } 1 \text{ or } 2 \\ \alpha - 1 & \text{if } \alpha \text{ is } 3 \text{ or } 4 \\ \alpha - 2 & \text{if } \alpha \text{ is even and } \alpha \geq 5 \\ \alpha - 3 & \text{if } \alpha \text{ is odd and } \alpha \geq 5 \end{cases}$$

Proof. Consider a as in the statement. By simple calculations, the follwing holds

$$\nu_{\tau'_{(8)}}(2^{\alpha}) = \begin{cases} 1 \text{ if } \alpha = 1 \\ 2 \text{ if } \alpha = 2 \\ 2 \text{ if } \alpha = 3 \\ 3 \text{ if } \alpha = 4 \\ 2 \text{ if } \alpha = 5 \end{cases}$$

For simplicity let $a' = 2^{\alpha}$, where $\alpha \geq 6$, and $b = 2^i$ a factor of a'. Clearly $1 = 2^0$ can not be a $\tau'_{(8)}$-factor of a', and 2^{α} is the trivial $\tau'_{(8)}$-factor. The proof investigate whether b is a $\tau'_{(8)}$-factor, when $1 \leq i \leq \alpha - 1$, analyzing several cases.

- If $i = 1$, then $\pm b * \underbrace{2 * \cdots * 2}_{\alpha-1}$ is a $\tau'_{(8)}$-factorization of a'. Thus, b is a $\tau'_{(8)}$-factor of a'.
- Assume $i = 2$. Consider the following cases:
  - Suppose α even. Then $\pm b * \underbrace{2^2 * \cdots * 2^2}_{\frac{\alpha}{2}}$ is a $\tau'_{(8)}$-factorization of a'. Thus, b is a $\tau'_{(8)}$-factor of a'.

- Suppose that α is an odd integer. Then b is not a $\tau'_{(8)}$-factor of a'. Otherwise if $\pm b * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(8)}$-factorization of a', then every $c_s \in [4]_{\tau'_{(8)}}$. By Lemma 3.4, every $c_s \equiv_8 \pm 2^2$; so $a' = b \cdot c_1 \cdots c_k \equiv_8 (\pm 2^2) \cdot \underbrace{(\pm 2^2) \cdots (\pm 2^2)}_{k} = \pm 2^{2k+2}$. This contradicts the hypothesis of α being odd integer.

- If $3 \leq i \leq \alpha - 3$, then $\pm b * 2^{\alpha - i}$ is a $\tau'_{(8)}$-factorization of a'. Hence, b is a $\tau'_{(8)}$-factor of a'.
- If $i = \alpha - 2$ (or $i = \alpha - 1$), then b can not be a $\tau'_{(8)}$-factor of a'. Otherwise if $\pm b * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(8)}$-factorization of a', then every $c_s \in [8]_{\tau'_{(8)}}$. But $c_1 \cdot c_2 \cdots c_k = 2^2 \notin [8]_{\tau'_{(8)}}$ (resp. $c_1 \cdot c_2 \cdots c_k = 2 \notin [8]_{\tau'_{(8)}}$).

Therefore, if $\alpha \geq 6$,

$$\nu_{\tau'_{(8)}}(2^\alpha) = \begin{cases} \alpha - 2 \text{ if } \alpha \text{ is even} \\ \\ \alpha - 3 \text{ if } \alpha \text{ is odd} \end{cases}$$

Now, assume that 2^i is a proper $\tau'_{(8)}$-factor of 2^α and c is a positive factor of $A = p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$. Then $\pm\big(2^i \cdot c\big) * \left(c_1 \cdot \frac{A}{c}\right) * c_2 * \cdots * c_k$ is a $\tau'_{(8)}$-factorization of a. Hence, $2^i \cdot c$ is a proper τ'_6-factor of a. Since c is any positive factor of A, for each proper $\tau'_{(8)}$-factor of 2^α, a has $\nu(A)$ $\tau'_{(8)}$-factors, plus 1, named a. Thus $\nu_{\tau'_{(8)}}(a) = \left(\nu_{\tau'_{(8)}}(2^\alpha) - 1\right)\nu(A) + 1$.

If $a \in [9]_{\tau'_{(8)}}$, then by Lemma 3.4 $\alpha = 0$ and $\sum_{j=1}^m \beta_j$ is even. Let b_1 be any integer greater than 1 with $a = \pm b_1 b_2$. In order for $a \in [9]_{\tau'_{(8)}}$, there are two cases.

- If $b_1 \in [9]_{\tau'_{(8)}}$, then either $b_2 \in [9]_{\tau'_{(8)}}$ or $b_2 = 1$. In both cases, b_1 is a $\tau'_{(8)}$–factor.
- If $b_1 \in [3]_{\tau'_{(8)}}$, then $b_2 \in [3]_{\tau'_{(8)}}$ and so b_1 is a $\tau'_{(8)}$–factor.

Since $b_1 \neq 1$ it represent any factor of a, this implies that $\nu_{\tau'_{(8)}}(a) = \nu(|a|) - 1$.

If $a \in [3]_{\tau'_{(8)}}$, then by Lemma 3.4 $\alpha = 0$ and $\sum_{j=1}^m \beta_j$ is odd. For simplicity let $Q = q_1^{\beta_1} \cdots q_m^{\beta_m}$, $P = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, and $b = q_1^{i_1} \cdots q_m^{i_m}$, where $0 \leq i_j \leq \beta_j$. If $\sum_{j=1}^m i_j$ is odd, then $b \in [3]_{\tau'_5}$, so

$$\pm b * \underbrace{q_1 * \cdots * q_1}_{\beta_1 - i_1} * \underbrace{q_2 * \cdots * q_2}_{\beta_2 - i_2} * \cdots * \underbrace{q_m * \cdots * q_m}_{\beta_m - i_m}$$

is a $\tau'_{(8)}$-factorization of Q. Thus, b is a $\tau'_{(8)}$-factor of Q. If $\sum_{j=1}^{m} i_j$ is even, then b is not a $\tau'_{(8)}$-factor of Q. Otherwise, if $\pm b * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(8)}$-factorization of Q, then every $c_s \in [9]_{\tau'_{(8)}}$. So $Q = b \cdot c_1 \cdots c_k \in [9]_{\tau'_{(8)}}$, which is a contradiction. By Lemma 3.2, the number of $\tau'_{(8)}$-factors of Q is $\left\lfloor \frac{\nu(Q)}{2} \right\rfloor = \frac{\nu(Q)}{2}$.

Now consider b a proper $\tau'_{(8)}$-factor of Q, and c any positive factor of P. If $\pm b * c_1 * \cdots * c_k$ is a $\tau'_{(8)}$-factorization of Q, then

$$\pm(b \cdot c) * \left(c_1 \cdot \frac{P}{c}\right) * c_2 * \cdots * c_k$$

is a $\tau'_{(8)}$-factorization of a. Hence, $b \cdot c$ is a proper $\tau'_{(8)}$-factor of a. Since c is any positive factor of P, for each proper $\tau'_{(8)}$-factor of Q, a has $\nu(P)$ $\tau'_{(8)}$-factors. Therefore,

$$\nu_{\tau'_{(8)}}(a) = \left(\frac{\nu(Q)}{2} - 1\right)\nu(P) + 1 = \frac{1}{2}\nu\big(|a|\big) - \nu(P) + 1.$$

This concludes the proof. $\square$

As an example of the Proposition 3.3, let's calculate the number of $\tau'_{(8)}$-factors of $2^3 \cdot 7^2 \cdot 11$. Since $7 \in [9]_{\tau'_{(8)}}$ and $11 \in [3]_{\tau'_{(8)}}$, then

$$\nu_{\tau'_{(8)}}(2^3 \cdot 7^2 \cdot 11) = \big(\nu_{\tau'_{(8)}}(2^3) - 1\big)\nu(7^2 \cdot 11) + 1 = (2-1)(3 \cdot 2) + 1 = 7.$$

All the $\tau'_{(8)}$-factors of $2^3 \cdot 7^2 \cdot 11$ are:

- 2
- $2 \cdot 7$
- $2 \cdot 7^2$
- $2 \cdot 11$
- $2 \cdot 7 \cdot 11$
- $2 \cdot 7^2 \cdot 11$
- $2^3 \cdot 7^2 \cdot 11$

It clear that any prime integer is a $\tau'_{(8)}$-atom. Note that the only not trivial factorization of 6 is $2 \cdot 3$, but 2 and 3 are not related under $\tau'_{(8)}$. So $2 \cdot 3$ is not a $\tau'_{(8)}$-factorization of 6. This implies that 6 is a $\tau'_{(8)}$-atom. A similar reasoning proves that $21 = 3 \cdot 7$ is a $\tau'_{(8)}$-atom. These are examples of three types of $\tau'_{(8)}$-atom that are characterized in the following corollary.

Corollary 3.3. *The set of $\tau'_{(8)}$-atoms is the set of elements of the form q, $\pm 2p_1 \cdots p_s$ and $\pm pq_1 \cdots q_s$, where p, q, each p_i and each q_i are positive primes with $p \in [3]_{\tau'_{(8)}}$, $p_i \neq 2$ and $q_i \in [9]_{\tau'_{(8)}}$.*

Proof. By the previous proposition each of these elements is a $\tau'_{(8)}$-atom. Let $a \in \mathbb{Z}^{\#}$ with canonical factorization $\pm 2^{\alpha} p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$ where $p_i \in [9]_{\tau'_{(8)}}$, $q_j \in [3]_{\tau'_{(8)}}$ are positive primes. If a is a $\tau'_{(8)}$-atom, then $\nu_{\tau'_{(8)}}(a) = 1$.

- If $\alpha \neq 0$, then $\Big(\nu_{\tau'_{(8)}}(2^{\alpha}) - 1\Big)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}) + 1 = 1$. Thus $\alpha = 1$ (because $\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}) \neq 0$). Hence, when $\alpha \neq 0$, the $\tau'_{(8)}$-atoms are of the form $\pm 2p_1 \cdots p_s$ where p_i's are positive primes distinct to two and $s \geq 0$.
- If $\alpha = 0$ and $\sum_{j=1}^{m} \beta_j$ is even, then $\nu(|a|) - 1 = 1$. Hence a must be a prime non-associated to 2.
- If $\alpha = 0$ and $\sum_{j=1}^{m} \beta_j$ is odd, then $\frac{1}{2}\nu\big(|a|\big) - \nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) + 1 = 1$. From previous theorem this happens only if $\frac{\nu(q_1^{\beta_1} \cdots q_m^{\beta_m})}{2} - 1 = 0$, because $\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) \neq 0$. Thus, the $\tau'_{(8)}$-atoms are of the form $\pm pp_1 \cdots p_s$, where $p \in [3]_{\tau'_{(8)}}$ and $p_i \in [9]_{\tau'_{(8)}}$.

$\square$

3.2.3 The number of $\tau_{(10)}$-factors

As expected $\mathbb{Z}^{\#}/\tau'_{(10)}$ has more elements, but $[10]_{\tau'_{(10)}}$, $[11]_{\tau'_{(10)}}$ and $[3]_{\tau'_{(10)}}$, behaves as $[5]_{\tau'_{(5)}}$, $[6]_{\tau'_{(5)}}$ and $[2]_{\tau'_{(5)}}$, respectively. Must notice that there is an unpleasent behaviour with respect to the special cases that arose from the classes $[2]_{\tau'_{(10)}}$ and $[4]_{\tau'_{(10)}}$. Lemma 3.5 and Table 3–4 would give a better idea of how the elements of

	$[10]_{\tau'_{(10)}}$	$[11]_{\tau'_{(10)}}$	$[2]_{\tau'_{(10)}}$	$[3]_{\tau'_{(10)}}$	$[4]_{\tau'_{(10)}}$	$[5]_{\tau'_{(10)}}$
$[10]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$
$[11]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[11]_{\tau'_{(10)}}$	$[2]_{\tau'_{(10)}}$	$[3]_{\tau'_{(10)}}$	$[4]_{\tau'_{(10)}}$	$[5]_{\tau'_{(10)}}$
$[2]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[2]_{\tau'_{(10)}}$	$[4]_{\tau'_{(10)}}$	$[4]_{\tau'_{(10)}}$	$[2]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$
$[3]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[3]_{\tau'_{(10)}}$	$[4]_{\tau'_{(10)}}$	$[11]_{\tau'_{(10)}}$	$[2]_{\tau'_{(10)}}$	$[5]_{\tau'_{(10)}}$
$[4]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[4]_{\tau'_{(10)}}$	$[2]_{\tau'_{(10)}}$	$[2]_{\tau'_{(10)}}$	$[4]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$
$[5]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[5]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[5]_{\tau'_{(10)}}$	$[10]_{\tau'_{(10)}}$	$[5]_{\tau'_{(10)}}$

Table 3–4: Cayley's multiplicative table for $\mathbb{Z}^\#/\tau'_{(10)}$

$\mathbb{Z}^\#/\tau'_{(10)}$ behaves.

Lemma 3.5. *Let* $a = \pm 2^\alpha 5^\beta p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m} \in \mathbb{Z}^\#$, *where* $\alpha, \beta, \alpha_i, \beta_j \in \mathbb{N} \cup \{0\}$, *each* $p_i \in [11]_{\tau'_{(10)}}$ *and each* $q_j \in [3]_{\tau'_{(10)}}$ *are non-associated positive primes. Then*

1. $a \in [10]_{\tau'_{(10)}}$ *if and only if* $\alpha \neq 0$ *and* $\beta \neq 0$,
2. $a \in [11]_{\tau'_{(10)}}$ *if and only if* $\alpha = \beta = 0$ *and* $\sum_{j=1}^m \beta_j$ *is even,*
3. $a \in [2]_{\tau'_{(10)}}$ *if and only if* $\alpha \neq 0$, $\alpha \not\equiv \sum_{j=1}^m \beta_j (mod\ 2)$ *and* $\beta = 0$,
4. $a \in [3]_{\tau'_{(10)}}$ *if and only if* $\alpha = \beta = 0$ *and* $\sum_{j=1}^m \beta_j$ *is odd,*
5. $a \in [4]_{\tau'_{(10)}}$ *if and only if* $\alpha \neq 0$, $\alpha \equiv \sum_{j=1}^m \beta_j (mod\ 2)$ *and* $\beta = 0$,
6. $a \in [5]_{\tau'_{(10)}}$ *if and only if* $\alpha = 0$ *and* $\beta \neq 0$.

Proof. Note that $\mathbb{Z}^\#/\tau'_{(10)} = \{[10]_{\tau'_{(10)}}, [11]_{\tau'_{(10)}}, [2]_{\tau'_{(10)}}, [3]_{\tau'_{(10)}}, [4]_{\tau'_{(10)}}, [5]_{\tau'_{(10)}}\}$ where $[10]_{\tau'_{(10)}}$ is the "0" and $[11]_{\tau'_{(10)}}$ is the identity element. The converse of each statement holds by Table 3–4 and the fact that

- $[2]^m_{\tau'_{(10)}} = \begin{cases} [4]_{\tau'_{(10)}} & \text{if } m \text{ is even} \\ [2]_{\tau'_{(10)}} & \text{if } m \text{ is odd} \end{cases}$,
- $[3]^m_{\tau'_{(10)}} = \begin{cases} [11]_{\tau'_{(10)}} & \text{if } m \text{ is even} \\ [3]_{\tau'_{(10)}} & \text{if } m \text{ is odd} \end{cases}$, and
- $[5]^m_{\tau'_{(10)}} = [5]_{\tau'_{(10)}}$ for all $m \in \mathbb{N}$.

Reciprocally, suppose that $a \in [10]_{\tau'_{(10)}}$, and either $\alpha = 0$ or $\beta = 0$. Assume $\alpha = 0$. If $\beta = 0$, then either $a \in [11]_{\tau'_{(10)}}$ or $a \in [3]_{\tau'_{(10)}}$, a contradicction (because $a \in [10]_{\tau'_{(10)}}$). If $\beta \neq 0$, then $a \in [5]_{\tau'_{(10)}}$, which is a contradicction too. The case $\alpha \neq 0$ and $\beta = 0$ implies that either $a \in [2]_{\tau'_{(10)}}$ or $a \in [4]_{\tau'_{(10)}}$, a contradicction. Therefore, if $a \in [10]_{\tau'_{(10)}}$, then $\alpha \neq 0$ and $\beta \neq 0$.

Assume that $a \in [11]_{\tau'_{(10)}}$, and $\alpha \neq 0$ or $\beta \neq 0$ or $\sum_{j=1}^{m} \beta_j$ is odd. If $\alpha \neq 0$, then (by the converse of each statement) $a \in [10]_{\tau'_{(10)}}$ or $a \in [2]_{\tau'_{(10)}}$ or $a \in [4]_{\tau'_{(10)}}$. This contradicts the fact that $a \in [11]_{\tau'_{(10)}}$. If $\beta \neq 0$, then by the converse of each statement $a \in [11]_{\tau'_{(10)}}$ or $a \in [5]_{\tau'_{(10)}}$, a contradiction. Now consider that $\sum_{j=1}^{m} \beta_j$ is odd, a can be in any class, except $[11]_{\tau'_{(10)}}$. Therefore, if $a \in [11]_{\tau'_{(10)}}$, then $\alpha = \beta = 0$ and $\sum_{j=1}^{m} \beta_j$ is even. A similar argument prove that if $a \in [3]_{\tau'_{(10)}}$, then $\alpha = \beta = 0$ and $\sum_{j=1}^{m} \beta_j$ is odd.

Now suppose that $a \in [2]_{\tau'_{(10)}}$, and $\alpha = 0$ or $\beta \neq 0$ or $\alpha \equiv \sum_{j=1}^{m} \beta_j (\text{mod } 2)$. If $\alpha = 0$, then a is in $[11]_{\tau'_{(10)}}$ or $[3]_{\tau'_{(10)}}$ or $[5]_{\tau'_{(10)}}$. This contradicts the fact that $a \in [2]_{\tau'_{(10)}}$. If $\beta \neq 0$, then either $a \in [10]_{\tau'_{(10)}}$ or $a \in [5]_{\tau'_{(10)}}$, which is a contradiction. If $\alpha \equiv \sum_{j=1}^{m} \beta_j (\text{mod } 2)$, then the converse of statement 3 guarantees that a can not be in $[2]_{\tau'_{(10)}}$. Therefore, if $a \in [2]_{\tau'_{(10)}}$, then $\alpha \neq 0$, $\beta = 0$ and $\alpha \not\equiv \sum_{j=1}^{m} \beta_j (\text{mod } 2)$. A similar argument prove that if $a \in [4]_{\tau'_{(10)}}$, then $\alpha \neq 0$, $\beta = 0$ and $\alpha \equiv \sum_{j=1}^{m} \beta_j (\text{mod } 2)$.

Finally, suppose $a \in [5]_{\tau'_{(10)}}$, and $\alpha \neq 0$ or $\beta = 0$. Note that if $\alpha \neq 0$, then the converse of each statement implies that $a \in [10]_{\tau'_{(10)}}$ or $a \in [2]_{\tau'_{(10)}}$ or $a \in [4]_{\tau'_{(10)}}$. If $\beta = 0$, then $a \in [11]_{\tau'_{(10)}}$ or $a \in [3]_{\tau'_{(10)}}$. This is a contradiction, because $a \in [5]_{\tau'_{(10)}}$. Therefore, if $a \in [5]_{\tau'_{(10)}}$, then $\alpha = 0$ and $\beta \neq 0$. $\square$

In the previous cases when $n = 5$ and $n = 8$, the first part of the formula of $\nu_{\tau'_{(n)}}(a)$ was very nicce. The behavior on the classes $[2]_{\tau'_{(10)}}$ and $[4]_{\tau'_{(10)}}$ forces to create a function that depends of a and not $2^{\alpha} \cdot 5^{\beta}$ (which it was the expected pattern). This happens because the equivalence class $[3]_{\tau'_{(10)}}$ permuts the equivalence classes $[2]_{\tau'_{(10)}}$ and $[4]_{\tau'_{(10)}}$. That is, by Table 3–4 $[3]_{\tau'_{(10)}} \cdot [2]_{\tau'_{(10)}} = [4]_{\tau'_{(10)}}$ and $[3]_{\tau'_{(10)}} \cdot [4]_{\tau'_{(10)}} = [2]_{\tau'_{(10)}}$. To see formally how it affect the formulas, let see the Proposition 3.4.

Proposition 3.4. *Let $a = \pm 2^{\alpha} 5^{\beta} p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m} \in \mathbb{Z}^{\#}$, where each $p_i \in [11]_{\tau'_{(10)}}$ and each $q_j \in [3]_{\tau'_{(10)}}$ are non-associated positive primes. Then*

$$\nu_{\tau'_{(10)}}(a) = \begin{cases} f(a)\nu(P \cdot Q) + 1, & \text{if } \alpha \neq 0 \text{ or } \beta \neq 0 \\ \nu(|a|) - 1, & \text{if } \alpha = \beta = 0 \text{ and } \displaystyle\sum_{j=1}^{m} \beta_j \text{ is even} \\ \dfrac{1}{2}\nu(|a|) - \nu(P) + 1, & \text{if } \alpha = \beta = 0 \text{ and } \displaystyle\sum_{j=1}^{m} \beta_j \text{ is odd} \end{cases}$$

where $P = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, $Q = q_1^{\beta_1} \cdots q_m^{\beta_m}$ and

$$f(a) = \begin{cases} |(\alpha - 1)(\beta - 1)|, & \text{if } \beta \neq 0 \\ 0, & \text{if } \beta = 0 \text{ and } \alpha = 1 \\ \dfrac{\left\lceil \frac{(\alpha-2)\nu(Q)}{2} \right\rceil}{\nu(Q)}, & \text{if } \beta = 0, \quad \alpha > 1 \text{ and } \alpha \not\equiv \displaystyle\sum_{j=1}^{m} \beta_j \text{ mod } 2 \\ \alpha - 1, & \text{if } \beta = 0, \quad \alpha > 1 \text{ and } \alpha \equiv \displaystyle\sum_{j=1}^{m} \beta_j \text{ mod } 2 \end{cases}$$

Proof. Consider $a' = 2^{\alpha} 5^{\beta}$, where $\alpha \geq 0$ and $\beta \neq 0$, and $b_1 = 2^i \cdot 5^j$ a proper factor of a'.

- Assume $\alpha = 0$ and $\beta \neq 0$. Since $b_1 * 5^{\beta-j}$ is a $\tau'_{(10)}$-factorization of a', b_1 is a $\tau'_{(10)}$-factor of a' for any $j \in \{1, \ldots, \beta-1\}$. Then $\nu_{\tau'_{(10)}}(5^\beta) = \beta$, because a' is also a $\tau'_{(10)}$-factor of a'.
- Assume $\alpha \neq 0$ and $\beta \neq 0$. If $i = 0$, then $b_1 = 2^0 5^j \in [5]_{\tau'_{(10)}}$ is not a $\tau'_{(10)}$-factor of a'. Otherwise, if $\pm b_1 * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(10)}$-factorization of a', then every $c_s \in [5]_{\tau'_{(10)}}$. But $c_1 \cdot c_2 \cdots c_k = 2^\alpha 5^{\beta-i} \notin [5]_{\tau'_{(10)}}$, which is not possible. For each fixed $i \in \{1, \ldots, \alpha-1\}$, the number of possible $\tau'_{(10)}$-factors of a' is $\beta - 1$, because $\pm 2^i 5^j * 2^{\alpha-i} 5^{\beta-j}$ is a $\tau'_{(10)}$-factorization of a', when $1 \leq j \leq \beta - 1$. Now $2^\alpha 5^j$ is not a $\tau'_{(10)}$-factor of $2^\alpha 5^\beta$ when $1 \leq j \leq \beta - 1$. Otherwise, if $\pm\big(2^\alpha 5^j\big) * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(10)}$-factorization of a', then every $c_s \in [10]_{\tau'_{(10)}}$; but $c_1 \cdot c_2 \cdots c_k = 5^{\beta-i} \notin [10]_{\tau'_{(10)}}$. Similarly 2^α is not a $\tau'_{(10)}$-factor of a'. On the other hand, $2^\alpha 5^\beta$ is a $\tau'_{(10)}$-factor of a'. Therefore, if $\alpha \neq 0$ and $\beta \neq 0$, $\nu_{\tau'_{(10)}}(2^\alpha 5^\beta) = (\alpha-1)(\beta-1) + 1$.

As a consequence, if $\beta \neq 0$, $\nu_{\tau'_{(10)}}(2^\alpha 5^\beta) = |(\alpha-1)(\beta-1)| + 1$.

It is now possible to analyze each class:

1. Consider $a \in [10]_{\tau'_{(10)}}$. By Lemma 3.5 $\alpha \neq 0$ and $\beta \neq 0$. It can be assumed that $\alpha, \beta \geq 2$, otherwise a is a $\tau'_{(10)}$-atom and will have only one $\tau'_{(10)}$-factor. Let $2^i 5^j \in [10]_{\tau'_{(10)}}$ be a proper $\tau'_{(10)}$-factor of $2^\alpha 5^\beta$ and c any positive factor of $A = p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$. If $\pm(2^i 5^j) * c_1 * \cdots * c_k$ is a $\tau'_{(10)}$-factorization of $2^\alpha 5^\beta$, then
$$\pm\Big(\big(2^i 5^j\big)c\Big) * \Big(c_1 \cdot \frac{A}{c}\Big) * c_2 * \cdots * c_k$$
is a $\tau'_{(10)}$-factorization of a. Hence, $\big(2^i 5^j\big) \cdot c$ is a proper $\tau'_{(10)}$-factor of a. Since c represents any positive factor of A, for each proper $\tau'_{(10)}$-factor of $2^\alpha 5^\beta$, a has $\nu(A)$ $\tau'_{(10)}$-factors. Again, counting a as a $\tau'_{(10)}$-factor, the number of $\tau'_{(10)}$-factors of a is $\nu_{\tau'_{(10)}}(a) = \big(\nu_{\tau'_{(10)}}(2^\alpha \cdot 5^\beta) - 1\big)\nu(A) + 1$.
2. If $a \in [11]_{\tau'_{(10)}}$, then by Lemma 3.5 $\alpha = \beta = 0$ and $\sum_{j=1}^m \beta_j$ is an even integer. Let $b_1 \in \mathbb{Z}^+ - \{1\}$ such that $a = \pm b_1 b_2$.

- Let $b_1 \in [11]_{\tau'_{(10)}}$. In order for $a \in [11]_{\tau'_{(10)}}$, b_2 is either 1 or lies in $[11]_{\tau'_{(10)}}$. So b_1 is a $\tau'_{(10)}$–factor.
- Similarly, if $b_1 \in [3]_{\tau'_{(10)}}$, then $b_2 \in [3]_{\tau'_{(10)}}$ and so b_1 is a $\tau'_{(10)}$–factor.

Since $b_1 \neq 1$ represents any factor of a, then $\nu_{\tau'_{(10)}}(a) = \nu(|a|) - 1$.

3. If $a \in [2]_{\tau'_{(10)}}$, then by Lemma 3.5 $\alpha \neq 0$, $\alpha \not\equiv \sum_{j=1}^{m} \beta_j \bmod 2$ and $\beta = 0$. Consider $b = 2^{\alpha} q_1^{\beta_1} \cdots q_m^{\beta_m} \in [2]_{\tau'_{(10)}}$ and let $b_1 = 2^{i} q_1^{i_1} \cdots q_m^{i_m} \in \mathbb{Z}^{+} - \{1\}$ be a proper factor of b. Consider b_2 to be the integer so that $b = b_1 b_2$.

   (a) Assume α is an odd integer and $\sum_{j=1}^{m} \beta_j$ is an even integer. If $\alpha = 1$, then $b_1 = 2q_1^{i_1} \cdots q_m^{i_m}$ or $b_1 = q_1^{i_1} \cdots q_m^{i_m}$.

   - If $b_1 = 2q_1^{i_1} \cdots q_m^{i_m}$, then $b_2 = q_1^{\beta_1 - i_1} \cdots q_m^{\beta_m - i_m}$. So b_1 is either in $[2]_{\tau'_{(10)}}$ or $[4]_{\tau'_{(10)}}$. On the other hand, every factor of b_2 is either in $[11]_{\tau'_{(10)}}$ or $[3]_{\tau'_{(10)}}$. Therefore, b_1 can not be a proper $\tau'_{(10)}$-factor of b.
   - If $b_1 = q_1^{i_1} \cdots q_m^{i_m}$, then $b_2 = 2q_1^{\beta_1 - i_1} \cdots q_m^{\beta_m - i_m}$. So b_1 is either in $[11]_{\tau'_{(10)}}$ or $[3]_{\tau'_{(10)}}$, while every factor of b_2 is either in $[2]_{\tau'_{(10)}}$ or $[4]_{\tau'_{(10)}}$. Therefore, b_1 is not a proper $\tau'_{(10)}$-factor of b.

   Hence, b is a $\tau'_{(10)}$-atom and $\nu_{\tau'_{(10)}}(b) = 1$.

   Suppose $\alpha \geq 3$.

   - Consider first that i and $\sum_{j=1}^{m} i_j$ to be both even integers. If $i = 0$, then $b_1 \in [11]_{\tau'_{(10)}}$ and $b_2 = 2^{\alpha} q_1^{\beta_1 - i_1} \cdots q_m^{\beta_m - i_m} \in [2]_{\tau'_{(10)}}$, but any $\tau'_{(10)}$-factor of b_2 is either in $[2]_{\tau'_{(10)}}$ or $[4]_{\tau'_{(10)}}$. Thus, b_1 can not be a $\tau'_{(10)}$-factor of b. Suppose $1 < i \leq \alpha - 1$. If $a = \pm b_1 * b_{21} * \cdots * b_{2k}$ is a $\tau'_{(10)}$-factorization of a, then each $b_{2s} \in [4]_{\tau'_{(10)}}$. Thus

     $$b_2 = b_{21} \cdots b_{2k} \equiv_{10} (\pm 4)^{k} \equiv_{10} \pm 4^{k} \equiv_{10} \pm 2^{2k} \equiv_{10} \pm 4.$$

     This contradicts the fact that $b_2 \in [2]_{\tau'_{(10)}}$. Therefore, b_1 is not a $\tau'_{(10)}$-factor of b.

- Suppose that i and $\sum_{j=1}^{m} i_j$ to be both odd integers. Then $b_1 \in [4]_{\tau'_{(10)}}$ and $b_2 \in [2]_{\tau'_{(10)}}$. A similar argument as in the previous case, shows that b_1 is not a $\tau'_{(10)}$-factor of b.
- Let i be an even integer and $\sum_{j=1}^{m} i_j$ be an odd integer. Then $b_1 \in [2]_{\tau'_{(10)}}$ and $b_2 \in [4]_{\tau'_{(10)}}$. If $i = 0$, then b_1 is not a $\tau'_{(10)}$-factor of b. Otherwise, if $b = \pm b_1 * b_{21} * \cdots * b_{2k}$ is a $\tau'_{(10)}$-factorization of b, then each $b_{2s} \in [3]_{\tau'_{(10)}}$. This forces b_2 to be either in $[3]_{\tau'_{(10)}}$ or $[11]_{\tau'_{(10)}}$. A contradiction.

  Now assume $1 < i < \alpha - 2$. Since $b_1, \frac{b_2}{2}, 2 \in [2]_{\tau'_{(10)}}$, then $b = \pm b_1 * \frac{b_2}{2} * 2$ is a $\tau'_{(10)}$-factorization of b. If $i = \alpha - 1$, then $b_2 = 2q_1^{\beta_1 - i_1} \cdots q_m^{\beta_m - i_m} \in [4]_{\tau'_{(10)}}$, which is a τ'_{10}-atom, as in a previous case. Therefore, b_1 is a $\tau'_{(10)}$-factor of b if $i \neq \alpha - 1$ is an even integer and $\sum_{j=1}^{m} i_j$ is an odd integer. By Lemma 3.2 (in this case), b has

$$\left(\left\lceil \frac{\alpha + 1}{2} \right\rceil - 2 \right) \left\lfloor \frac{\nu(Q)}{2} \right\rfloor$$

  proper $\tau'_{(10)}$-factors.
- Let i be an odd integer and $\sum_{j=1}^{m} i_j$ be an even integer. Then $b_1 \in [2]_{\tau'_{(10)}}$ and $b_2 \in [4]_{\tau'_{(10)}}$. By a similar argument, b_1 is a proper $\tau'_{(10)}$-factor of b, only if $1 \leq i \leq \alpha - 2$. By Lemma 3.2 (in this case), b has

$$\left(\left\lfloor \frac{\alpha + 1}{2} \right\rfloor - 1 \right) \left\lceil \frac{\nu(Q)}{2} \right\rceil$$

  proper $\tau'_{(10)}$-factors.

Since α is an odd integer, the number of $\tau'_{(10)}$-factors of b is

$$\begin{aligned}\nu_{\tau'_{(10)}}(b) &= \left(\left\lceil\frac{\alpha+1}{2}\right\rceil - 2\right)\left\lfloor\frac{\nu(Q)}{2}\right\rfloor \\ &+ \left(\left\lfloor\frac{\alpha+1}{2}\right\rfloor - 1\right)\left\lceil\frac{\nu(Q)}{2}\right\rceil + 1 \\ &= \left(\frac{\alpha+1}{2} - 2\right)\left\lfloor\frac{\nu(Q)}{2}\right\rfloor \\ &+ \left(\frac{\alpha+1}{2} - 1\right)\left\lceil\frac{\nu(Q)}{2}\right\rceil + 1 \\ &= \frac{\alpha-3}{2}\left\lfloor\frac{\nu(Q)}{2}\right\rfloor + \frac{\alpha-1}{2}\left\lceil\frac{\nu(Q)}{2}\right\rceil + 1\end{aligned} \tag{3.3}$$

If $\nu(Q)$ is an even integer, then Equation 3.3 can be re-written as follows:

$$\begin{aligned}\nu_{\tau'_{(10)}}(b) &= \frac{\alpha-3}{2}\frac{\nu(Q)}{2} + \frac{\alpha-1}{2}\frac{\nu(Q)}{2} + 1 \\ &= \frac{(2\alpha-4)\nu(Q)}{4} + 1 = \frac{(\alpha-2)\nu(Q)}{2} + 1 \\ &= \left\lceil\frac{(\alpha-2)\nu(Q)}{2}\right\rceil + 1.\end{aligned} \tag{3.4}$$

If $\nu(Q)$ is an odd integer, then (by Proposition 2.2) Equation 3.3 can be re-written as follows:

$$\begin{aligned}\nu_{\tau'_{(10)}}(b) &= \frac{\alpha-3}{2}\frac{\nu(Q)-1}{2} + \frac{\alpha-1}{2}\frac{\nu(Q)+1}{2} + 1 \\ &= \frac{(\alpha-3)(\nu(Q)-1) + (\alpha-1)(\nu(Q)+1)}{4} + 1 \\ &= \frac{2\alpha\nu(Q) - 4\nu(Q) + 2}{4} + 1 = \frac{\alpha\nu(Q) - 2\nu(Q) + 1}{2} + 1 \\ &= \frac{(\alpha-2)\nu(Q)+1}{2} + 1 = \left\lceil\frac{(\alpha-2)\nu(Q)}{2}\right\rceil + 1.\end{aligned} \tag{3.5}$$

Therefore, $\nu_{\tau'_{(10)}}(b) = \left\lceil\frac{(\alpha-2)\nu(Q)}{2}\right\rceil + 1.$

(b) Assume α to be an even integer and $\sum_{j=1}^{m}\beta_j$ an odd integer. A similar argument as in the previous case, shows that b_1 is a proper $\tau'_{(10)}$-factor when i and $\sum_{j=1}^{m} i_j$ have different parity. Therefore, the number of $\tau'_{(10)}$-factors of

b is

$$\begin{aligned}\nu_{\tau'_{(10)}}(b) = &\left(\left\lceil \frac{\alpha+1}{2} \right\rceil - 2\right)\left\lfloor \frac{\nu(Q)}{2} \right\rfloor \\ &+ \left(\left\lfloor \frac{\alpha+1}{2} \right\rfloor - 1\right)\left\lceil \frac{\nu(Q)}{2} \right\rceil + 1\end{aligned}$$

Since α is an even integer, then by Proposition 2.2

$$\begin{aligned}\nu_{\tau'_{(10)}}(b) &= \left(\frac{\alpha+1+1}{2} - 2\right)\left\lfloor \frac{\nu(Q)}{2} \right\rfloor \\ &\quad + \left(\frac{\alpha+1-1}{2} - 1\right)\left\lceil \frac{\nu(Q)}{2} \right\rceil + 1 \\ &= \frac{\alpha-2}{2}\left\lfloor \frac{\nu(Q)}{2} \right\rfloor + \frac{\alpha-2}{2}\left\lceil \frac{\nu(Q)}{2} \right\rceil + 1 \\ &= \frac{\alpha-2}{2}\left(\left\lfloor \frac{\nu(Q)}{2} \right\rfloor + \left\lceil \frac{\nu(Q)}{2} \right\rceil\right) + 1 \\ &= \frac{\alpha-2}{2}\nu(Q) + 1 = \left\lceil \frac{(\alpha-2)\nu(Q)}{2} \right\rceil + 1.\end{aligned}$$

As a consequence, if $\alpha \not\equiv \sum_{j=1}^{m} \beta_j \pmod 2$,

$$\nu_{\tau'_{(10)}}(a) = \begin{cases} 1 & \text{, if } \alpha = 1 \\ \left\lceil \dfrac{(\alpha-2)\nu(Q)}{2} \right\rceil \nu(P) + 1, & \text{if } \alpha > 1. \end{cases}$$

4. If $a \in [3]_{\tau'_{(10)}}$, then by Lemma 3.5 $\alpha = \beta = 0$ and $\sum_{j=1}^{m} \beta_j$ is an odd integer. Let $b = q_1^{i_1} \cdots q_m^{i_m}$ where $0 \le i_j \le \beta_j$. If $\sum_{j=1}^{m} i_j$ is odd, then $b \in [3]_{\tau'_{(10)}}$. So

$$\pm b * \underbrace{q_1 * \cdots * q_1}_{\beta_1 - i_1} * \underbrace{q_2 * \cdots * q_2}_{\beta_2 - i_2} * \cdots * \underbrace{q_m * \cdots * q_m}_{\beta_m - i_m}$$

is a $\tau'_{(10)}$-factorization of Q. Thus b is a $\tau'_{(10)}$-factor of Q. If $\sum_{j=1}^{m} i_j$ is even, then b is not a $\tau'_{(10)}$-factor of Q. Otherwise, if $\pm b * c_1 * \cdots * c_k$ is a $\tau'_{(10)}$-factorization of Q, then every $c_s \in [11]_{\tau'_{(10)}}$. But this implies that $Q = b \cdot c_1 \cdots c_k \in [11]_{\tau'_{(10)}}$ which is a contradiction. By Lemma 3.2, the number of $\tau'_{(10)}$-factors of Q is $\left\lfloor \frac{\nu(Q)}{2} \right\rfloor = \frac{\nu(Q)}{2}$.

Consider b to be a proper $\tau'_{(10)}$-factor of Q and c a positive factor of P. For any $\pm b * c_1 * \cdots * c_k$ $\tau'_{(10)}$-factorization of Q,

$$\pm\big(b \cdot c\big) * \left(c_1 \cdot \frac{P}{c}\right) * c_2 * \cdots * c_k$$

is a $\tau'_{(10)}$-factorization of a. Hence, $b \cdot c$ is a proper $\tau'_{(10)}$-factor of a. Since c is any positive factor of P, for each proper $\tau'_{(10)}$-factor of Q, a has $\nu(A)$ $\tau'_{(10)}$-factors. Therefore, $\nu_{\tau'_{(10)}}(a) = \big(\nu_{\tau'_{(10)}}(Q) - 1\big)\nu(P) + 1 = \frac{1}{2}\nu(A) - \nu(P) + 1$.

5. If $a \in [4]_{\tau'_{(10)}}$, then by Lemma 3.5 $\alpha \neq 0$, $\alpha \equiv \sum_{j=1}^{m} \beta_j \pmod 2$ and $\beta = 0$. Consider $b = 2^{\alpha} q_1^{\beta_1} \cdots q_m^{\beta_m} \in [4]_{\tau'_{(10)}}$ and let $b_1 = 2^{i} q_1^{i_1} \cdots q_m^{i_m} \in \mathbb{Z}^{+} - \{1\}$ be a proper factor of b. So that $b = b_1 b_2$.

   (a) Assume that α and $\sum_{j=1}^{m} \beta_j$ are both even integers.

   - Suppose first that i and $\sum_{j=1}^{m} i_j$ to be both even integers. If $i = 0$, then $b_1 \in [11]_{\tau'_{(10)}}$ and $b_2 = 2^{\alpha} q_1^{\beta_1 - i_1} \cdots q_m^{\beta_m - i_m} \in [4]_{\tau'_{(10)}}$. Since any $\tau'_{(10)}$-factor of b_2 is either in $[2]_{\tau'_{(10)}}$ or $[4]_{\tau'_{(10)}}$, b_1 is not a $\tau'_{(10)}$-factor of b. If $1 \leq i \leq \alpha - 1$, then both b_1, $b_2 \in [4]_{\tau'_{(10)}}$. Since $b = \pm b_1 * b_2$ is a $\tau'_{(10)}$-factorization of b, then b_1 is a $\tau'_{(10)}$-factor of b. If $i = \alpha$, then the only possible way for b_1 to be a $\tau'_{(10)}$-factor of b is when $b_1 = b$. Otherwise $b_2 = q_1^{\beta_1 - i_1} \cdots q_m^{\beta_m - i_m}$. Since any factor of b_2 is either in $[11]_{\tau'_{(10)}}$ or $[3]_{\tau'_{(10)}}$, b_1 can not be a $\tau'_{(10)}$-factor of b.

   - Let i be an even integer and $\sum_{j=1}^{m} i_j$ an odd integer. If $i = 0$, then $b_1 \in [3]_{\tau'_{(10)}}$ and $b_2 = 2^{\alpha} q_1^{\beta_1 - i_1} \cdots q_m^{\beta_m - i_m} \in [2]_{\tau'_{(10)}}$. Since any factor of b_2 is either in $[2]_{\tau'_{(10)}}$ or $[4]_{\tau'_{(10)}}$, b_1 can not be a $\tau'_{(10)}$-factor of b. If $1 \leq i \leq \alpha - 1$, then both b_1, $b_2 \in [2]_{\tau'_{(10)}}$. Since $b = \pm b_1 * b_2$ is a $\tau'_{(10)}$-factorization of b, then b_1 is a $\tau'_{(10)}$-factor of b. If $i = \alpha$, b_1 is not a $\tau'_{(10)}$-factor of b, because $b_1 \in [2]_{\tau'_{(10)}}$, $b_2 = q_1^{\beta_1 - i_1} \cdots q_m^{\beta_m - i_m} \in [3]_{\tau'_{(10)}}$ and any factor of b_2 is either in $[11]_{\tau'_{(10)}}$ or $[3]_{\tau'_{(10)}}$.

- Consider i an odd integer and $\sum_{j=1}^{m} i_j$ an even integer. Then $b_1 \in [2]_{\tau'_{(10)}}$ and $b_2 \in [2]_{\tau'_{(10)}}$. Thus for $1 \le i \le \alpha-1$, $b = \pm b_1 * b_2$ is a $\tau'_{(10)}$-factorization of b.
- Suppose that i and $\sum_{j=1}^{m} i_j$ are both odd integers. If $1 \le i \le \alpha - 1$, then $b_1 \in [4]_{\tau'_{(10)}}$ and $b_2 \in [4]_{\tau'_{(10)}}$. Therefore $b = \pm b_1 * b_2$ is a $\tau'_{(10)}$-factorization of b and b_1 is a $\tau'_{(10)}$-factor of b.

Hence, b_1 is a proper $\tau'_{(10)}$-factor of b when $1 \le i \le \alpha - 1$. Therefore

$$\nu_{\tau'_{(10)}}(b) = (\alpha - 1)\nu(q_1^{\beta_1} \cdots q_m^{\beta_m}) + 1.$$

(b) Assume that α and $\sum_{j=1}^{m} \beta_j$ are both odd integers. A similar argument as in the previous case, shows that b_1 is a $\tau'_{(10)}$-factor when $1 \le i \le \alpha-1$. Therefore, $\nu_{\tau'_{(10)}}(b) = (\alpha - 1)\nu(q_1^{\beta_1} \cdots q_m^{\beta_m}) + 1$.

Hence, if $\alpha \equiv \sum_{j=1}^{m} \beta_j \pmod 2$, then $\nu_{\tau'_{(10)}}(a) = (\alpha - 1)\nu(PQ) + 1$.

6. If $a \in [5]_{\tau'_{(10)}}$, then by Lemma 3.5 $\alpha = 0$ and $\beta \neq 0$. Let 5^j be a proper $\tau'_{(10)}$-factor of 5^β and c any positive factor of $A = p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$. If $\pm 5^j * c_1 * \cdots * c_k$ is a $\tau'_{(10)}$-factorization of 5^β, then $\pm\big(5^j \cdot c\big) * \left(c_1 \cdot \frac{A}{c}\right) * c_2 * \cdots * c_k$ is a $\tau'_{(10)}$-factorization of a. Hence, $5^j \cdot c$ is a proper $\tau'_{(10)}$-factor of a. Since c represents any positive factor of A, for each proper $\tau'_{(10)}$-factor of 5^β, a has $\nu(A)$ $\tau'_{(10)}$-factors. Therefore, $\nu_{\tau'_{(10)}}(a) = \big(\nu_{\tau'_{(10)}}(5^\beta) - 1\big)\nu(A) + 1$.

□

As an example of the Proposition 3.4, let's calculate the number of $\tau'_{(10)}$-factors of $2^3 \cdot 7^2 \cdot 11$. Since $7 \in [3]_{\tau'_{(10)}}$, $11 \in [11]_{\tau'_{(10)}}$ and $3 \not\equiv_2 2$, then

$$\nu_{\tau'_{(10)}}(2^3 \cdot 7^2 \cdot 11) = \lceil \frac{(3-2)\nu(7^2)}{2} \rceil \nu(11) + 1 = \lceil \frac{3}{2} \rceil \cdot 2 + 1 = 5.$$

All the $\tau'_{(10)}$-factors of $2^3 \cdot 7^2 \cdot 11$ are:

- 2
- $2 \cdot 7^2$
- $2 \cdot 11$
- $2 \cdot 7^2 \cdot 11$
- $2^3 \cdot 7^2 \cdot 11$

It clear that any prime integer is a $\tau'_{(10)}$-atom. Note that the only not trivial factorization of 10 is $2 \cdot 5$, but 2 and 5 are not related under $\tau'_{(10)}$. So $2 \cdot 5$ is not a $\tau'_{(10)}$-factorization of 10. This implies that 10 is a $\tau'_{(10)}$-atom. A similar reasoning proves that $33 = 3 \cdot 11$ is a $\tau'_{(10)}$-atom. The following corollary shows the form of all the $\tau_{(10)}$-atom.

Corollary 3.4. *The set of $\tau'_{(10)}$-atoms is given by the elements of form*

1. *The primes of $\mathbb{Z}$.*
2. *$\pm 2p_1 \cdots p_s$, where p_i are positive primes distinct to two.*
3. *$\pm 5p_1 \cdots p_s$, where p_i are positive primes distinct to five.*
4. *$\pm 2^2 p_1 \cdots p_s q_1 \cdots q_r$ where $p_i \in [11]_{\tau'_{(10)}}$ and $q_j \in [3]_{\tau'_{(10)}}$ are positive primes with r odd.*
5. *$\pm pp_1 \cdots p_s$ where $p \in [3]_{\tau'_{(10)}}$ and each $p_i \in [11]_{\tau'_{(10)}}$ are positive primes.*

Proof. By Proposition 3.4 each of these elements is a $\tau'_{(10)}$-atom. Let $a \in \mathbb{Z}^{\#}$ with canonical factorization $\pm 2^{\alpha} 5^{\beta} p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$ where $p_i \in [11]_{\tau'_{(10)}}$ and $q_j \in [3]_{\tau'_{(10)}}$ are positive distinct primes. If a is a $\tau'_{(10)}$-atom, satisfies that $\nu_{\tau'_{(10)}}(a) = 1$.

- If $\alpha \neq 0$ or $\beta \neq 0$, then $\big(f(\alpha, \beta) - 1\big)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}) + 1 = 1$. Therefore $f(\alpha, \beta) = 1$, because $\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}) \neq 0$. By the definition of $f(\alpha, \beta)$, four cases are verified:
  - If $\beta \neq 0$, then $(\alpha - 1)(\beta - 1) = 0$ if and only if either $\alpha = 1$ or $\beta = 1$. Therefore, the $\tau'_{(10)}$-atoms with this conditions have the form $\pm 2 \cdot 5^{\beta} \cdot p_1 \cdots p_s$ and $\pm 2^{\alpha} \cdot 5 \cdot p_1 \cdots p_s$, where each p_i is relative prime to 2 and 5, and $s \geq 0$.
  - If $\beta = 0$ and $\alpha = 1$, then the $\tau'_{(10)}$-atoms have the form $\pm 2 \cdot p_1 \cdots p_s$, where each p_i is relative prime to 2 and 5, and $s \geq 0$.
  - If $\beta = 0$, $\alpha > 1$ and $\alpha \not\equiv_2 \sum_{j=1}^{m} \beta_j$, then $\left\lceil \frac{(\alpha-2)\nu(q_1^{\beta_1} \cdots q_m^{\beta_m})}{2} \right\rceil = 0$ implies $\alpha = 2$, because $\nu(q_1^{\beta_1} \cdots q_m^{\beta_m}) \geq 0$. In order for a to be a $\tau'_{(10)}$-atom, it must have the

form $\pm 2^2 \cdot p_1 \cdots p_s \cdot q_1 \cdots q_r$, where $p_i \in [11]_{\tau'_{(10)}}$ and $q_j \in [3]_{\tau'_{(10)}}$ are positive primes, r is an odd integer and $s \geq 0$.

– If $\beta = 0$, $\alpha > 1$ and $\alpha \equiv_2 \sum_{j=1}^{m} \beta_j$, then $\alpha = 1$. Therefore, the $\tau'_{(10)}$-atoms have the form $\pm 2 \cdot p_1 \cdots p_s q_1 \cdots q_r$, where $p_i \in [11]_{\tau'_{(10)}}$ and $q_j \in [3]_{\tau'_{(10)}}$ are positive primes, r is an even integer and $s \geq 0$.

- If $\alpha = \beta = 0$ and $\sum_{j=1}^{m} \beta_j$ is an even integer, then $\nu(|a|) - 1 = 1$. In this case, the $\tau'_{(10)}$-atoms are prime integer non-asociated to 2 or 5.
- If $\alpha = \beta = 0$ and $\sum_{j=1}^{m} \beta_j$ is an odd integer, then $\frac{1}{2}\nu\big(|a|\big) - \nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) + 1 = 1$. Hence $\frac{\nu(q_1^{\beta_1} \cdots q_m^{\beta_m})}{2} = 1$, because $\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) \neq 0$. In this case the $\tau'_{(10)}$-atoms are of the form $\pm p p_1 \cdots p_s$, where p and the p_i's are positive primes with $p \in [3]_{\tau'_{(10)}}$ and each $p_i \in [11]_{\tau'_{(10)}}$.

□

3.2.4 The number of $\tau_{(12)}$-factors

Must noted that even though $\mathbb{Z}^{\#}/\tau'_{(12)}$ have seven equivalence class, the structure behaves nicer than $\mathbb{Z}^{\#}/\tau'_{(10)}$. Table 3–5 and Lemma 3.6 helps the reader to have a better to understand how the classes interact among them.

Lemma 3.6. *Let $a = \pm 2^{\alpha} 3^{\beta} p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m} \in \mathbb{Z}^{\#}$, where the p_i's $\in [13]_{\tau'_{(12)}}$ and q_j's $\in [5]_{\tau'_{(12)}}$ are non-associated positive primes . Then*

1. *$a \in [12]_{\tau'_{(12)}}$ if and only if $\alpha \geq 2$ and $\beta \neq 0$,*
2. *$a \in [13]_{\tau'_{(12)}}$ if and only if $\alpha = \beta = 0$ and $\sum_{i=1}^{m} \beta_i$ is even,*
3. *$a \in [2]_{\tau'_{(12)}}$ if and only if $\alpha = 1$ and $\beta = 0$,*
4. *$a \in [3]_{\tau'_{(12)}}$ if and only if $\alpha = 0$ and $\beta \neq 0$,*
5. *$a \in [4]_{\tau'_{(12)}}$ if and only if $\alpha \geq 2$ and $\beta = 0$,*
6. *$a \in [5]_{\tau'_{(12)}}$ if and only if $\alpha = \beta = 0$ and $\sum_{i=1}^{m} \beta_i$ is odd,*
7. *$a \in [6]_{\tau'_{(12)}}$ if and only if $\alpha = 1$ and $\beta \neq 0$.*

	$[12]_{\tau'_{(12)}}$	$[13]_{\tau'_{(12)}}$	$[2]_{\tau'_{(12)}}$	$[3]_{\tau'_{(12)}}$	$[4]_{\tau'_{(12)}}$	$[5]_{\tau'_{(12)}}$	$[6]_{\tau'_{(12)}}$
$[12]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$
$[13]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[13]_{\tau'_{(12)}}$	$[2]_{\tau'_{(12)}}$	$[3]_{\tau'_{(12)}}$	$[4]_{\tau'_{(12)}}$	$[5]_{\tau'_{(12)}}$	$[6]_{\tau'_{(12)}}$
$[2]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[2]_{\tau'_{(12)}}$	$[4]_{\tau'_{(12)}}$	$[6]_{\tau'_{(12)}}$	$[4]_{\tau'_{(12)}}$	$[2]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$
$[3]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[3]_{\tau'_{(12)}}$	$[6]_{\tau'_{(12)}}$	$[3]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[3]_{\tau'_{(12)}}$	$[6]_{\tau'_{(12)}}$
$[4]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[4]_{\tau'_{(12)}}$	$[4]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[4]_{\tau'_{(12)}}$	$[4]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$
$[5]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[5]_{\tau'_{(12)}}$	$[2]_{\tau'_{(12)}}$	$[3]_{\tau'_{(12)}}$	$[4]_{\tau'_{(12)}}$	$[13]_{\tau'_{(12)}}$	$[6]_{\tau'_{(12)}}$
$[6]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[6]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[6]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$	$[6]_{\tau'_{(12)}}$	$[12]_{\tau'_{(12)}}$

Table 3–5: Cayley's multiplicative table for $\mathbb{Z}^{\#}/\tau'_{(12)}$

Proof. Note that $\mathbb{Z}^{\#}/\tau'_{(12)} = \{[12]_{\tau'_{(12)}}, [13]_{\tau'_{(12)}}, [2]_{\tau'_{(12)}}, [3]_{\tau'_{(12)}}, [4]_{\tau'_{(12)}}, [5]_{\tau'_{(12)}}, [6]_{\tau'_{(12)}}\}$. By Table 3–5,we have that:

- $[2]^m_{\tau'_{(12)}} = [4]_{\tau'_{(12)}}$ for all $m \in \mathbb{N}$ with $m \geq 2$,
- $[3]^m_{\tau'_{(12)}} = [3]_{\tau'_{(12)}}$ for all $m \in \mathbb{N}$,
- $[5]^m_{\tau'_{(12)}} = \begin{cases} [13]_{\tau'_{(12)}}, \text{ if } m \text{ is even} \\ [5]_{\tau'_{(12)}}, \text{ if } m \text{ is odd} \end{cases}$

Hence, simple calculations shows that the converse of each statement holds.

Reciprocally, suppose that $a \in [12]_{\tau'_{(12)}}$, and either $\alpha < 2$ or $\beta = 0$. If $\alpha = 0$, then (by the converse of each statement) $a \in [13]_{\tau'_{(12)}}$ or $a \in [3]_{\tau'_{(12)}}$ or $a \in [5]_{\tau'_{(12)}}$. If $\alpha = 1$, then $a \in [2]_{\tau'_{(12)}}$ or $a \in [6]_{\tau'_{(12)}}$. Therefore, if $\alpha < 2$, then it contradicts the fact that $a \in [12]_{\tau'_{(12)}}$. Now, if $\beta = 0$, then can be in the classes $[13]_{\tau'_{(12)}}$, $[2]_{\tau'_{(12)}}$, $[4]_{\tau'_{(12)}}$ or $[5]_{\tau'_{(12)}}$, a contradiction. Therefore, if $a \in [12]_{\tau'_{(12)}}$, then $\alpha \geq 2$ and $\beta \neq 0$. A similar argument prove that if $a \in [4]_{\tau'_{(12)}}$, then $\alpha \geq 2$ and $\beta = 0$.

Assume $a \in [13]_{\tau'_{(12)}}$, and at least one of the following holds: $\alpha \neq 0$, $\beta \neq 0$ or $\sum_{i=1}^{m} \beta_i$ is odd. If $\alpha \neq 0$, then by the converse of each statement a can be in the classes $[12]_{\tau'_{(12)}}$, $[2]_{\tau'_{(12)}}$, $[4]_{\tau'_{(12)}}$ or $[6]_{\tau'_{(12)}}$. This contradicts that $a \in [13]_{\tau'_{(12)}}$. If $\beta \neq 0$, then a can be in $[12]_{\tau'_{(12)}}$, $[3]_{\tau'_{(12)}}$ or $[6]_{\tau'_{(12)}}$, which is a contradiction. If $\sum_{i=1}^{m} \beta_i$ is odd, then a can be in any class, except $[13]_{\tau'_{(12)}}$. Therefore, if $a \in [13]_{\tau'_{(12)}}$, then $\alpha = \beta = 0$ and $\sum_{j=1}^{m} \beta_j$ is even. A similar argument prove that if $a \in [5]_{\tau'_{(12)}}$, then $\alpha = \beta = 0$ and $\sum_{j=1}^{m} \beta_j$ is odd.

Now suppose that $a \in [2]_{\tau'_{(12)}}$, and either $\alpha \neq 1$ or $\beta \neq 0$. If $\alpha \neq 1$, then a can be in any class, except the clases $[2]_{\tau'_{(12)}}$ and $[6]_{\tau'_{(12)}}$. This contradicts that $a \in [2]_{\tau'_{(12)}}$. If $\beta \neq 0$, then a can be in $[12]_{\tau'_{(12)}}$, $[3]_{\tau'_{(12)}}$ or $[6]_{\tau'_{(12)}}$. Therefore, if $a \in [2]_{\tau'_{(12)}}$, then $\alpha = 1$ and $\beta = 0$. A similar argument prove that if $a \in [6]_{\tau'_{(12)}}$, then $\alpha = 1$ and $\beta \neq 0$.

Finally, assume that $a \in [3]_{\tau'_{(12)}}$, and $\alpha \neq 0$ or $\beta = 0$. If $\alpha \neq 0$, then a can be in the classes $[12]_{\tau'_{(12)}}$, $[2]_{\tau'_{(12)}}$, $[4]_{\tau'_{(12)}}$ or $[6]_{\tau'_{(12)}}$. This contradicts the fact that $a \in [3]_{\tau'_{(12)}}$. If $\beta = 0$, then can be in the classes $[13]_{\tau'_{(12)}}$, $[2]_{\tau'_{(12)}}$, $[4]_{\tau'_{(12)}}$ or $[5]_{\tau'_{(12)}}$, a contradiction. Therefore, if $a \in [3]_{\tau'_{(12)}}$, then $\alpha = 0$ and $\beta \neq 0$. $\square$

Now it is possible to count the number of $\tau'_{(12)}$-factors of an element $a \in \mathbb{Z}^{\#}$.

Proposition 3.5. *Let $a = \pm 2^{\alpha} 3^{\beta} p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m} \in \mathbb{Z}^{\#}$ where each $p_i \in [13]_{\tau'_{(12)}}$ and each $q_j \in [5]_{\tau'_{(12)}}$ are non-associated positive primes. Then*

$$\nu_{\tau'_{(12)}}(a) = \begin{cases} \Big(\nu_{\tau'_{(12)}}(2^{\alpha}3^{\beta}) - 1\Big)\nu(PQ) + 1, & \text{if } \alpha \neq 0 \text{ or } \beta \neq 0 \\ \nu\big(|a|\big) - 1, & \text{if } \alpha = \beta = 0 \text{ and } \displaystyle\sum_{j=1}^{m} \beta_j \text{ is even} \\ \dfrac{1}{2}\nu(|a|) - \nu(P) + 1, & \text{if } \alpha = \beta = 0 \text{ and } \displaystyle\sum_{j=1}^{m} \beta_j \text{ is odd} \end{cases}$$

where $P = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, $Q = q_1^{\beta_1} \cdots q_m^{\beta_m}$ *and*

$$\nu_{\tau'_{(12)}}(2^\alpha 3^\beta) = \begin{cases} \beta, & \textit{if } \alpha = 0 \textit{ and } \beta \neq 0. \\ 1, & \textit{if } \alpha = 1 \textit{ and } \beta \geq 0 \\ 2, & \textit{if } \alpha = 2 \textit{ and } \beta = 0 \\ \beta, & \textit{if } \alpha = 2 \textit{ and } \beta \neq 0 \\ \alpha - 1, & \textit{if } \alpha \geq 3 \textit{ and } \beta = 0 \\ (\alpha-3)(\beta-1)+1, & \textit{if } \alpha \geq 3 \textit{ and } 0 \neq \beta < \alpha \\ (\alpha-2)(\beta-2)+1, & \textit{if } \beta \geq \alpha \geq 3 \end{cases}$$

Proof. In order to calculate $\nu_{\tau'_{(12)}}(a)$, when $\alpha \neq 0$ or $\beta \neq 0$, need to calculate $\nu_{\tau'_{(12)}}(2^\alpha 3^\beta)$. For simplicity denote $a' = 2^\alpha 3^\beta$. If $\alpha = 0$, then by the clousure of $[3]_{\tau'_{(12)}}$ the number of $\tau'_{(12)}$-factor of 3^β is β. If $\alpha = 1$, then a' is a $\tau'_{(12)}$-atom. Hence $\nu_{\tau'_{(12)}}(a') = 1$.

Now assume $\alpha = 2$. If $\beta = 1$, then a' is a $\tau'_{(12)}$-atom. Therefore, the number of $\tau'_{(12)}$-factors of 12 is 1. Suppose $\beta \geq 2$. Since $3^n \equiv_{\tau'_{(12)}} 3$ for all $n \in \mathbb{N}$, then 3^i, is not a $\tau'_{(12)}$-factor of a'. Otherwise, if $a' = \pm 3^i * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of a', then every $c_j \in [3]_{\tau'_{(12)}}$, which contradicts that $a' \in [12]_{\tau'_{(12)}}$. Note that $3^0 = 1$ is not a $\tau'_{(12)}$-factor of a'. On the other hand, $2 \cdot 3^i$ is a $\tau'_{(12)}$-factor of a', because $\pm(2 \cdot 3^i) * (2 \cdot 3^{\beta-i})$ is a $\tau'_{(12)}$-factorization of a' for $1 \leq i \leq \beta - 1$. Note that 2 and $2 \cdot 3^\beta$ are not $\tau'_{(12)}$-factors of a'. Finally, $2^2 \cdot 3^i$ is not a $\tau'_{(12)}$-factor of a' for any $1 \leq i \leq \beta - 1$. Otherwise, if $\pm(2^2 \cdot 3^i) * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of a', then every $c_j \in [12]_{\tau'_{(12)}}$. But $c_1 \cdot c_2 \cdots c_k = 3^{\beta-i} \notin [12]_{\tau'_{(12)}}$. It's clear that 2^2 is not a $\tau'_{(12)}$-factor of a'. Therefore, if $\beta \geq 2$, $\nu_{\tau'_{(12)}}(2^2 \cdot 3^\beta) = (\beta - 1) + 1 = \beta$.

Assume $\alpha = 3$. If $\beta \in \{1, 2\}$, then a' is a $\tau'_{(12)}$-atom. Therefore, $\nu_{\tau'_{(12)}}(2^3 \cdot 3^\beta) = 1$, when $\beta \in \{1, 2\}$. Now consider $\beta \geq 3$.

- Note 3^i can not be a $\tau'_{(12)}$-factor of a', for $1 \leq i \leq \beta$. Otherwise, if $\pm 3^i * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of a', then every $c_j \in [3]_{\tau'_{(12)}}$. This contradicts the fact that $a' \in [12]_{\tau'_{(12)}}$. Clearly, $3^0 = 1$ is not a $\tau'_{(12)}$-factor of a'.
- Since $(2 \cdot 3^i) * (2 \cdot 3^{\beta-i-1}) * (2 \cdot 3)$ is a $\tau'_{(12)}$-factorization of a', when $1 \leq i \leq \beta - 2$, then $2 \cdot 3^i$ is a $\tau'_{(12)}$-factor of a'. Note that 2, $2 \cdot 3^{\beta-1}$ and $2 \cdot 3^{\beta}$ are not $\tau'_{(12)}$-factors of a'.
- Let $j \in \{2, 3\}$. Let see why does $2^j \cdot 3^i$ is not a $\tau'_{(12)}$-factor of a' when $1 \leq i \leq \beta - 1$. Otherwise, if $\pm(2^j \cdot 3^i) * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of a', then every $c_s \in [12]_{\tau'_{(12)}}$; but $c_1 \cdot c_2 \cdots c_k = 2^{3-j} \cdot 3^{\beta-i} \notin [12]_{\tau'_{(12)}}$. Similarly, 2^2, 2^3 and $2^2 \cdot 3^{\beta}$ are not a $\tau'_{(12)}$-factors of a', but $2^3 \cdot 3^{\beta}$ is. Therefore, if $\beta \geq 3$

$$\nu_{\tau'_{(12)}}(2^3 \cdot 3^{\beta}) = (\beta - 2) + 1 = \beta - 1.$$

Finally, assume $\alpha \geq 4$. Re-write $a' = b_1 \cdot b_2$ where $b_1 = 2^i 3^j$ with $i \leq \alpha$ and $j \leq \beta$.

- If $i = 0$, then $b_1 = 3^j$ is not a $\tau'_{(12)}$-factor of a'. Otherwise, if $\pm b_1 * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of a', then every $c_s \in [3]_{\tau'_{(12)}}$, which contradicts that $a' \in [12]_{\tau'_{(12)}}$. Note that $j \neq 0$, because 1 can not be a $\tau'_{(12)}$-factor of a'.
- Let $i = 1$. If $\beta < \alpha$, then b_1 can not be a $\tau'_{(12)}$-factor of a', for any $1 \leq j \leq \beta - 1$. Otherwise, if $\pm b_1 * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of a', then each $c_s \in [6]_{\tau'_{(12)}}$. By Lemma 3.6, each c_s must have the form $2 \cdot 3^{n_s} \cdot Q_s$, where Q_s is an odd integer and $n_s \geq 1$. Thus

$$2^{\alpha-1} 3^{\beta-j} = c_1 \cdot c_2 \cdots c_k = \left(2 \cdot 3^{n_1} \cdot Q_1\right) \cdots \left(2 \cdot 3^{n_k} \cdot Q_k\right) = 2^k \cdot 3^{n_1 + \cdots + n_k} Q_1 \cdots Q_k.$$

  Therefore, $k = \alpha - 1$ and $\beta > \beta - j \geq n_1 + \cdots + n_k \geq k$. Thus $\alpha - 1 = k < \beta$ Since $\beta < \alpha$, then $\alpha - 1 < \beta < \alpha$. This is a contradiction because $\alpha - 1$, α and β are integers. Neither 2 nor $2 \cdot \beta$ are $\tau'_{(12)}$-factors of a'. Now, assume $\beta \geq \alpha$. Let $j \leq \beta - \alpha + 1$. Since $\beta - j - \alpha + 2 \geq 1$, then $b_1 * \left(2 \cdot 3^{\beta-j-\alpha+2}\right) * \underbrace{(2 \cdot 3) * \cdots * (2 \cdot 3)}_{\alpha-2}$ is a $\tau'_{(12)}$-factorization of a'. Hence b_1 is a $\tau'_{(12)}$-factor of a'. If $\beta - \alpha + 1 < j$, then

$b_1 \in [6]_{\tau'_{(12)}}$ can not be a $\tau'_{(12)}$-factor of a'. Otherwise, if $\pm b_1 * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of a', then every $c_s \in [6]_{\tau'_{(12)}}$. But by Lemma 3.6, every c_s must have the form $2 \cdot 3^{n_s} \cdot Q_s$, where each Q_s is an odd integer and each $n_s \geq 1$. Thus, $2^{\alpha-1}3^{\beta-j} = c_1 \cdot c_2 \cdots c_k = \left(2 \cdot 3^{n_1} \cdot Q_1\right) \cdots \left(2 \cdot 3^{n_k} \cdot Q_k\right) = 2^k \cdot 3^{n_1+\cdots+n_k} Q_1 \cdots Q_k$. This forces k to be equal to $\alpha - 1$ and $\beta - j \geq n_1 + \cdots + n_k \geq k$. On the other hand, $\beta - \alpha + 1 < j$ implies $\beta - j < \alpha - 1$; a contradiction. Similar arguments show it is impossible for 2 and $2 \cdot 3^\beta$ to be $\tau'_{(12)}$-factors of a'.

- If $2 \leq i \leq \alpha - 2$ and $1 \leq j \leq \beta - 1$, then $b_1 \in [12]_{\tau'_{(12)}}$ is a $\tau'_{(12)}$-factor of a'. Because b_1 and b_2 both belong to $[12]_{\tau'_{(12)}}$. Note that neither 2^i nor $2^i 3^\beta$ can be $\tau'_{(12)}$-factors of a'.
- If $i = \alpha - 1$, then b_1 can not be a $\tau'_{(12)}$-factor of a'. Otherwise, if $1 \leq j \leq \beta$ and $\pm b_1 * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of a', then every $c_s \in [12]_{\tau'_{(12)}}$. But $c_1 \cdot c_2 \cdots c_k = 2 \cdot 3^{\beta-j} \notin [12]_{\tau'_{(12)}}$. Note that $2^{\alpha-1}$ is not a $\tau'_{(12)}$-factor of a'.
- Let $i = \alpha$. Then b_1 is not a $\tau'_{(12)}$-factor of a', for any $1 \leq j \leq \beta - 1$. Otherwise, if $\pm b_1 * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of a'. This forces every $c_s \in [12]_{\tau'_{(12)}}$, but $c_1 \cdot c_2 \cdots c_k = 3^{\beta-j} \notin [12]_{\tau'_{(12)}}$. The only $\tau'_{(12)}$-factor of a' with $i = \alpha$, is a' itself.

Therefore, if $\alpha \geq 4$,

$$\nu_{\tau'_{(12)}}(2^\alpha \cdot 3^\beta) = \begin{cases} (\alpha-3)(\beta-1)+1, \text{ if } \beta < \alpha \\ (\alpha-2)(\beta-2)+1, \text{ if } \beta \geq \alpha \end{cases}$$

For $\beta \neq 0$, the above facts can be summarized in the following formula

$$\nu_{\tau'_{(12)}}(2^\alpha \cdot 3^\beta) = \begin{cases} 1, & \text{if } \alpha = 1 \\ \beta, & \text{if } \alpha = 2 \\ (\alpha-3)(\beta-1)+1, & \text{if } \alpha \geq 3 \text{ and } \beta < \alpha \\ (\alpha-2)(\beta-2)+1, & \text{if } \alpha \geq 3 \text{ and } \beta \geq \alpha \end{cases}$$

Since $[2]^m_{\tau'_{(12)}} = [4]_{\tau'_{(12)}}$ for all $m \geq 2$, and $[3]^t_{\tau'_{(12)}} = [3]_{\tau'_{(12)}}$ for all t, then it follows that $\nu_{\tau'_{(12)}}(3^\beta) = \beta$ and

$$\nu_{\tau'_{(12)}}(2^\alpha) = \begin{cases} 1, & \text{if } \alpha = 1 \\ 2, & \text{if } \alpha = 2 \\ \alpha - 1, & \text{if } \alpha \geq 3 \end{cases}$$

Consider $a = \pm 2^\alpha 3^\beta p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$ where $p_i \in [13]_{\tau'_{(12)}}$ and $q_j \in [5]_{\tau'_{(12)}}$ are positive distinct primes. Let $2^i 3^j$ be a proper $\tau'_{(12)}$-factor of $2^\alpha 3^\beta$ and c a positive factor of $A = p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$. If $\pm(2^i 3^j) * c_1 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of $2^\alpha 3^\beta$, then by the properties of the elements of $\mathbb{Z}^\#/\tau'_{(12)}$,

$$\pm\Big(\big(2^i 3^j\big)c\Big) * \Big(c_1 \cdot \frac{A}{c}\Big) * c_2 * \cdots * c_k$$

must be a $\tau'_{(12)}$-factorization of a. Hence, $\big(2^i 3^j\big)c$ is a proper $\tau'_{(12)}$-factor of a. Since c is any positive factor of A, then for each proper $\tau'_{(12)}$-factor of $2^\alpha 3^\beta$, a has $\nu(A)$ $\tau'_{(12)}$-factors. Since a is also a $\tau'_{(12)}$-factor of a, $\nu_{\tau'_{(12)}}(a) = \big(\nu_{\tau'_{(12)}}(2^\alpha \cdot 3^\beta) - 1\big)\nu(A) + 1$.

By Lemma 3.6, the above equation calculates the number of $\tau'_{(12)}$-factors if $a \in \{[12]_{\tau'_{(12)}}, [2]_{\tau'_{(12)}}, [3]_{\tau'_{(12)}}, [4]_{\tau'_{(12)}}, [6]_{\tau'_{(12)}}\}$.

Now consider the cases when a is relative prime to 12. First, let $a \in [13]_{\tau'_{(12)}}$. By Lemma 3.6, $\alpha = \beta = 0$ and $\sum_{j=1}^m \beta_j$ is even. Consider $b_1 \in \mathbb{Z}^+ - \{1\}$ to be a factor of a. Say, $a = \pm b_1 \cdot b_2$ where $b_2 \in \mathbb{Z}$. Then in order to obtain a $\tau'_{(12)}$-factorization it suffices to see the following cases:

1. If $b_1 \in [13]_{\tau'_{(12)}}$, then either $b_2 \in [13]_{\tau'_{(12)}}$ or $b_2 = 1$. Therefore, $a = \pm b_1 * b_2$ or $a = \pm b_1$, is a $\tau'_{(12)}$-factorization of a. Hence b_1 is a $\tau'_{(12)}$-factor of a.
2. If $b_1 \in [5]_{\tau'_{(12)}}$, then $b_2 \in [5]_{\tau'_{(12)}}$. Thus, b_1 is a $\tau'_{(12)}$-factor of a, because $\pm b_1 * b_2$ is a $\tau'_{(12)}$-factorization of a.

Since $b_1 \neq 1$ represents any factor of a, $\nu_{\tau'_{(12)}}(a) = \nu(|a|) - 1$.

Now, suppose $a \in [5]_{\tau'_{(12)}}$. By Lemma 3.6, $\alpha = \beta = 0$ and $\sum_{j=1}^{m} \beta_j$ is odd. Let $b_1 \in \mathbb{Z}^+ - \{1\}$ a factor of Q. Say, $Q = b_1 \cdot b_2$, where $b_2 \in \mathbb{Z}$. Then in order to obtain a $\tau'_{(12)}$-factorization of Q it suffices to see the following cases:

1. If $b_1 \in [13]_{\tau'_{(12)}}$, then b_1 can not be a $\tau'_{(12)}$-factor of Q. Otherwise, if $\pm b_1 * c_1 * c_2 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of Q, then every $c_s \in [13]_{\tau'_{(12)}}$. This forces that $Q \in [5]_{\tau'_{(12)}}$, a contradiction.
2. If $b_1 \in [5]_{\tau'_{(12)}}$, then $\pm b_1 * b_{2(1)} * b_{2(2)} * \cdots * b_{2(t)}$ is a τ'_5-factorization of Q, where each $b_{2(i)}$ is a (not necessarily distinct) prime factor of b_2. Therefore, b_1 is a $\tau'_{(12)}$-factor of Q.

Now, observe that if $b_1 \in [5]_{\tau'_{(12)}}$, then $b_1 = q_1^{\beta'_1} q_2^{\beta'_2} \cdots q_m^{\beta'_m}$, where $\sum_{j=1}^{m} \beta'_j$ is an odd integer. By Lemma 3.2, there are $\lfloor \frac{\nu(Q)}{2} \rfloor = \frac{\nu(Q)}{2}$ factors with this condition. If b_1 a proper $\tau'_{(12)}$-factor of Q, c a positive factor of P and $\pm b_1 * c_1 * \cdots * c_k$ is a $\tau'_{(12)}$-factorization of Q, then

$$\pm(b_1 \cdot c) * \left(c_1 \cdot \frac{P}{c}\right) * c_2 * \cdots * c_k$$

is a $\tau'_{(12)}$-factorization of a. Hence, $b_1 \cdot c$ is a proper $\tau'_{(12)}$-factor of a. Since c is any positive factor of P, for each proper $\tau'_{(12)}$-factor of Q, a has $\nu(P)$ $\tau'_{(12)}$-factors. Thus $\nu_{\tau'_{(12)}}(a) = \left(\frac{\nu(Q)}{2} - 1\right)\nu(P) + 1 = \frac{\nu(|a|)}{2} - \nu(P) + 1$. This completes the proof. $\square$

As an example of the Proposition 3.5, let's calculate the number of $\tau'_{(12)}$-factors of $2^3 \cdot 7^2 \cdot 11$. Since $7 \in [5]_{\tau'_{(12)}}$, $11 \in [13]_{\tau'_{(12)}}$ and $\nu_{\tau'_{(12)}}(2^3) = 2$, then

$$\nu_{\tau'_{(12)}}(2^3 \cdot 7^2 \cdot 11) = \left(\nu_{\tau'_{(12)}}(2^3) - 1\right)\nu(7^2 \cdot 11) + 1 = (2-1)(3 \cdot 2) + 1 = 7.$$

The following are all $\tau'_{(12)}$-factors of $2^3 \cdot 7^2 \cdot 11$.

- 2
- $2 \cdot 7$

- $2 \cdot 7^2$
- $2 \cdot 11$
- $2 \cdot 7 \cdot 11$
- $2 \cdot 7^2 \cdot 11$
- $2^3 \cdot 7^2 \cdot 11$

It is clear that any prime integer is a $\tau'_{(12)}$-atom. Note that $6 = 2 \cdot 3$ have only two proper factors, and none of them is a $\tau'_{(12)}$-factor of 6. So, 6 is a $\tau'_{(12)}$-atom. Similarly, $55 = 5 \cdot 11$ is a $\tau'_{(12)}$-atom. The following corollary show the form of all the $\tau'_{(12)}$-atoms. Now consider $72 = 2^3 \cdot 3^2$. The proper factor of 72 are 3, 9, 2, 4, 8, 6, 12, 18, 24 and 36. By Table 3–5 and Lemma 3.6, 3^k and $2^i \cdot 3^j$ are not related under $\tau'_{(12)}$. This implies that the only not trivial factorizations of 72 which could occur are $6 \cdot 12$ and $2 \cdot 6 \cdot 6$. These factorizations are not $\tau'_{(12)}$-factorizations, because 6 are not related with 2 nor 12 under $\tau'_{(12)}$.

Corollary 3.5. *The set of $\tau'_{(12)}$-atoms is given by the elements of the following form:*

1. *The primes of $\mathbb{Z}$.*
2. $\pm 2p_1 \cdots p_s$, *where each p_i is a positive prime distinct to two.*
3. $\pm 3p_1 \cdots p_s$, *where each p_i is a positive prime distinct to three.*
4. $\pm 72p_1 \cdots p_s$, *where each $p_i > 3$ is a primes, and $s \geq 0$.*
5. $\pm pp_1 \cdots p_s$, *where $p \in [5]_{\tau'_{(12)}}$ and each $p_i \in [13]_{\tau'_{(12)}}$ is a positive prime.*

Proof. By Proposition 3.5 each of these elements is a $\tau'_{(12)}$-atom. Let $a \in \mathbb{Z}^{\#}$ with canonical factorization $\pm 2^\alpha 3^\beta p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}$ where each $p_i \in [13]_{\tau'_{(12)}}$ and each $q_j \in [5]_{\tau'_{(12)}}$ are distinct positive primes. If a is a $\tau'_{(12)}$-atom, satisfies that $\nu_{\tau'_{(12)}}(a) = 1$.

- If $\alpha \neq 0$ or $\beta \neq 0$, then $\left(\nu_{\tau'_{(12)}}(2^\alpha 3^\beta) - 1\right)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}) + 1 = 1$ from where $\nu_{\tau'_{(12)}}(2^\alpha 3^\beta) = 1$, because $\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l} q_1^{\beta_1} \cdots q_m^{\beta_m}) \neq 0$. By the definition of $\nu_{\tau'_{(12)}}(2^\alpha 3^\beta)$
  - If $\alpha = 0$ and $\beta \neq 0$, then $\beta = 1$. Therefore, the $\tau'_{(12)}$-atoms have the form $\pm 3 \cdot p_1 \cdots p_s$, where each $p_i > 3$ is a prime and $s \geq 0$.

- If $\alpha = 1$ and $\beta \geq 0$, then the $\tau'_{(12)}$-atoms have the form $\pm 2 \cdot 3^{\beta} \cdot p_1 \cdots p_s$, where each $p_i > 3$ is a prime and $s \geq 0$.
- If $\alpha = 2$ and $\beta = 0$, then $\nu_{\tau'_{(12)}}(2^2) = 2$. Thus, there are no atoms of this form.
- If $\alpha = 2$ and $\beta \neq 0$, then $\beta = 1$. Therefore, the $\tau'_{(12)}$-atoms have the form $\pm 2^2 \cdot 3 \cdot p_1 \cdots p_s$, where each $p_i > 3$ is a prime and $s \geq 0$.
- If $\alpha \geq 3$ and $\beta = 0$, then $\alpha = 2$. This is a contradicction. Therefore, there are no atoms of this form.
- If $\alpha \geq 3$ and $0 \neq \beta < \alpha$, then $(\alpha - 3)(\beta - 1) = 0$. Therefore the $\tau'_{(12)}$-atoms have the form $\pm 2^3 \cdot 3^{\beta} \cdot p_1 \cdots p_s$, with $\beta \in \{1, 2\}$, or $\pm 2^{\alpha} \cdot 3 \cdot p_1 \cdots p_s$, where each $p_i > 3$ is a prime and $s \geq 0$.
- If $\beta \geq \alpha \geq 3$, then $(\alpha - 2)(\beta - 2) = 0$. So, $\alpha = 2$ or $2 = \beta \geq \alpha$. This is impossible, because $\beta \geq \alpha \geq 3$. Therefore, there are no atoms of this form.

- If $\alpha = \beta = 0$ and $\sum_{j=1}^{m} \beta_j$ is an even integer, then $\nu(|a|) + 1 = 1$. Therefore, the $\tau'_{(12)}$-atoms are the primes greater than three.
- If $\alpha = \beta = 0$ and $\sum_{j=1}^{m} \beta_j$ is an odd integer, then $\frac{1}{2}\nu\big(|a|\big) - \nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) + 1 = 1$. Since $\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) \neq 0$, $\frac{\nu(q_1^{\beta_1} \cdots q_m^{\beta_m})}{2} = 1$. Therefore the $\tau'_{(12)}$-atoms are of the form $\pm p p_1 \cdots p_s$ where p and p_i are positive primes with $p \in [5]_{\tau'_{(12)}}$ and the p_i's $\in [13]_{\tau'_{(12)}}$.

Therefore, the τ'_{12}-atoms are

1. the primes of $\mathbb{Z}$.
2. $\pm 2 p_1 \cdots p_s$, where each p_i is a positive prime distinct to two.
3. $\pm 3 p_1 \cdots p_s$, where each p_i is a positive prime distinct to three.
4. $\pm 2^3 \cdot 3^2 \cdot p_1 \cdots p_s$, where each $p_i > 3$ is a primes, and $s \geq 0$.
5. $\pm p p_1 \cdots p_s$, where $p \in [5]_{\tau'_{(12)}}$ and each $p_i \in [13]_{\tau'_{(12)}}$ is a positive prime.

$\square$

This chapter found formulas to count the number of $\tau_{(n)}$-factors when $\phi(n) = 2$ or $\phi(n) = 4$. In these cases the number of prime relative classes to n are 1 or 2, which simplify the calculations. Using these formulas was possible to find the form of $\tau_{(n)}$-atoms. This form confirms the $\tau_{(n)}$-atoms for $n = 5$ and $n = 6$ found in previous research work done by Frazier, Hamon and Lanterman. For the first time a research work provides forms of the $\tau_{(8)}$-atoms, $\tau_{(10)}$-atoms and $\tau_{(12)}$-atoms.

When $\phi(n) \geq 6$, the number of prime relative classes (with respect to $\tau'_{(n)}$) is greater than 2. This makes the structure of $\mathbb{Z}^{\#}/\tau'_{(n)}$ more complicated, and therefore it is necessary to find another form to count the number of $\tau_{(n)}$-factors of an element in $\mathbb{Z}^{\#}$.

CHAPTER 4
SOME GENERAL EQUATIONS OF $\tau_{(n)}$

This chapter presents some general equations to count the number of $\tau'_{(n)}$-factors of a nonzero nonunit integer. These formulas were suggested by the proofs and patterns found in the cases showed in Chapter 3.

The proof of case $a \in [5]_{\tau'_{(5)}}$ in Proposition 3.2 suggests that for any prime p and $a = p^{\alpha} p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ with $p \not| p_i$, then $\nu_{\tau'_{(p)}} = (\alpha - 1)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l})$. For example, $7^4 \cdot 3$ have six $\tau'_{(7)}$-factors given by 7, $7 \cdot 3$, $7^2 \cdot 3$, 7^2, $7^3 \cdot 3$ and 7^3. This result is proved in general in the following proposition.

Proposition 4.1. *Let p a positive prime integer and consider $a = \pm p^{\alpha} p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, where the p_i's are positive primes distinct to p. Then*

$$\nu_{\tau'_{(p)}}(a) = (\alpha - 1)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) + 1.$$

Proof. Let $a' = p^{\alpha}$. Then $a' = p^i * p^{\alpha - i}$ is a $\tau'_{(p)}$-factorization of a' when $1 \leq i \leq \alpha - 1$. Recall that 1 can not be a $\tau'_{(p)}$-factor, but p^{α} is. Therefore, $\nu_{\tau'_{(p)}}(a') = \alpha$.

Now assume $P = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$. If $p^i * p^{\alpha - i}$ is a proper $\tau'_{(p)}$-factorization of a', and c is a positive factor of P, then $\pm\left(p^i \cdot c\right) * \left(p^{\alpha - i} \cdot \frac{P}{c}\right)$ is a $\tau'_{(p)}$-factorization of a. Since, c represents any positive factor of P, then $p^i \cdot c$ is a proper $\tau'_{(p)}$-factor of a, for any $1 \leq i \leq \alpha - 1$. Therefore, $\nu_{\tau'_{(p)}}(a) = (\alpha - 1)\nu(P) + 1$. $\square$

The following proposition shows that the number of $\tau'_{(n)}$-factors of an element $a \in [n+1]_{\tau'_{(n)}}$, where a the product of primes integer in $[n+1]_{\tau'_{(n)}}$ only, is the number of positive factor (distinct to 1) of a.

Proposition 4.2. *Let $n \in \mathbb{N}$ and consider $a = \pm p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, where each p_i is a positive prime in $[n+1]_{\tau'_{(n)}}$. Then $\nu_{\tau'_{(n)}}(a) = \nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) - 1$.*

Proof. Note that $[n+1]_{\tau'_{(n)}}$ is the identity element of $\mathbb{Z}^{\#}/\tau'_{(n)}$. Thus, any factor of a is in $[n+1]_{\tau'_{(n)}}$. Therefore, for every proper positive factor c of a, $(\pm 1)c * \frac{a}{c}$ is a $\tau'_{(n)}$-factorization of a. Since 1 is not a $\tau'_{(n)}$-factor of a, then $\nu_{\tau'_{(n)}}(a) = \nu(a) - 1$. $\square$

The next proposition generalized Proposition 4.2, and part of Proposition 3.2.

Proposition 4.3. *Let $n \in \mathbb{N}$ and $a = \pm p_1^{\alpha_1} \cdots p_l^{\alpha_l} \cdot q_1^{\beta_1} \cdots q_m^{\beta_m}$, where p_i and q_j are positive primes in $[n+1]_{\tau'_{(n)}}$ and $[q]_{\tau'_{(n)}}$ respectively. Suppose that $\left|[q]_{\tau'_{(n)}}\right| = 2$. Then*

$$\nu_{\tau'_{(n)}}(a) = \begin{cases} \nu(|a|) - 1 & \text{if } \sum_{i=1}^{m} \beta_i \text{ is even} \\ \frac{1}{2}\nu(|a|) - \nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) + 1 & \text{if } \sum_{i=1}^{m} \beta_i \text{ is odd.} \end{cases}$$

Proof. Let a satisfy the given hypothesis. First note that

$$[q]^s_{\tau'_{(n)}} = \begin{cases} [n+1]_{\tau'_{(n)}} & \text{if } s \text{ is even} \\ [q]_{\tau'_{(n)}} & \text{if } s \text{ is odd} \end{cases}$$

Let $b \neq 1$ a positive factor of a. If $\sum_{i=1}^{m} \beta_i$ is an even integer, then $a \in [n+1]_{\tau'_{(n)}}$. Thus $\pm b * \frac{a}{b}$ is a $\tau'_{(n)}$-factorization of a, because b and $\frac{a}{b}$ are either both in $[n+1]_{\tau'_{(n)}}$ or both in $[q]_{\tau'_{(n)}}$. Therefore, if $\sum_{i=1}^{m} \beta_i$ is even, then $\nu_{\tau'_{(n)}}(a) = \nu(|a|) - 1$.

On the other hand, if $\sum_{i=1}^{m}\beta_i$ is an odd integer, then $Q = q_1^{\beta_1}\cdots q_m^{\beta_m} \in [q]_{\tau'_{(n)}}$. Assume that $b' = q_1^{\delta_1}\cdots q_m^{\delta_m}$, where each $0 \le \delta_i \le \beta_i$. If $\sum_{i=1}^{m}\delta_i$ is an odd integer, then $b' \in [q]_{\tau'_{(n)}}$. Thus

$$\pm b' * \underbrace{q_1 * \cdots * q_1}_{\beta_1-\delta_1} * \cdots * \underbrace{q_m * \cdots * q_m}_{\beta_m-\delta_m}$$

is a $\tau'_{(n)}$-factorization of a. If $\sum_{i=1}^{m}\delta_i$ is an even integer and $\pm b' * c_1 * c_2 * \cdots * c_t$ is a $\tau'_{(n)}$-factorization of a, then each $c_i \in [n+1]_{\tau'_{(n)}}$ (because $b' \in [n+1]_{\tau'_{(n)}}$). Thus $\frac{Q}{b'} = c_1 \cdot c_2 \cdots c_t \in [n+1]_{\tau'_{(n)}}$, which implies that $Q = b' \cdot \frac{Q}{b'} \in [n+1]_{\tau'_{(n)}}$. This is a contradiction. By Lemma 3.2, $\nu_{\tau'_{(n)}}(Q) = \lfloor \frac{\nu(Q)}{2} \rfloor = \frac{\nu(Q)}{2}$. Now, let b' a proper $\tau'_{(n)}$-factor of Q and c a positive factor of $P = p_1^{\alpha_1}\cdots p_l^{\alpha_l}$. If $\pm b' * c_1 * \cdots * c_k$ is a $\tau'_{(n)}$-factorization of Q, then

$$\pm\big(b' \cdot c\big) * \left(c_1 \cdot \frac{P}{c}\right) * c_2 * \cdots * c_k$$

is a $\tau'_{(n)}$-factorization of a. Hence, $b' \cdot c$ is a proper $\tau'_{(n)}$-factor of a. Since c is any positive factor of P, for each proper $\tau'_{(n)}$-factor of b', a has $\nu(P)$ $\tau'_{(n)}$-factors. Thus $\nu_{\tau'_{(n)}}(a) = \big(\nu_{\tau'_{(n)}}(Q) - 1\big)\nu(P) + 1 = (\frac{\nu(Q)}{2} - 1)\nu(P) + 1 = \frac{1}{2}\nu(|a|) - \nu(P) + 1$.
Therefore,

$$\nu_{\tau'_{(n)}}(a) = \begin{cases} \nu(|a|) - 1 & \text{if } \displaystyle\sum_{i=1}^{m}\beta_i \text{ is even} \\ \dfrac{1}{2}\nu(|a|) - \nu(p_1^{\alpha_1}\cdots p_l^{\alpha_l}) + 1 & \text{if } \displaystyle\sum_{i=1}^{m}\beta_i \text{ is odd.} \end{cases}$$

□

The proof above suggests the following proposition.

Proposition 4.4. *Let $n \in \mathbb{N}$ and $a = b \cdot p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, where the p_i's are primes in $[n+1]_{\tau'_{(n)}}$. Then*

$$\nu_{\tau'_{(n)}}(a) = \begin{cases} \big(\nu_{\tau'_{(n)}}(b) - 1\big)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) + 1 \ \textit{if}\ b \notin [n+1]_{\tau'_{(n)}} \\ \big(\nu_{\tau'_{(n)}}(b) + 1\big)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) - 1 \ \textit{if}\ b \in [n+1]_{\tau'_{(n)}} \end{cases}$$

Proof. Consider $P = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, c to be a positive factor of P and $\pm c_1 * c_2 * \cdots * c_t$ a proper $\tau'_{(n)}$-factorization of b. Since each $c_i \in [q]_{\tau'_{(n)}}$ (for some $q \in \mathbb{Z}^{\#}$) and $[q]_{\tau'_{(n)}} \cdot [n+1]_{\tau'_{(n)}} = [q]_{\tau'_{(n)}}$, then $\pm(c_1 \cdot c) * \big(c_2 \cdot \frac{P}{c}\big) * \cdots * c_t$ is a $\tau'_{(n)}$-factorization of a. Thus, $c_1 \cdot c$ is a proper $\tau'_{(n)}$-factor of a. Since b does not have $\tau'_{(n)}$-factors in $[n+1]_{\tau'_{(n)}}$ (otherwise $b \in [n+1]_{\tau'_{(n)}}$), then a has $(\nu_{\tau'_{(n)}}(b) - 1)\nu(P)$ proper $\tau'_{(n)}$-factors. Therefore $\nu_{\tau'_{(n)}}(a) = \big(\nu_{\tau'_{(n)}}(b) - 1\big)\nu(P) + 1$.

Now suppose $b \in [n+1]_{\tau'_{(n)}}$. Since $c \in [n+1]_{\tau'_{(n)}}$, then $\pm b * c * \frac{P}{c}$ and $\pm(b \cdot c) * \frac{P}{c}$ are $\tau'_{(n)}$-factorizations of a (except the factorization $(b \cdot P) \cdot 1$). Therefore, $c_1 \cdot c$ (with c_1 a proper $\tau'_{(n)}$-factor of b), $b \cdot c$ and c (except $c = 1$) are $\tau_{(n)}$-factors of a. Thus $\nu_{\tau'_{(n)}}(a) = \big(\nu_{\tau'_{(n)}}(b) - 1\big)\nu(P) + \nu(P) + \nu(P) - 1 = \big(\nu_{\tau'_{(n)}}(b) + 1\big)\nu(P) - 1$. $\square$

Note that first equation generalized the way we count the number of $\tau'_{(n)}$-factors of an element $a \in \mathbb{Z}^{\#}$, when $gcd(a, n) \neq 1$. The second equation was not used in any of the proofs given in Chapter 3. But, for example, it can be used to count the number of $\tau'_{(7)}$-factor of $a \in [8]_{\tau'_{(7)}}$.

By definition of $\tau'_{(n)}$, if $q \in \mathbb{Z}^{\#}$, $a \in [q]_{\tau'_{(n)}}$ if and only if $a = nz \pm q$ for some $z \in \mathbb{Z}$. The following fact is useful to simplify the proof of the proposition below.

Lemma 4.1. *Let $d, n \in \mathbb{N}$ such that $d|n$, and $[a]_{\tau'_{(n)}} \in U\Big(\mathbb{Z}^{\#}/\tau'_{(n)}\Big)$. Then*

$$[d \cdot a]_{\tau'_{(n)}} = [d]_{\tau'_{(n)}} \ \textit{if and only if}\ \Big|U\big(\mathbb{Z}^{\#}/\tau'_{(\frac{n}{d})}\big)\Big| = 1.$$

Proof. First observe that for all $m \in \mathbb{N}$ with $m > 1$, $\left|U\left(\mathbb{Z}^{\#}/\tau'_{(m)}\right)\right| \geq 1$ (because $[m+1]_{\tau'_{(m)}}$ is the identity element). If $\left|U\left(\mathbb{Z}^{\#}/\tau'_{(\frac{n}{d})}\right)\right| = 1$, then the only element of $\mathbb{Z}^{\#}/\tau'_{(\frac{n}{d})}$ is $[\frac{n}{d}+1]_{\tau'_{(\frac{n}{d})}}$. If $a \in [\frac{n}{d}+1]_{\tau'_{(\frac{n}{d})}}$, then $a = \frac{n}{d}z \pm \left(\frac{n}{d}+1\right)$ for some $z \in \mathbb{Z}$. This implies that $d \cdot a = nz \pm (n+d) = nz' \pm d$, where $z' = z \pm 1$. Thus, $d \cdot a \in [d]_{\tau'_{(n)}}$. Since $[d \cdot a]_{\tau'_{(n)}} \cap [d]_{\tau'_{(n)}} \neq \emptyset$, then $[d \cdot a]_{\tau'_{(n)}} = [d]_{\tau'_{(n)}}$.

For the converse, suppose that $\left|U\left(\mathbb{Z}^{\#}/\tau'_{(\frac{n}{d})}\right)\right| > 1$. Let $[s]_{\tau'_{(\frac{n}{d})}} \in U\left(\mathbb{Z}^{\#}/\tau'_{(\frac{n}{d})}\right)$ such that $[s]_{\tau'_{(\frac{n}{d})}} \neq [\frac{n}{d}+1]_{\tau'_{(\frac{n}{d})}}$. If $[d \cdot s]_{\tau'_{(n)}} = [d]_{\tau'_{(n)}}$, then $d \cdot s \in [d]_{\tau'_{(n)}}$. By definition $d \cdot s = nz \pm d$ for some $z \in \mathbb{Z}$. Thus, $s = \frac{n}{d}z' \pm (\frac{n}{d}+1)$ where $z' = z \mp 1$. Therefore, $s \in [\frac{n}{d}+1]_{\tau'_{(\frac{n}{d})}}$, and so $[s]_{\tau'_{(\frac{n}{d})}} = [\frac{n}{d}+1]_{\tau'_{(\frac{n}{d})}}$. This is a contradiction. In consecuence, $[d \cdot s]_{\tau'_{(n)}} \neq [d]_{\tau'_{(n)}}$. □

Proposition 4.5. *Let $d, n \in \mathbb{N}$ such that $d|n$, and $[a]_{\tau'_{(n)}} \in U\Big(\mathbb{Z}^{\#}/\tau'_{(n)}\Big)$. Then, $[d \cdot a]_{\tau'_{(n)}} = [d]_{\tau'_{(n)}}$ if and only if $\frac{\phi(n)}{2} \leq d$.*

Proof. By Lemma 4.1 is sufficient to prove that $\left|U\left(\mathbb{Z}^{\#}/\tau'_{(\frac{n}{d})}\right)\right| = 1$ if and only if $\frac{\phi(n)}{2} \leq d$. Let $n = n_1^{\alpha_1} \cdots n_k^{\alpha_k}$ the canonical factorization of n, and $d = n_1^{\beta_1} \cdots n_k^{\beta_k}$, where $0 \leq \beta_i \leq \alpha_i$. Observe that $\frac{\phi(n)}{2} = \frac{\phi(n_1^{\alpha_1} \cdots n_k^{\alpha_k})}{2} = \frac{n_1^{\alpha_1-1} \cdots n_k^{\alpha_k-1}(n_1-1)\cdots(n_k-1)}{2}$ and

$$\begin{aligned}\left|U\left(\mathbb{Z}^{\#}/\tau'_{(\frac{n}{d})}\right)\right| &= \left\lceil \frac{\phi(n_1^{\alpha-\beta_1} \cdots n_k^{\alpha-\beta_k})}{2} \right\rceil \\ &= \left\lceil \frac{n_1^{\alpha-\beta_1-1} \cdots n_k^{\alpha-\beta_k-1}(n_1-1)\cdots(n_k-1)}{2} \right\rceil\end{aligned}$$

If $\frac{\phi(n)}{2} \leq d$, then $\frac{n_1^{\alpha_1-1} \cdots n_k^{\alpha_k-1}(n_1-1)\cdots(n_k-1)}{2} \leq n_1^{\beta_1} \cdots n_k^{\beta_k}$. This implies that

$$\frac{n_1^{\alpha-\beta_1-1} \cdots n_k^{\alpha-\beta_k-1}(n_1-1)\cdots(n_k-1)}{2} \leq 1.$$

Therefore, $\left|U\left(\mathbb{Z}^{\#}/\tau'_{(\frac{n}{d})}\right)\right| = 1$.

Conversely, if $\frac{\phi(n)}{2} > d$, then $\frac{n_1^{\alpha_1 - 1} \cdots n_k^{\alpha_k - 1}(n_1-1)\cdots(n_k-1)}{2} > n_1^{\beta_1} \cdots n_k^{\beta_k}$. This implies that

$$\frac{n_1^{\alpha-\beta_1-1} \cdots n_k^{\alpha-\beta_k-1}(n_1-1)\cdots(n_k-1)}{2} > 1.$$

Therefore, $\left|U\left(\mathbb{Z}^{\#}/\tau'_{(\frac{n}{d})}\right)\right| > 1$. $\square$

Corollary 4.1. *Let $n \in \mathbb{N}$, p a prime such that $p|n$, and $a = \pm p^{\alpha} p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, where the p_i's are primes in the equivalence class of $U\left(\mathbb{Z}^{\#}/\tau'_{(n)}\right)$. If $\frac{\phi(n)}{2} \leq p$, then*

$$\nu_{\tau'_{(n)}}(a) = \left(\nu_{\tau'_{(n)}}(p^{\alpha}) - 1\right)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) + 1.$$

Proof. Proposition 4.5 implies that $[p]_{\tau'_{(n)}} \cdot [q]_{\tau'_{(n)}} = [p]_{\tau'_{(n)}}$ for all $[q]_{\tau'_{(n)}} \in U\left(\mathbb{Z}^{\#}/\tau'_{(n)}\right)$. Therefore $[q]_{\tau'_{(n)}}$ functions as $[n+1]_{\tau'_{(n)}}$ with respect to any power of $[p]_{\tau'_{(n)}}$. So, the proof of Proposition 4.4 shows that

$$\nu_{\tau'_{(n)}}(a) = \left(\nu_{\tau'_{(n)}}(p^{\alpha}) - 1\right)\nu(p_1^{\alpha_1} \cdots p_l^{\alpha_l}) + 1.$$

$\square$

In order to find more general results we need to understand the cases when $\phi(n) \geq 6$. We expect to used the results in Chapter 3 and this chapter to find such formulas or described them.

CHAPTER 5
CONCLUSION AND FUTURE WORK

This chapter presents a summary of the main results of this research and some considerations to approach the study of $\tau'_{(7)}$.

5.1 Conclusion

The results found in this work can be shown in two parts. The first part is when $\left|U\big(\mathbb{Z}^{\#}/\tau'_{(n)}\big)\right| = 1$. Recall that $\left|U\big(\mathbb{Z}^{\#}/\tau'_{(n)}\big)\right| = 1$ if and only if $n \in \{2,3,4,6\}$. The formulas given by Figueroa for $\nu_{\tau'_{(n)}}(a)$ when $n \in \{2,3,4\}$, are summarized in Table 2–2. By the Table 2–2 and Proposition 3.1, $\nu_{\tau'_{(n)}}(a)$ (when $\phi(n) = 2$) can be written as

$$\nu_{\tau'_{(n)}}(a) = \begin{cases} \big(\nu_{\tau'_{(n)}}(a') - 1\big)\nu(P) + 1 \text{ if } gcd(a,n) \neq 1 \\ \nu(|a|) - 1 \text{ if } gcd(a,n) = 1. \end{cases}$$

where a' is the positive integer which is composed by prime factors of n with $a = \pm a'P$ and $gcd(P,n) = 1$.

The second part is when $\left|U\big(\mathbb{Z}^{\#}/\tau'_{(n)}\big)\right| = 2$ (or $\phi(n) = 4$). Therefore, in these cases $U\big(\mathbb{Z}^{\#}/\tau'_{(n)}\big) = \{[n+1]_{\tau'_{(n)}}, [q]_{\tau'_{(n)}}\}$, where q is an integer with $gcd(q,n) = 1$ and $q \notin [n+1]_{\tau'_{(n)}}$. This happens when $n \in \{5,8,10,12\}$. By Propositions 3.2, 3.3, 3.4 and 3.5, $\nu_{\tau'_{(n)}}(a)$ (when $\phi(n) = 4$) can be written as

$$\nu_{\tau'_{(n)}}(a) = \begin{cases} f_n(a)\nu(PQ) + 1 \text{ if } gcd(a,n) \neq 1 \\ \nu(|a|) - 1 \text{ if } a \in [n+1]_{\tau'_{(n)}} \\ \frac{1}{2}\nu(|a|) - \nu(P) + 1 \text{ if } a \in [q]_{\tau'_{(n)}} \end{cases}$$

where $f_n(a')$ is a function, which in most cases is $\nu_{\tau'_{(n)}}(a')-1$ where a' is the positive integer which is composed by prime factors of n so that $a=\pm a'PQ$, $gcd(PQ,n)=1$, and P (respectively, Q) is the product of the prime factors of a in $[n+1]_{\tau'_{(n)}}$ (respectively, in $[q]_{\tau'_{(n)}}$).

5.2 Future work

The formulas for $\nu_{\tau'_{(n)}}$ when the order of $U\big(\mathbb{Z}^{\#}/\tau'_{(n)}\big)$ is 1 or 2 are complete. Now, the next step is to verify what happens in other cases when $\phi(n)\geq 6$. For example, $\left|U\big(\mathbb{Z}^{\#}/\tau'_{(n)}\big)\right| = 3$ if and only if $n\in\{7,9,14,18\}$. It is possible to prove that in each one of these cases, $U\big(\mathbb{Z}^{\#}/\tau'_{(n)}\big)=\{[n+1]_{\tau'_{(n)}},[q]_{\tau'_{(n)}},[r]_{\tau'_{(n)}}\}$ where q and r are integers satisfying that

$$\begin{aligned}
[q]_{\tau'_{(n)}}\cdot[r]_{\tau'_{(n)}} &= [n+1]_{\tau'_{(n)}},\\
[q]^2_{\tau'_{(n)}} &= [r]_{\tau'_{(n)}} \text{ and}\\
[r]^2_{\tau'_{(n)}} &= [q]_{\tau'_{(n)}}.
\end{aligned}\tag{5.1}$$

In particular, for $n=7$, $U\big(\mathbb{Z}^{\#}/\tau'_{(7)}\big)=\{[8]_{\tau'_{(7)}},[2]_{\tau'_{(7)}},[3]_{\tau'_{(7)}}\}$. Let

$$a=\pm 7^{\alpha}PQR,$$

where $P=p_1^{\alpha_1}\cdots p_l^{\alpha_l}$, $Q=q_1^{\beta_1}\cdots q_m^{\beta_m}$, $R=r_1^{\delta_1}\cdots r_s^{\delta_s}$, and p_i, q_j and r_k are primes integers in $[8]_{\tau'_{(7)}}$, $[2]_{\tau'_{(7)}}$ and $[3]_{\tau'_{(7)}}$, respectively.

	$[7]_{\tau'_{(7)}}$	$[8]_{\tau'_{(7)}}$	$[2]_{\tau'_{(7)}}$	$[3]_{\tau'_{(7)}}$
$[7]_{\tau'_{(7)}}$	$[7]_{\tau'_{(7)}}$	$[7]_{\tau'_{(7)}}$	$[7]_{\tau'_{(7)}}$	$[7]_{\tau'_{(7)}}$
$[8]_{\tau'_{(7)}}$	$[7]_{\tau'_{(7)}}$	$[8]_{\tau'_{(7)}}$	$[2]_{\tau'_{(7)}}$	$[3]_{\tau'_{(7)}}$
$[2]_{\tau'_{(7)}}$	$[7]_{\tau'_{(7)}}$	$[2]_{\tau'_{(7)}}$	$[3]_{\tau'_{(7)}}$	$[8]_{\tau'_{(7)}}$
$[3]_{\tau'_{(7)}}$	$[7]_{\tau'_{(7)}}$	$[3]_{\tau'_{(7)}}$	$[8]_{\tau'_{(7)}}$	$[2]_{\tau'_{(7)}}$

Table 5–1: Cayley's multiplicative table for $\mathbb{Z}^{\#}/\tau'_{(7)}$.

By Table 5–1 and some basic calculations, we obtain properties that determine how does the $\tau'_{(7)}$-products behaves. These are summarized in the following four facts:

$$a \in [7]_{\tau'_{(7)}} \text{ if and only if } \alpha \neq 0,$$

$$a \in [8]_{\tau'_{(7)}} \text{ if and only if } \sum_{j=1}^{m} \beta_j - \sum_{k=1}^{s} \delta_k \equiv_3 0,$$

$$a \in [2]_{\tau'_{(7)}} \text{ if and only if } \sum_{j=1}^{m} \beta_j - \sum_{k=1}^{s} \delta_k \equiv_3 1,$$

and

$$a \in [3]_{\tau'_{(7)}} \text{ if and only if } \sum_{j=1}^{m} \beta_j - \sum_{k=1}^{s} \delta_k \equiv_3 2.$$

By Proposition 4.1, we know that $\nu_{\tau'_{(7)}}(a) = (\alpha - 1)\nu(\left|\frac{a}{7^\alpha}\right|) + 1$, if $a \in [7]_{\tau'_{(7)}}$. So need to study what happen if $a \notin [7]_{\tau'_{(7)}}$. When $a \in [8]_{\tau'_{(7)}}$, we consider the expresion in Table 5–2 that shows explicitly the $\tau'_{(7)}$-factors of $QR = q_1 q_2 q_3^2 r_1^2 r_2^2 \in [8]_{\tau'_{(7)}}$. The first and second column shows the number of primes in the class $[2]_{\tau'_{(7)}}$ or $[3]_{\tau'_{(7)}}$, respectively, of a factor b of QR. The third column shows a possible form of b. The sixth column shows a possible $\tau'_{(7)}$-factorization for QR where b is a $\tau'_{(7)}$-factor.

Note that in Table 5–2, the only factors of factor QR that are not $\tau'_{(7)}$-factors (except when $b = 1$), occur when $\frac{QR}{b}$ is a prime in $[2]_{\tau'_{(7)}}$ or $[3]_{\tau'_{(7)}}$. Therefore, after a review of cases that must occur for QR to admit a proper $\tau'_{(7)}$-factorization is possible to prove that $\nu_{\tau'_{(7)}}(QR) = \nu(QR) - (m + s + 1)$, and so by Proposition 4.4,

$$\begin{aligned}
\nu_{\tau'_{(7)}}(a) &= \big(\nu_{\tau'_{(7)}}(QR) + 1\big)\nu(P) - 1 \\
&= \big(\nu(QR) - (m + s)\big)\nu(P) - 1 \\
&= \nu(|a|) - (m + s)\nu(P) - 1.
\end{aligned}$$

In $[2]_{\tau'_{(7)}}$	In $[3]_{\tau'_{(7)}}$	Factor b	$\tau'_{(7)}$-factor	Class	$\tau'_{(7)}$-fact.
0	0	1	NO	–	–
0	1	r_1	YES	[3]	$r_1 * r_1 * q_1q_2q_3^2r_2^2$
0	2	r_1^2	YES	[2]	$r_1^2 * q_1 * q_2q_3^2r_2^2$
0	3	$r_1^2r_2$	YES	[8]	$r_1^2r_2 * q_1q_2q_3^2r_2$
0	4	$r_1^2r_2^2$	YES	[3]	$r_1^2r_2^2 * q_1q_2 * q_3^2$
1	0	q_1	YES	[2]	$q_1 * q_2 * q_3^2r_1^2r_2^2$
1	1	q_1r_1	YES	[8]	$q_1r_1 * q_2q_3^2r_1r_2^2$
1	2	$q_1r_1^2$	YES	[3]	$q_1r_1^2 * r_2 * r_2q_2q_3^2$
1	3	$q_1r_1^2r_2$	YES	[2]	$q_1r_1^2r_2 * q_2 * q_3^2r_2$
1	4	$q_1r_1^2r_2^2$	YES	[8]	$q_1r_1^2r_2^2 * q_2q_3^2$
2	0	q_3^2	YES	[3]	$q_3^2 * r_1 * q_1q_2r_1r_2^2$
2	1	$q_3^2r_1$	YES	[2]	$q_3^2r_1 * q_1 * q_2r_1r_2^2$
2	2	$q_3^2r_1^2$	YES	[8]	$q_3^2r_1^2 * q_1q_2r_2^2$
2	3	$q_3^2r_1^2r_2$	YES	[3]	$q_3^2r_1^2r_2 * r_2 * q_1q_2$
2	4	$q_3^2r_1^2r_2^2$	YES	[2]	$q_3^2r_1^2r_2^2 * q_1 * q_2$
3	0	$q_1q_3^2$	YES	[8]	$q_1q_3^2 * q_1r_1^2r_2^2$
3	1	$q_1q_3^2r_1$	YES	[3]	$q_1q_3^2r_1 * r_1 * q_1r_2^2$
3	2	$q_1q_3^2r_1^2$	YES	[2]	$q_1q_3^2r_1^2 * q_2 * r_2^2$
3	3	$q_1q_3^2r_1^2r_2$	YES	[8]	$q_1q_3^2r_1^2r_2 * q_2 * r_2$
3	4	$q_1q_3^2r_1^2r_2^2$	NO	[3]	$q_1q_3^2r_1^2r_2^2 \cdot q_2$
4	0	$q_1q_2q_3^2$	YES	[2]	$q_1q_2q_3^2 * r_1^2 * r_2^2$
4	1	$q_1q_2q_3^2r_1$	YES	[8]	$q_1q_2q_3^2r_1 * r_1r_2^2$
4	2	$q_1q_2q_3^2r_1^2$	YES	[3]	$q_1q_2q_3^2r_1^2 * r_2 * r_2$
4	3	$q_1q_2q_3^2r_1^2r_2$	NO	[2]	$q_1q_2q_3^2r_1^2r_2 \cdot r_2$
4	4	$q_1q_2q_3^2r_1^2r_2^2$	YES	[8]	$q_1q_2q_3^2r_1^2r_2^2 * 1$

Table 5–2: The $\tau'_{(7)}$-factors of $q_1q_2q_3^2r_1^2r_2^2$.

Determining the behavior of how P, Q and R is more difficult than expected. In order to move in this direction is required to have more tools, are required such as representation into primitive roots or dividing the problem into substructures of $\mathbb{Z}^{\#}/\tau'_{(n)}$.

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ON THE NUMBER OF $\tau_{(n)}$-FACTORS

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