# On the formalism of quantum mechanics constructed with the dispersion relation of deformed Special Relativity

by

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Abstract of Thesis Presented to the Graduate School of the University of Puerto Rico in Partial Fulfillment of the Requirements for the Degree of Master of Science

**ON THE FORMALISM OF QUANTUM MECHANICS CONSTRUCTED WITH THE DISPERSION RELATION OF DEFORMED SPECIAL RELATIVITY**

By

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Over the last ten years there has been a significant effort to develop Deformed Special Relativity (DSR). This theory has two observer invariant scales: the speed of light,($c$), and the Planck energy,($E_p$). It is argued by F. Girelli, E.R. Livine, and D. Oriti that this type of theory is an effective flat limit of quantum gravity. The idea behind the formulation of such theories is that at energies comparable to $E_P$ the energy-momentum dispersion relation $E^2 = p^2c^2 + m^2c^4$ must be modified by *quantum gravitational effects*. As shown by Amelino-Camelia and Piran the dispersion relation will have the general form: $E^2 = p^2c^2 + m^2c^4 + \lambda E^3 + \cdots$ where $\lambda$ is of the order of Planck length (inverse of $E_p$) This modification of the dispersion relation induces changes in the structure of the relativistic wave equations and the Schrödinger equation. The main goal of this project is to determine the form of the DSR-modified Schrödinger equation and to explicitly calculate the DSR corrections to its solutions for $d = 1$ and $d = 3$ problems. The Free Particle, the Harmonic Oscillator, and the Hydrogen Atom were studied. The study also includes applications of the Lagrangian formalism and the effects of local gauge transformations on the Lagrangian of the theory.

Resumen de Disertación Presentado a la Escuela Graduada de la Universidad de Puerto Rico como Requisito Parcial para el grado de Maestría en Ciencias

# SOBRE EL FORMALISMO DE MECÁNICA CUÁNTICA CONSTRUIDA CON LA RELACIÓN DE DISPERSION DE RELATIVIDAD ESPECIAL DEFORMADA

By

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Durante los últimos diez años se ha observado un esfuerzo importante para desarrollar la teora llamada Deformed Special Relativity (DSR). Esta teoría tiene dos escalas invariantes: la velocidad de la luz ($c$) y la energía de Planck ($E_p$). Se argumenta por F. Girelli, Livine E.R., y D. Oriti que este tipo de teoría es un límite efectivo para el espacio-tiempo plano de la gravedad cuántica. La idea detrás de la formulación de estas teorías es que a energías comparables a, la relación de energía y momentum $E^2 = p^2c^2+m^2c^4$ debe ser modificada por efectos gravitacionales cuánticos. Como ha sido demostrado por Amelino-Camelia y Piran la relación de dispersión tendrá la forma general:$E^2 = p^2c^2 + m^2c^4 + \lambda E^3$, donde $\lambda$ es del orden de la longitud de Planck (el inverso de $E_P$). Esta modificación de la relación de dispersión induce cambios en la estructura de las ecuaciones de onda relativista y la ecuación de Schrödinger . El objetivo principal de este trabajo es determinar la forma de la ecuación de Schrödinger modificada por los efectos de DSR y calcular explícitamente las correcciones inducidas por DSR a las soluciones

de la ecuación de Schrödinger para problemas de una y tres dimensiones. Se estudiaron la partícula libre, el oscilador armónico, y el átomo de hidrógeno. El estudio también incluye aplicaciones del formalismo de Lagrange y los efectos de las transformaciones de calibre local en la función de Lagrange de la teoría.

# Contents

# Chapter 1

# Introduction

## 1.1 Overview

In the late 19th and early 20th centuries, with the discovery of quantum theory and special relativity, the world of physics would not be the same. Soon after these theories were discovered and experimentally tested, it did not took long to combine them. The first person to do this was P.A.M Dirac with the theory of the positron. Later, when other fundamental particles and forces were discovered and studied by physicists, we had a model that explained the dynamics of subatomic particles known as the Standard Model. With these outstanding results of quantum theory it didn't took long for physicists to investigate if gravity, well described by Einstein's general relativity, could be quantized. These investigations led to the birth of quantum gravity. In the late 90's, there were some work by Gambini and Pullin [1] in the propagation of light of semiclassical space-time. While investigating corrections the modification of Maxwell's equations due to quantum gravity they found that there were not Lorentz covariant. This result suggested that there will be a modification to the dispersion relation of light propagation. Later on, in 2002 in an article by Alfaro, Morales-Técolt and Urrutia [2], studying a approximation to the Einstein-Maxwell equations, they found a correction of Maxwell equations in flat space that exibit Planck scale corrections. They started with a modified dispersion relation for the energy of a photon.

On the experimental side, in 2004 it was reported that cosmic rays with an estimated energy of approximately $3 \times 10^{20}$ electronvolts were observed [3]. These results were in disagreement with the so called Greisen-Zatsepin-Kuzmin limit (GZK limit) [4], which in turn is based on Einstein's special relativity. Although the validity of the GZK has been confirmed in 2007 by the Pierre Auger Observatory [5], the initial disagreement prompted the development of alternative theories aimed at explaining it. Giovanni Amelino-Camelia was the first to develop a theory called Deformed-Special Relativity or Doubly Special Relativity(DSR). It is a modified version of Einstein's special relativity that includes two observer-independent quantities: the speed of light and the Planck energy. It is expected to have corrections to special relativity due to quantum gravitational effects [6]. This theory was constructed both to account for the experimental data as a modification demanded by theoretical development in quantum gravity.

Our interest in this thesis is to investigate the non-relativistic limit of the DSR dispersion relation and apply it to quantum mechanics. We will explore if DSR can give new interactions or corrections to the theory in the non-relativistic limit. In the next chapter we will briefly discuss how Galilean relativity is incompatible with electromagnetic theory, Einsteins solution to the problem, and the development of DSR. DSR tries to find a nonlinear action in momentum space that leaves the Lorentz group invariant. This leads to the deformation of the usual dispersion relation. The modified dispersion relation that can be constructed is not unique, there are several theories of DSR and they are discussed in this chapter.

In the third chapter we use two approximations of the energy-momentum relation proposed by Amelino-Camelia. The first approximation will be applied to the modified dispersion relation proposed by Amelino-Camelia, where we have extended to second order on our expansion parameter. This approximation yields a more manageable equation that can be solved. The second approximation will be the non-relativistic limit. After these approximations are completed we will write the Schrödinger equation with the DSR correction.

Once we have the DSR Schrödinger equation the most natural step is to solve some elementary quantum mechanics problems in the DSR context and compare them to the ordinary quantum problems. This is done in chapter four. We will study the harmonic oscillator, the free particle, the Green function for the free particle and the hydrogen atom. Because the Lagrangian formalism must be equal to the ordinary Hamiltonian approach, we explore its application in chapter five. We will briefly discuss the Lagrangian for the ordinary Schrödinger field and write this equation in operator form. Later we will find the Lagrangian for our DSR-modified theory in the low momentum(energy) approximation.

In Chapter six we investigate the principle of local gauge transformation. It is known that local gauge invariance gives rise to the electromagnetic interaction. We do this gauge transformation by two procedures. One of these methods is by using the transfomation directly on the Schrödinger equation. This is usually done in the non-relativistic case. The other method uses the gauge transformation directly from the Lagrangian. For the case of the gauge transformation of the Lagrangian we briefly discuss the ordinary quantum mechanics case. Later we discuss the case for the DSR-modified Lagrangian.

## 1.2 Revision of Bibliography

Since its creation in 2000, there has been a lot of activity in DSR research. In this section we briefly discuss the advances in this field.

1. **Planck-scale deformation of Lorentz symmetry as a solution to the ultrahigh energy cosmic ray and the TeV-photon paradoxes.** [7] Two threshold anomalies are discussed, both arise from astronomical observations. The first is the observation of ultrahigh energy cosmic rays with energies above the Greisen-Zatsepin-Kuzmin threshold. The second is photons with high energy coming from the galaxy Mk 501. Amelino-Camelia and Piran proposed that a violation of the ordinary Lorentz invariance would solve this problem. For the

first time a deviation from the dispersion relation from special relativity was written to solve this “paradox” . Also the author discussed a five parameter formalism were they generalized the dispersion relations to accommodate these parameters in the theory. Although later it was shown that there was no such GZK violation, this initial aparent disagreement prompted the development of DSR.

2. **Measurement of the energy spectrum of cosmic rays above** $10^{18}$ **eV using the Pierre Auger Observatory.** [5] The flux of cosmic rays is measured with the Pierre Auger Observatory. The energy range from $10^{18}$ eV to $10^{20}$ eV. The observatory used a combination of two detectors, a flourescence detector and a surface detector array. A power law extrapolation was found for energies above $4 \times 10^{20}$ eV. A break in the power law was observed at energy of $3 \times 10^{18}$ eV. This is called the ankle. This break in the energy spectrum has been attributed to the transition from the galactic component of the cosmic ray flux to a flux dominated by extragalactic sources. This finding confirmed the validity of the GZK limit. The original prediction of DSR on the study of cosmic rays is not valid.

3. **Kinematical solution of the UHE-cosmic-ray puzzle without a preferred class of inertial observers.** [8]

   Amelino-Camelia proposes a kinematical solution of the ultra-high-energy(UHE) cosmic-rays “paradox”. He deformed the Lorentz symmetry so that Planck energy $E_p$ was also an invariant. To analyze the kinematical problem three functions were introduced. These functions in general depend in energy, momentum and mass of the particle and were constructed from the rapidity relation used by DSR theory. These functions satisfy the usual dispersion relation of special relativity. These auxiliary functions transform under rotations and boosts in the familiar way. The cosmic-ray problem is studied using these functions as one normally does, but at the end one substitutes what this function is in terms of the physical momentum, energy and mass.

4. **Non-commutative space-time of Doubly Special Relativity theory.** [9]

   There are infinitely many constructions of DSR in energy-momentum space and any of them can be promoted to a quantum group. With this group one can derive non-commutative space-time relations that describe the DSR theory. This is done by using the co-products of the algebra and the $\kappa$ deformed phase space via Heisenberg double. Although we have an ambiguity in the energy-momentum sector the space-time of DSR theory is unique. This non-commutative version of Minkowski space-time has ordinary Lorentz symmetry.

5. **Generalized Lorentz invariance with an invariant energy scale.** [6]

   A general method for implementing nonlinear actions of the Lorentz group is discussed for a general dispersion relation. Varying speed of light theory are also discussed within the context of the deformed dispersion relations of DSR. These theories are presented as an alternative to cosmological inflation. Composite system and conservation laws are also discussed.

6. **Transformations of coordinates and Hamiltonian formalism in deformed special relativity.** [10]

   The coordinate transformation law for writing a covariant Hamiltonian formalism for a DSR theory is discussed. These transformations were investigated using commutative and noncommutative coordinates with deformed Poisson brackets. The coordinate transformations laws were momentum dependent. These transformations were applied to the Magueijo-Smolin model of DSR.

7. **Dirac spinors for Doubly special relativity and $\kappa$-Minkowski noncommutative spacetime.** [11]

   The Dirac equation is derived in the momentum representation for the Amelino-Camelia scheme of DSR. The derivation of the Dirac equation uses the group properties of the generator of rotation and boost. There is a modification of differential operators on energy-momentum space of the generators of boosts and rotations to introduce the DSR terms. The generators still satisfy the Lorentz algebra. This technique also works for the DSR scheme of Smolin, Magueijo, Kowalski-Glikman and Lukierski. Later the Dirac equation is derived in coordinate space for a non-commutative space-time.

8. **Canonical doubly Special Relativity theory.** [12]

   The Lorentz transformations for spacetime are obtained from momentum space by canonical methods for a two observer-independent theory.

Later on a space-time metric is found. This spacetime metric depends on energy-momentum.

9. **Position space versions of the Magueijo-Smolin doubly special relativity proposal and the problem of total momentum.** [13] The construction of a coordinate space for the Magueijo-Smolin theory of DSR is proposed. This is done by two different procedures, defining conservation of momentum for ordinary special relativity in a certain way. This shall lead to the dispersion relation of Magueijo-Smolin version of DSR theory and a nonlinear transformation on momentum space. The standard Lorentz transformations for coordinate space generates these nonlinear momentum transformations. The other procedure is to use the usual definition of conservation of momentum, this will deform the Lorentz group in position space.

10. **Quantum uncertainty in doubly special relativity.** [14] A nonlinear realization of the Lorentz transformation in momentum space is parameterized by an invariant length. This parameterization involves auxiliary linear transformation variables which define the nonlinear Lorentz transformation. This parameterization is used to find four commutators in phase space. These results are for a general theory of DSR. This commutators are between time and space, time and momentum, coordinates and energy and coordinates and momentum. Later on the author found these commutators for Magueijo-Smolin and Amelino-Camelia versions of DSR.

11. **Deformed special relativity in position space.** [15]

    The deformation of special relativity was achieved in coordinate space such that the contraction of the wave-vector and coordinate-vector remains invariant. This was done by using the speed of light to be energy dependent and an energy dependent Planck constant. With this it is possible to determine the active transformations in position space.

12. **Conservation Laws in Doubly Special Relativity.** [16]

    Conservation laws are obtained for both energy and momentum for two types of DSR theory. These two theories of DSR is the one proposed by Amelino-Camelio(DSR1) and the other one by Smolin and Magueijo(DSR2). The conservation laws are found using two approaches: one investigated the nature of the nonlinear realization of the symmetry group and used its properties as a constraints on the conservation laws for composite system. The second approach directly used the transformation laws and applied physical restrictions to deduce the conservation laws. Although this is done for DSR1 and DSR2 these methods apply to any DSR theory.

13. **Berry phase effectes in the dynamics of Dirac electrons in Doubly special relativity framework.** [17] The Dirac equation is found for the Magueijo-Smolin theory of DSR. This equation is derived by algebraic methods, first using classical Poisson braket and later quantizing the equation. The author worked out the energy eigenvalues of the Dirac equation using the Foldy-Wouthuysen transformation.

In summary in all articles except article [5] there is a modification of the dispersion relation of special relativity. This modification of the dispersion relation will account in a deformation of the Lorentz symmetry, on spacetime, conservation laws, etc. To the best of our knowledge there is no work on the quantum mechanical non-relativistic limit of DSR using the generalized dispersion relation studied in this work.

# Chapter 2

# Special Relativity and Deformed Special Relativity

## 2.1 Special Relativity

In 1687 Sir Isaac Newton published Philosophiæ Naturalis Principia Mathematica [18] where he established three laws of motion and the law of universal gravitation. These laws provided satisfying results between observation and theory of celestial bodies for over two hundred years[1]. In 1905, Albert Einstein published the theory of special relativity driven by Maxwell's theory of electromagnetism and the idea of the luminiferous aether[2]. Although, Newton's laws have their range of validity, Einstein concluded that Newton's laws in general must be modified for bodies that have speeds near the speed of light. The failure of Newton's laws lay in the transformation of

[1]Newtonian physics still did not correctly predict Mercury's period, later this will be predicted correctly by Albert Einstein with his theory of General Relativity.

[2]The Luminiferous aether was the medium physicists thought light propagated.

coordinates between two inertial frames, know as the Galilean transformation.

When Galilean transformations are used on Newton's second law the principle of relativity holds. Newton stated the principle of relativity in one of his corollaries of the laws of motion "*The motions of bodies included in a given space are the same among themselves, whether that space is at rest or moves uniformly forward in a straight line.*" [18] This means that two observers that are in two different inertial frames, moving at uniform speed or at rest, will experience the same phenomena as if the inertial frame is at rest. Mathematically this means that if we perform a Galilean transformation on Newton's second law the form of the equation remains the same. The Galilean transformation is:

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{x} - \boldsymbol{v}t, \\ t' &= t, \end{aligned} \tag{2.1}$$

where $\boldsymbol{x}'$ is the coordinate vector of the event in the $S'$ frame, $\boldsymbol{x}$ in the $S$ frame and $\boldsymbol{v}$ is the relative velocity between them, provided that the origins in space and time are chosen suitably. If a group of particles are interacting via a two-body central potential,$V_{ij}$, Newton's second law can be written in a moving inertial frame as

$$m_i \frac{d\boldsymbol{v}'_i}{dt'} = -\boldsymbol{\nabla}'_i \sum_j V_{ij} \left( \left| \boldsymbol{x}'_i - \boldsymbol{x}'_j \right| \right). \tag{2.2}$$

Were $m_i$ is the más of a particle under the potential energy $V_{ij}$ and

$d\boldsymbol{v}_i/dt$ is the acceleration. We can go from one inertial frame to another using the equations the Galilean transformation (2.1). Newton's second law (2.2) becomes

$$m_i \frac{d\boldsymbol{v}_i}{dt} = -\boldsymbol{\nabla}_i \sum_j V_{ij} \left( |\boldsymbol{x}_i - \boldsymbol{x}_j| \right). \tag{2.3}$$

This means that Newton's law of force is invariant under a Galilean transformation (2.1). The principle of relativity doesn't hold true for the Maxwell equations if one uses a Galilean transformation. Maxwell equations lead to the following differential operator

$$\hat{W}' = \nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}, \tag{2.4}$$

but if one use equation (2.1) to transform to another inertial frame our differential operator changes to

$$\hat{W} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2}{c^2}\boldsymbol{v}\cdot\boldsymbol{\nabla}\frac{\partial}{\partial t} - \frac{1}{c^2}\left(\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\left(\ \boldsymbol{v}\cdot\boldsymbol{\nabla}\right). \tag{2.5}$$

So Maxwell equations are not invariant under this transformation and don't obey the principle of relativity [19].

At the time of Einstein, there existed few possibilities concerning the theory of electromagnetism. One of them was that Maxwell equations were incorrect, because there weren't invariant under a Galilean transformation. This line of thought was shown to be incorrect because of the successes of experimentation. The second alternative was that Galilean relativity applied to Newtonian mechanics, but not electromagnetism. Electromagnetism seemed to have a preferred reference frame, the frame in which the luminiferous aether was at rest. Some physicists prefered this alternative, but some experiments like the Michelson-Morley experiment and other performed by Hippolyte Fizeau questioned its existence [19]. A later alternative was the one that Einstein accepted: that there existed a principle of relativity that both electromagnetism and Newtonian mechanics will share. Newton's laws and the transformations between reference frame must be modified.

In 1905 Einstein derived the transformations between two reference frames found previously by Lorentz, Lamor, and later Poincaré. Einstein derived these equations from his two postulates of relativity:

1. *The Principle of Relativity*: The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems in uniform translatory motion relative to each other [20].

2. *The Principle of Invariant Light Speed* "... light is always propagated in empty space with a definite speed $c$ which is independent of the state of motion of the emitting body.". That is, light in vacuum propagates with the speed $c$ (a fixed constant, independent of direction) in all inertial systems, regardless of the state of motion of the light source [20].

The Lorentz Transformations for two inertial reference frames in which their axes are parallel, but the relative velocity $\boldsymbol{v}$ among the frames has arbitrary direction to the other is

$$\begin{aligned} t' &= \gamma(t - \boldsymbol{\beta} \cdot \boldsymbol{x}), \\ \boldsymbol{x}' &= \boldsymbol{x} + \frac{(\gamma - 1)}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}t, \end{aligned} \tag{2.6}$$

where we have used the following definitions:

$$\begin{aligned} \boldsymbol{\beta} &= \frac{\boldsymbol{v}}{c}, \\ \gamma &= (1 - \beta^2)^{-1/2}. \end{aligned} \tag{2.7}$$

At this point, it is helpful to introduce the notation that will be used in this work and was introduced by Einstein. In light of special relativity we are introduced to the symmetry of space and time. An event takes place on a pseudo-Euclidean space, known as the Minkowski space. Because we treat space and time on equal footing we will use the same variable for coordinates and time but different indices. We will use the following definition for the coordinates $x^0 = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$. Because of the symmetry between space and time it is convenient to introduce the notion of a. A four vector with supper index is called a contravariant vector. A contravariant vector for coordinates on the Minkowski space-time is

$$\begin{aligned} x &= x^\mu \\ &= (x^0, \boldsymbol{x}) \\ &= (x^0, x^1, x^2, x^3) \\ &= (ct, x, y, z) \end{aligned} \tag{2.8}$$

We define the covariant vector $x_\mu = (x^0, -\boldsymbol{x}) = g^{\mu\nu}x_\nu$ where $g^{\mu\nu}$ is know as the metric tensor. The metric tensor contains all the information about the geometry of the space we are working with. In special relativity, in which we are using a flat hyperbolic space, the metric tensor is represented by a diagonal matrix with diagonal elements given by $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$. Other vectorial quantities such as momentum, current density and differential operator can be written in this form.

Because of the principle of relativity all equations that we write must be invariant under a Lorentz transformation. The inner product between a covariant and a contravariant vector is called a Lorentz scalar. This inner product is written in the following way $a_\mu b^\mu = a^\mu b_\mu = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b}$, where we have use Einstein's summation convention: repeated indeces, one as a subscript and the other as a superscript imply a sum over them. An important example of a Lorentz scalar is the inner product between two momentum 4-vectors[3] :

$$p_\mu p^\mu = E^2 - p^2 c^4 = m^2 c^4. \tag{2.9}$$

Were $E$, $m$ and $p$ is the energy, mass and momentum of the particle respectively and $c$ is the speed of light in vacuum. This is the dispersion relation for special relativity.

[3]The momentum four-vector is defined as $p^\mu = (E/c, \boldsymbol{p})$.

## 2.2 Deformed Special Relativity

When DSR was born, one of the motiviation was an experimental one. Amelino-Camelia and Piran [7] first proposed a deformation of the dispesion relation of special relativity when investigating the results of ultrahigh energy cosmic rays that arrive at Earth with energy above the Greisen-Zatsepin-Kuzmin threshold. This limit is based on the interaction between the cosmic rays and the photons of the cosmic microwave background (CMB) [21]. There was an aparent contradiction with theory because this predicts that cosmic rays with energies over the threshold energy[4] of $5 \times 10^{19}\, eV$ would interact with the CMB photons to produce pions via a $\Delta$ resonance. The interaction is of the form

$$\gamma_{CMB} + p \to p + \pi. \tag{2.10}$$

This process continues until the cosmic ray energy falls below the pion production threshold. If this threshold is surpassed there will be violation of special relativity. DSR theory tries to solve this anomaly by finding new Lorentz transformations in momentum space and a new dispersion relation. Later on some new observations were done with greater precision using the Pierre Auger Observatory. Using a fluorescence detector and a surface detector. it was concluded that such aparent violation of the GZK limit were taken place.

[4]Threshold energy is the minimum photon energy required for the creation of a pair of fermion-antifermion pair. This energy must be greater than the total rest energy of the particles created.

There is also a theoretical motivation for this modification and it is found in some calculations in loop quantum gravity were new forms of the dispersion relation were found. It has been pointed out that although the relation between energy and momentum can change, it is still possible to keep the principle of the relativity of inertial frames. We simply modify the laws by which energy and momentum measured by different inertial observer are related to each other. This is done by adding nonlinear terms to the action of the Lorentz transformation on momentum space [6]. The quadratic invariant is replaced by a nonlinear invariant which in turn leads to the form for the energy-momentum function. [22]

DSR is based on four principles [6]:

1. *The relativity of inertial frames* -"When gravitational effects can be neglected, all observers in free, inertial motion are equivalent."

2. *The equivalence principle* -"Under the effect of gravity, free falling observers are all equivalent to each other and are equivalent to inertial observers."

3. *The observer independence of the Planck energy*[5] -"All observers agree that there is an invariant energy scale $E_p$."

4. *The correspondence principle* -"At energy scale much smaller that $E_p$, conventional special relativity and general relativity are true; that is,

[5]The Planck energy is a unit of energy in the" natural unit system" and yield $E_p = \sqrt{\frac{\hbar c^5}{G}} = 1.9561 \times 10^9 J$.

they hold to first order in the ratio of energy scale to $E_p$."

The third principle gives rise to a deformation of the usual dispersion relation of special relativity: $E^2 = p^2c^2 + m^2c^4$. Amelino-Camelia [22] proposed the following deformed dispersion relation based on his early work [7]:

$$E^2 = p^2c^2 + m^2c^4 + F\left(E, p, m; E_p^{-1}\right), \tag{2.11}$$

where $E_p$ is the Planck energy and $E_p^{-1}$ is the order of $10^{-9}J^{-1}$. Some authors have proposed some dispersion relations that maintain the speed of light and the Planck energy a constant. The first theory, proposed by Amelino-Camelia [22], is characterized by the following energy relation

$$2E_p^2\left[\cosh\left(\frac{E}{E_p}\right) - \cosh\left(\frac{m}{E_p}\right)\right] = p^2\, e^{E/E_p}. \tag{2.12}$$

In the literature theories that use this dispersion relation fall under the category of DSR1. More recently Smolin and Maguejio have propose a second example, known as DSR2, with the dispersion relation given by

$$\frac{m^2}{\left(1 - m/E_p\right)^2} = \frac{E^2 - p^2}{\left(1 - E/E_p\right)^2}. \tag{2.13}$$

This dispersion relations have been generalized by Magueijo and Smolin in [6]. The generalization of equation (2.13) is

$$E^2 f_1^2\left(E, m; \lambda\right) - p^2 f_2^2\left(E, m; \lambda\right) = m^2, \tag{2.14}$$

where $\lambda$ is the inverse of a observer independent energy scale of order of the Planck energy. For this equation to satisfy the principles of DSR there should be a suitable transformation for the boost generators. This new boost generators are given by following a similarity transformation [6].

$$K^i = U^{-1}(p_0)\, L_0^i U(p_0), \tag{2.15}$$

where $K^i$ is the boost generator and the operator $\hat{L}$ is the Lorentz generator for rotation. Equation (2.13) implies that the mapping of $U$ have the following properties

$$(E', \boldsymbol{p}) = U \circ (E, \boldsymbol{p}) = (E f_1(E, \lambda), \boldsymbol{p} f_2(E, \lambda)). \tag{2.16}$$

For example, Magueijo and Smolin have shown that if $U$ have the following property [6]

$$U \circ p_a = \frac{p_a}{1 - \lambda p_0},$$

this will lead to the relation dispersion proposed by them in equation (2.13). For the Planck energy $E_p$ to be an invariant quantity under the action of the Lorentz group $U$ must be singular at $E_p$. Unless $f_1 = f_2$ this theory exhibit a frequency dependent speed of light. Defining $f_3 = f_2/f_1$ we have that the speed of light is given by [6]

$$c = \frac{dE}{dp} = \frac{f_3}{1 - \frac{E f_3'}{f_3}}, \tag{2.17}$$

so this formalism may be adapted for varying speed of light theories (VSL) that are found in cosmological models.

The following example is an alternative to cosmological inflation [6]. If $f_1 = 1$ and $f_2 = 1 + \lambda E$, the resulting model is known to have an energy dependence on the speed of light and is given by

$$c\,(E) = \frac{dE}{dp} = c\,(1 + \lambda E)^2 \,. \tag{2.18}$$

In this case all momentum must be smaller than the maximum momentum $p = \lambda^{-1}$, which can only be reached by photons with infinite energy. Quantum gravity also has two independent constants, the speed of light and the Planck length[6]. In quantum gravity important processes like the quantization of space-time and formation of black holes should take place at the Planck length [24]. It was been some claim that DSR theories can be interpreted as a flat space limit of Quantum Gravity or a Quantum Special Relativity. [24, 25]. The existence of a length scale is directly linked with the breakdown of the space-time continuum and the emergence of a non-commutative space-time [9]. The specific form of the non-commutative space time that one supposedly encounters in the context of DSR is known as $\kappa$-Minkowski space-time. [9] There have been constructions of some DSR dispersion relations which appear as Casimir operators of distinct $\kappa$-Poincaré algebras[7] [23]. From a quantum group theoretic point of view, it has been argued

[6]The Planck length is a natural unit that is defined as $l_p = \sqrt{\frac{\hbar G}{c^3}} = 16.16252 \times 10^{-36}\, m$.

[7]The Poincaré algebra is the algebra that follows the Poincaré group, this is the full symmetry group of quantum field theory.

that these $\kappa$-Poincare algebra are dual to different forms of $\kappa$-Minkowski non-commutative phase spaces, all of which, indeed, have the same $\kappa$-Minkowski space time structure [23].

The commutation relations for space-time is given by [11]

$$[x_j, x_0] = i\lambda\, x_j,$$

where $\lambda = 1/\kappa$ and is the deformation parameter.

# Chapter 3

# Dispersion Relation for DSR

It has been stated by Smolin and Magueijo in [6],based on [7], that their representation for the dispersion relation of DSR can be written as an infinite series as

$$E^2 = p^2c^2 + m^2c^4 + \lambda E^3 + \cdots ,$$

where $\lambda$ is the expansion parameter. $\lambda$ is of the of order of the inverse of the Plank energy, and has a value of $10^{-10}J^{-1}$. In this work we generalized this idea to second order in $\lambda$. We want to investigate if there are any correction to the energy-eigenvalues or wave functions to non-relativistic quantum mechanics. Our energy-momentum relation is of the form

$$E^2 = \mathcal{E}^2 + \lambda E^3 + a^2\lambda^2 E^4, \tag{3.1}$$

where $\mathcal{E}^2 = p^2c^2 + m^2c^4$ and $a$ is a constant . We assume that this energy-momentum relation holds for DSR. We are interested in studying the non-relativistic limit of (3.1) and look for it's application to Quantum Mechanics. To this end we use a method of recursion called the method of successive aproximations or Picard iteration method. In this method we subtitute equation (3.1) in every part of the right side of the equation where $E^3$ or higher terms appears. After this is done we eliminate terms of $\mathcal{O}(\lambda^3)$ and higher. Applied to (3.1), this method yields:

$$E^2 = \mathcal{E}^2 + \lambda E^3 + a^2\lambda^2 E^4, \tag{3.2}$$

$$E^2 = \mathcal{E}^2 + \lambda\left(\mathcal{E}^2 + \lambda E^3 + a^2\lambda^2E^4\right)E + a^2\lambda^2\left(\mathcal{E}^2 + \lambda E^3 + a^2\lambda^2E^4\right)E^2, \tag{3.3}$$

$$E^2 = \mathcal{E}^2 + \lambda\mathcal{E}^2E + \lambda^2E^4 + a^2\lambda^2\mathcal{E}^2E^2 + \mathcal{O}\left(\lambda^3\right) + \mathcal{O}\left(\lambda^4\right), \tag{3.4}$$

$$E^2 \approx \mathcal{E}^2 + \lambda\mathcal{E}^2E + \lambda^2E^4 + a^2\lambda^2\mathcal{E}^2E^2. \tag{3.5}$$

In the second line we have separated the terms of $E^3$ and $E^4$ as $E^2E$ and $E^2E^2$ and substituted in the original equation where the term $E^2$ appears. If we use the recursion again we end up with

$$E^2 = \mathcal{E}^2 + \lambda\mathcal{E}^2E + \lambda^2E^2\mathcal{E}^2 + a^2\lambda^2\mathcal{E}^2E^2. \tag{3.6}$$

This equation can be a candidate for the energy-momentum relation for the Klein-Gordon equation but with DSR corrections. Because we want a Schrödinger like equation we must have the energy term to be linear. To this end, we factor the $E^2$ term in equation (3.2) and write it as follows

$$\left\{1-\lambda^2\mathcal{E}^2\left(1+a^2\right)\right\}E^2-\lambda\mathcal{E}^2E-\mathcal{E}^2=0. \tag{3.7}$$

Now that we have the term $E^2$ we can use the quadratic equation to have a solution for the energy. This will be used for the Hamiltonian of the Schrödinger equation. The simplified solution for the energy is

$$E=\frac{\lambda\mathcal{E}^2\pm 2\mathcal{E}\sqrt{1-b^2\lambda^2\mathcal{E}^2}}{2\left\{1-\lambda^2\mathcal{E}^2\left(1+a^2\right)\right\}}, \tag{3.8}$$

were $b^2 = a^2 + 3/4$. We note that there are two problems with equation (3.4). One is that there is a square root in the equation and that inside this square root there is the term $\mathcal{E}$, that for quantum mechanics is a differential operator. In this case we have the same problem that the Klein-Gordon equation will have if we solve for $E^2$ directly. The other problem we have is the appearance of a differential operator in the denominator. To solve these problems we approximate the equation further by using the binomial series in the numerator and the geometric series in the denominator. All terms with $\mathcal{O}(\lambda^3)$ and larger are ignored so as the negative solution. We have eliminated negative solution to cosider only particles and not antiparticles. Thus we have

$$E = \frac{1}{2}\lambda\mathcal{E}^2+\mathcal{E}\left(1+\alpha^2\lambda^2\mathcal{E}^2\right), \tag{3.9}$$

$$E = \frac{1}{2}\lambda p^2c^2+\frac{1}{2}\lambda m^2c^4+mc^2\left(1+\left(\frac{p}{mc^2}\right)^2\right)^{1/2}+\alpha^2\lambda^2m^3c^6\left(1+\left(\frac{p}{mc^2}\right)^2\right)^{3/2} \tag{3,10}$$

where $\alpha^2 = \frac{a^2}{2} + \frac{5}{8}$. The approximation yield equations (3.5) and (3.6), the first equation is in terms of $\mathcal{E}$ and the second is an explicit equation in terms of momentum and mass. Equation (3.5) and (3.6) is the simplest relativistic dispersion relation we can find that is linear in $\mathcal{E}$. One should notice that in the limit $\lambda \longrightarrow 0$ we recover the usual relation between energy and momentum of the mass-shell. Because we want to construct a Schrödinger-like equation this must be nonrelativistic. Then equation (3.6) must be aproximated but this time in the low momentum(energy) domain. As usual with an equation that involves a square root we use the binomial series and ignore all terms higher than $\mathcal{O}(p^2)$. The simplified nonrelativistic relation between energy and momentum is

$$E = \left(\lambda mc^2 + 3\alpha^2\lambda^2 m^2 c^4 + 1\right)\frac{p^2}{2m} + \frac{1}{2}\lambda m^2 c^4\left(1 + 2mc^2\lambda\alpha^2\right) + mc^2. \quad (3.11)$$

The final term in equation (3.7) is the rest energy. This term must be subtracted from the total energy $E$ to have the kinetic energy of the particle. The kinetic energy $T$ is given by

$$T = \eta_\lambda\left(\lambda, m; \alpha^2\right)\frac{p^2}{2m} + f_\lambda\left(\lambda, m; \alpha^2\right), \quad (3.12)$$

where we have two new constants that arise from DSR and that are unique for this type of dispersion relation (3.1),

$$\eta_\lambda\left(\lambda, m; \alpha^2\right) = \lambda mc^2 + 3\alpha^2\lambda^2 m^2 c^4 + 1, \quad (3.13)$$

and

$$f_\lambda\left(\lambda, m; \alpha^2\right) = \frac{1}{2}\lambda m^2 c^4 \left(1 + 2mc^2\lambda\alpha^2\right). \tag{3.14}$$

We observe that $\eta_0 \equiv \eta_\lambda\left(0, m; \alpha^2\right) = 1$ and $f_0 \equiv f_\lambda\left(0, m; \alpha^2\right) = 0$. In this limit we recover the usual non-relativistic kinetic energy term $T = p^2/2m$. The Hamiltonian for our deformed Schrödinger equation is

$$\hat{H}_\lambda = \eta_\lambda \frac{\hat{p}^2}{2m} + f_\lambda + \hat{V}\left(\boldsymbol{x}\right). \tag{3.15}$$

Performing the standard operator substitution $\hat{H} \rightarrow i\frac{\partial}{\partial t}$ we have

$$\left\{\eta_\lambda \frac{\hat{p}^2}{2m} + f_\lambda + \hat{V}\left(\boldsymbol{x}\right)\right\}\psi\left(\boldsymbol{x}, t\right) \;=\; i\hbar\frac{\partial}{\partial t}\psi\left(\boldsymbol{x}, t\right). \tag{3.16}$$

Multipliying by $\eta_\lambda$ in both sides we have

$$\left\{\frac{\hat{p}_\lambda^2}{2m} + \eta_\lambda f_\lambda + \hat{V}_\lambda\left(\boldsymbol{x}\right)\right\}\psi\left(\boldsymbol{x}, t\right) \;=\; i\hbar_\lambda\frac{\partial}{\partial t}\psi\left(\boldsymbol{x}, t\right), \tag{3.17}$$

where we have done the following definitions

$$\hbar_\lambda = \eta_\lambda \hbar, \tag{3.18}$$

$$\hat{p}_\lambda = \eta_\lambda \hat{p} = -i\hbar_\lambda \boldsymbol{\nabla}, \tag{3.19}$$

$$\hat{V}_\lambda\left(\boldsymbol{x}\right) = \eta_\lambda \hat{V}\left(\boldsymbol{x}\right). \tag{3.20}$$

Because equation (3.11) has the same structure of the ordinary Schrödinger equation; this equation also obey the time independent Schrödinger equation:

$$\begin{aligned} \left\{\frac{\hat{p}_\lambda^2}{2m} + \hat{V}_\lambda\left(\boldsymbol{x}\right)\right\}\psi\left(\boldsymbol{x}\right) = & \quad E_\lambda \psi\left(\boldsymbol{x}\right), \\ \hat{H}_\lambda^c \psi\left(\boldsymbol{x}\right) = & \quad E_\lambda \psi\left(\boldsymbol{x}\right), \end{aligned} \tag{3.21}$$

where we have redefined the energy eigenvalues as, $E_\lambda\left(\lambda, m; \alpha^2\right) = E - \eta_\lambda f_\lambda$. We note that the DSR dispersion relation in the non-relativistic limit has the effect of lowering the energy eigenvalues by a constant term.

# Chapter 4

# Examples

## 4.1 The Harmonic Oscillator in 1-D

In this chapter we investigate one of the “classic” problems of physics: the harmonic oscillator. We will investigate the energy eigenvalues and what effect does the DSR correction have on physical quantities such mass, frequency, occupation number etc. Because our time independent Schrödinger equation for DSR is similar to the time independent Schrödinger equation of ordinary quantum mechanics we will try the same technique to find the eigenvalues. Our hamiltonian of DSR with the harmonic oscillator potential is written in the following manner:

$$\hat{H}_\lambda = \frac{\hat{p}_\lambda^2}{2m} + \frac{1}{2} m \omega_\lambda^2 \hat{x}^2, \tag{4.1}$$

where we have absorbed the DSR constant $\eta_\lambda$ in the potential term $\omega_\lambda^2 = \eta_\lambda \omega^2$. This is done to perserve the symmetry in the creation and anhiliation operators[1]. This symmetry between ladder operators would be broken if we absorb the DSR constant $\eta_\lambda$ in the mass term, because the mass term that also apperars in the kinetic part of the hamiltonian and in the denominator the ladder operators will not be the same, in one of the ladder operator we will have a term $m_\lambda = \eta_\lambda m$ and in the other a term for the mass $m$. The ladder operators are then defined as:

$$\hat{a} = \sqrt{\frac{m\omega_\lambda}{2\hbar_\lambda}}\left(x + \frac{ip_\lambda}{m\omega_\lambda}\right), \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega_\lambda}{2\hbar_\lambda}}\left(x - \frac{ip_\lambda}{m\omega_\lambda}\right). \tag{4.2}$$

Because the ladder operators are the same in form, in term of variables, the conmutation relations are similar. We find that for the conmutator between the coordinate operator $\hat{x}$ and momentum operator $\hat{p}_\lambda$ for DSR differ only by the constant $\eta_\lambda$, when compared with the "undeformed" Schrödinger equation:

$$[\hat{x}, \hat{p}_\lambda] = i\hbar_\lambda = i\hbar(\lambda mc^2 + 3\alpha^2\lambda^2 m^2 c^4 + 1), \tag{4.3}$$

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{4.4}$$

Because these conmutation relations are similar to that of ordinary quantum mechanics we can further define the number operator as $\hat{N} = a^\dagger a$. We can write the hamiltonian operator in terms of the number operator

$$\hat{H}_\lambda = \left(\hat{N} + \frac{1}{2}\right)\hbar_\lambda\omega_\lambda. \tag{4.5}$$

[1]The creation and anhiliation operators are also known as ladder operators. These operator when acting on a state vector($|n\rangle$) will increase or decrease the eigenvalue of the state by one quantum unit of $\hbar\omega$.

Written in this form we find that the DSR energy eigenvalues are

$$E_\lambda^n = \left(n + \frac{1}{2}\right) \hbar_\lambda \omega_\lambda. \tag{4.6}$$

In terms of the original energy $E$ w obtain

$$\begin{aligned} E_n\left(\lambda, m; \alpha^2\right) &= \left(n + \frac{1}{2}\right) \hbar_\lambda \omega_\lambda + \eta_\lambda f_\lambda, \\ &= \eta_\lambda^{3/2}\left(n + \frac{1}{2}\right) \hbar\omega + \eta_\lambda f_\lambda, \qquad (4.7) \\ &= \left(\lambda mc^2 + 3\alpha^2\lambda^2 m^2 c^4 + 1\right)^{3/2}\left(n + \frac{1}{2}\right)\hbar\omega + \frac{1}{2}\lambda^2 m^3 c^6 + f_\lambda, \end{aligned}$$

where in the last equation we have done the multiplication of $f_\lambda \eta_\lambda$ and eliminated terms of order $\mathcal{O}\left(\lambda^3\right)$. If one approximate this equation to order $\mathcal{O}(\lambda)$ we have.

$$\begin{aligned} E_n &\cong \left[1 + \frac{3}{2}\lambda mc^2\right]\left(n + \frac{1}{2}\right)\hbar\omega + \frac{1}{2}\lambda mc^2, \qquad (4.8) \\ &= (n + \frac{1}{2})\hbar\omega + \frac{3}{2}\lambda mc^2 (n + \frac{1}{2})\hbar\omega + \frac{1}{2}\lambda m^2 c^4, \qquad (4.9) \\ &= E_n^0 + \frac{3}{2}\lambda mc^2 E_n^0 + \frac{1}{2}\lambda m^2 c^4, \qquad (4.10) \end{aligned}$$

where in the first term of equation (4.8) we have use the binomial expantion and $E_n^0 = (n + \frac{1}{2})\hbar\omega$. The last term on equation (4.10) $(\frac{1}{2}\lambda m^2 c^4)$ come out from the aproximation to order $\prime(\lambda)$ of $f_\lambda$. This term is of order of $10^{-36} J$ and the second constant term of equation (4.10) is of order of $10^{-23}$. We note that the DSR correction to the dispersion relation have the effect of an energy shift because of the two term added to $E_n^0$. Our wave function also gets corrections:

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{m\omega}{\sqrt{\eta_\lambda}\pi\hbar} \right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar\sqrt{\eta_\lambda}}} H_n \left( \sqrt{\frac{m\omega}{\sqrt{\eta_\lambda}\hbar}} x \right). \tag{4.11}$$

## 4.2 The Hydrogen Atom

To futher investigate our non-relativistic limit of the DSR theory we investigate the problem of the hydrogen atom. We can write the time independent Schrödinger equation as:

$$\left\{\frac{-\hbar_\lambda^2}{2m_e}\nabla^2 + V_\lambda\left(\boldsymbol{x}\right)\right\}\psi\left(\boldsymbol{x}\right) = E_\lambda\psi\left(\boldsymbol{x}\right).$$

After we change to spherical coordinates the Schrödinger equation is

$$\begin{aligned}\frac{-\hbar_\lambda^2}{2m_e r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi}\right]+V_\lambda\psi &= E_\lambda\psi, \quad (4.12)\\ \frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)-\frac{2m_e r^2}{\hbar_\lambda^2}\left[V_\lambda\left(r,\theta,\phi\right)-E_\lambda\right]\psi-\frac{1}{\hbar^2}\hat{L}^2\psi &= 0.\end{aligned}$$

For potentials with spherical symmetry the first two terms in the last equation correspond to the radial part of the Laplacian and the last term corresponds to its angular part. $\hat{L}^2$ is the usual angular momentum operator. We notice that the correction of DSR is included only to the radial part of the equation. After the separation of variables, $\psi(r,\theta,\phi) = R_{nl}(r)Y_m^l(\theta,\phi)$, we will end up with two equation one for the angular part that doesn't get corrections of DSR and the radial part. The radial part of the equation is

$$\frac{-\hbar_\lambda^2}{2m_e}\frac{d^2R}{dr^2}+\left[V_\lambda\left(r\right)+\frac{\hbar_\lambda^2}{2m_e}\frac{l(l+1)}{r^2}\right]R = E_\lambda R.$$

For the case of interest,that is the hydrogen atom, we include the Coulomb potential in the last equation and our final differential equation is

$$\frac{-\hbar_\lambda^2 d^2 R}{2m_e dr^2} + \left[ -\frac{e_\lambda^2}{4\pi\epsilon_0}\frac{1}{r^2} + \frac{\hbar_\lambda^2}{2m_e}\frac{l(l+1)}{r^2} \right] R = E_\lambda R, \tag{4.13}$$

where $e_\lambda^2 = \eta_\lambda e^2$. Because this is the same equation as in ordinary quantum mechanics the eigenenergy will have the same form:

$$E_\lambda^n = -\left[ \frac{m}{2\hbar_\lambda^2} \left( \frac{e_\lambda^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2}.$$

After inserting $e_\lambda^2$ and $\hbar_\lambda^2$ in the last equation we find that

$$E_\lambda^n = -\left[ \frac{m}{2\hbar^2} \left( \frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2}. \tag{4.14}$$

The energy (not cosidering the shift $\eta_\lambda f_\lambda$) is the same as the ordinary of quantum mechanics. This is different from the harmonic oscillator were the energy had a correction in a effective Planck constant and angular frequency, $E_\lambda^n = \left(n + \frac{1}{2}\right) \hbar_\lambda \omega_\lambda$. Investigating the energy-eigenvalues for the hydrogen atom we also observed an important result for quantum mechanics and this is the Bohr's radius($a_0$). Because the differential equation is the same, the constants that define's the Bohr's radius are the same. The Bohr's radius is given by

$$a_\lambda = \frac{4\pi\epsilon_0\hbar_\lambda^2}{me_\lambda^2} = \eta_\lambda a_0 = (1 + \lambda mc^2 + 3\lambda^2\alpha^2 m^2 c^4) a_0 \tag{4.15}$$

The term $\lambda mc^2$ for the electron yields a value of $4.185\,054\,537 \times 10^{-23}$.

To conclude this section we write the wave function and this is given by

$$\psi_{nlm}(r,\theta,\phi)=\sqrt{\left(\frac{2}{na_\lambda}\right)^3\frac{(n-l-1)!}{2n\left[(n+l)!\right]^3}}\; e^{-r/na_\lambda}\left(\frac{2r}{na_\lambda}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_\lambda}\right)Y_l^m(\theta,\phi)\,, \tag{4.16}$$

where the function $L_{n-l-1}^{2l+1}\left(\frac{2r}{na_\lambda}\right)$ is the assosiated Laguerre polynomial.

## 4.3 The Free Particle in 1-D

The Schrödinger equation for the free particle is:

$$-\frac{\hbar_\lambda^2}{2m}\frac{\partial^2\psi}{\partial x^2}+f_\lambda\eta_\lambda\psi=i\hbar_\lambda\frac{\partial\psi}{\partial t}. \tag{4.17}$$

To find the solution of this equation we used the Fourier transform method. We asume that the solution of the equation is

$$\psi\left(x,t\right)=\frac{1}{\sqrt{2\pi\hbar_\lambda}}\int dp\; e^{ipx/\hbar_\lambda}\,\Gamma\left(p,t\right), \tag{4.18}$$

Now we insert the last equation on the original differential equation take the appropriate derivative. When this is done we will have a new equation but now for the unknown function $\Gamma(p,t)$. The differential equation that $\Gamma(p,t)$ satisfies is

$$\frac{\partial}{\partial t}\Gamma\left(p,t\right)=\frac{-i}{\hbar_\lambda}\left(\frac{p^2}{2m}+f_\lambda\eta_\lambda\right)\Gamma\left(p,t\right). \tag{4.19}$$

This equation has solution

$$\Gamma\left(p,t\right)=\Phi(p)\,e^{-i\left(\frac{p^2}{2m}+f_\lambda\eta_\lambda\right)t/\hbar_\lambda}, \tag{4.20}$$

where $\Phi$ is an unknown function that arises from the integration of the differential equation and is found from the time independent Schrödinger equation. By substituting our result for $\Gamma$ on equation (4.15) the solution for the free particle is found to be:

$$\Psi\left(x,t\right)=\frac{e^{-if_\lambda t/\hbar}}{\sqrt{2\pi\hbar_\lambda}}\int dp\;\Phi\left(p\right)e^{i\left[px-\frac{p^2t}{2m}\right]/\hbar_\lambda}. \tag{4.21}$$

We observed that the phase factor in front of equation (4.18) doesn't depend of the factor $\eta_\lambda$. This occurred because of the factor $\hbar_\lambda=\eta_\lambda\hbar$ cancel the constant $\eta_\lambda$.

## 4.4 The Green Function for the 1-D Free Particle

In the usual formulation of quantum mechanics one can adopt the Heinsenberg or the Schrödinger picture. In both views of quantum mechanics, coordinates and momentum are replaced with operators that follows a commutation relation. Another formulation of quantum mechanics uses the idea of Green functions or propagators. By invoking the Huygen's principle the solution of the time dependent wave functions can also be written in the following manner

$$\psi(x,t) = \int dx'\, K(x',x;t)\, \psi(x',0). \tag{4.22}$$

This equation explicitly express the principle of causality. The function $K(x',x;t)$ is known as the Green function or the propagator. The propagator represents the probability amplitude for transition between two points of space-time. There are many ways to calculate the Green function but taking advantage of the last section results we will find the Green function by writing the appropriate differential equation that the Green function satisfies. We will write the differential equation following the same technique used in the previous section:

$$\left[-\frac{\hbar_\lambda^2}{2m}\frac{\partial^2}{\partial x'^2} + f_\lambda \eta_\lambda - i\hbar_\lambda \frac{\partial}{\partial t}\right] K(x',x;t) = -i\hbar_\lambda \delta(x'-x)\delta(t). \tag{4.23}$$

Using the Fourier transform method we assume that the solution has the following form

$$K(x', x; t) = \frac{1}{2\pi\hbar_\lambda} \int dp\, e^{ip(x'-x)/\hbar_\lambda}\, G(p,t), \tag{4.24}$$

and because the Fourier transform of the delta function $\delta(x'-x)$ is one, we will end up we the following equation for $G(p,t)$:

$$\frac{\partial G}{\partial t} + \frac{i}{\hbar_\lambda}\left[\frac{p^2}{2m} + f_\lambda \eta_\lambda\right] G = \delta(t). \tag{4.25}$$

We can solve this equation for $t \neq 0$ because the delta function is cero. The solution of this equation is

$$G(p,t) = N(p)\, e^{-\frac{i}{\hbar_\lambda}\left[\frac{p^2}{2m} + f_\lambda \eta_\lambda\right] t}. \tag{4.26}$$

This is the Green function for momentum space. If we want the Green function in coordinate space we have to integrate equation(4.20). To find the integration function $N(p)$ we evaluate $K(x', x; 0)$. It turns out that $N(p) = 1$. The integration of equation(4.20) yields

$$K(x', x; t) = \sqrt{\frac{m}{2\pi i \hbar_\lambda t}}\, e^{\frac{im(x'-x)^2}{2t\hbar_\lambda} - \frac{if_\lambda t}{\hbar}}. \tag{4.27}$$

The DSR effects add the change of $\hbar$ to $\hbar_\lambda$ and the second term of the exponential $e^{-if_\lambda t/\hbar}$. This term also appear in the wave function of the free particle, but this term are not observable. In both cases, for the wave function and Green function, are probability amplitude we have to take modulus squared this term will vanish.

# Chapter 5

# Lagrangian approach to the DSR Schrödinger equation

The Lagrangian formalism can give us another way of writing the equation of motion for DSR, and can futher give us a better insight on the dynamics of DSR. The advantage that the Lagrangian formalism has, is that some interaction can be added by the symmetry principle. We will explore this in the next chapter. First we will give a brief review of the Lagrangian formalism of the Schrödinger equation. Later we will derive the Lagrangian function for DSR.

## 5.1 Lagrangian of Quantum Mechanics

In quantum mechanics there exists a Lagrangian density function[1] that can be used to write the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi\left(\boldsymbol{x},t\right)+V\left(\boldsymbol{x}\right)\psi\left(\boldsymbol{x},t\right)=i\hbar\frac{\partial}{\partial t}\psi\left(\boldsymbol{x},t\right). \tag{5.1}$$

This Lagrangian function is [26]

$$\mathcal{L}=-\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla}\psi\right)\cdot\left(\boldsymbol{\nabla}\psi^*\right)-V\psi^*\psi+\frac{i\hbar}{2}\left(\psi^*\partial_t\psi-\psi\partial_t\psi^*\right). \tag{5.2}$$

We can write the Schrödinger equation using the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\psi^*}-\partial_t\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_t\psi^*\right)}\right)-\sum_{i=1}^{3}\partial_i\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_i\psi^*\right)}\right)=0. \tag{5.3}$$

The Lagrangian for the Schrödinger equation (5.1) can be written in terms of the operators $\hat{\boldsymbol{p}}$ y $\hat{H}$

$$\mathcal{L}=\frac{\hat{\boldsymbol{p}}\psi\cdot\hat{\boldsymbol{p}}\psi^*}{2m}-\hat{V}\psi^*\psi+\frac{1}{2}\left(\psi^*\hat{H}\psi-\psi\hat{H}\psi^*\right). \tag{5.4}$$

In this equation we observe that the Lagrangian consists of a kinetic, a potential, and temporal part of the Schrödinger equation. Because we wrote the Lagrangian in term of operators it is convenient to write the Euler-

[1]The Lagrangian density is defined by the following equation $L=\int d^3x\mathcal{L}$ were $L$ is the Lagrangian. The Lagrangian density can be expressed in term of tha Hamiltonian density as $\mathcal{L}=\sum\pi\dot{\eta}-\mathcal{H}$. This Hamiltonian density is not the total energy of the system.

Lagrange equation in terms of these operators as well

$$\frac{\partial \mathcal{L}}{\partial \psi^*} - \hat{H}\frac{\partial \mathcal{L}}{\partial (\hat{H}\psi^*)} - \sum_{i=1}^{3} \hat{p}_i \frac{\partial \mathcal{L}}{\partial \left(\hat{p}_i \psi^*\right)} = 0. \tag{5.5}$$

Remembering that the operator for the Hamiltonian corresponds to the temporal part ($\hat{H} \rightarrow i\hbar\partial_t$). It can be verified that using the Lagrangian written in that form that the Euler-Lagrange equation gives us

$$\hat{H}\psi = \frac{\hat{p}^2\psi}{2m} + \hat{V}\psi. \tag{5.6}$$

If we take the correspondence of the operators $\hat{H} \rightarrow i\hbar\partial_t$ and $\hat{p} \rightarrow -i\hbar\partial_i$ we obtain the time dependent Schrödinger equation. Therefore, the Euler-Lagrange equation(5.4) give us the most general expression for the Hamiltonian function. We will use this equation to find the Hamiltonian for DSR

$$\mathcal{H}_\lambda = \alpha^2\lambda^2\mathcal{E}^3 + \frac{1}{2}\lambda\mathcal{E}^2 + \mathcal{E}.$$

## 5.2 Lagrangian for DSR

To derive the non-relativistic DSR Schrödinger equation we will first give an ansatz of how the relativistic equation will look. Then we will use the Euler-Lagrange equation, and will then approximate this equation in the low momentum(energy) domain. Our ansatz for the Lagrangian function is

$$\mathcal{L} = \frac{1}{2}\left(\psi^{*}\hat{\mathcal{H}}_{\lambda}\psi - \psi\hat{\mathcal{H}}_{\lambda}\psi^{*}\right) + \hat{p}\psi^{*}\cdot\hat{\Omega}\psi + \hat{p}\psi\cdot\hat{\Omega}\psi^{*} + \psi^{*}\hat{V}\left(\vec{x}\right)\psi \tag{5.7}$$

The first and last term of equation (5.6) are familiar to us and appeared in our discussion of ordinary quantum theory. They represent the temporal part of the Hamiltonian and its classical interactions. The two middle terms arise from DSR. One expects that the operator $\hat{\boldsymbol{\Omega}}$ will contain the kinetic part of the Lagrangian similar to one discuss in the previous section and the DSR interaction. The operator $\hat{\boldsymbol{\Omega}}$ acts over the wave function and gives these new interactions. Our goal is to find what this new operator is. After taking the corresponding derivative in the Euler-Lagrange equation we end up with

$$\hat{\mathcal{H}}_{\lambda}\psi = \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\Omega}}\psi + \hat{V}\psi. \tag{5.8}$$

We identify the inner product of the momentum and the vector $\hat{\boldsymbol{\Omega}}\psi$ with

$$\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\Omega}}\psi = \alpha^{2}\lambda^{2}\mathcal{E}^{3}\psi + \frac{1}{2}\lambda\mathcal{E}^{2}\psi + \mathcal{E}\psi. \tag{5.9}$$

From this equation we will find the operator $\hat{\boldsymbol{\Omega}}$ under the low momentum(energy) aproximation[2]. To this end we will go to the coordinate representation of the momentum operator

$$-i\hbar\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\Omega}}\psi = \alpha^{2}\lambda^{2}\mathcal{E}^{3}\psi + \frac{1}{2}\lambda\mathcal{E}^{2}\psi + \mathcal{E}\psi.$$

[2] $\left(p^{2}+m^{2}\right)^{n/2} \approx m^{n}\left(1+\frac{np^{2}}{2m^{2}}\right)$

After our aproximation we obtain

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\Omega}}\psi = \frac{i}{\hbar}\left(f_\lambda + m + \eta_\lambda \frac{p^2}{2m}\right)\psi. \tag{5.10}$$

We want to find what is the operator $\hat{\boldsymbol{\Omega}}$ acting over the wave function. Because what we have in equation (5.9) is the divergence of vector $(\hat{\boldsymbol{\Omega}}\psi)$, we will use the Divergence Theorem. For this, we will integrate over the volumen in both sides of equation (5.9):

$$\begin{aligned}\int \boldsymbol{\nabla}\cdot\hat{\boldsymbol{\Omega}}\psi\, d^3x &= \frac{i}{\hbar}\int \left(f_\lambda + m + \eta_\lambda \frac{p^2}{2m}\right)\psi\, d^3x,\\ &= \frac{i}{\hbar}\int \left(f_\lambda + m - \frac{\eta_\lambda \hbar^2}{2m}\nabla^2\right)\psi\, d^3x.\end{aligned}$$

Using the Divergence Theorem on the left side yields

$$\oint \hat{\boldsymbol{\Omega}}\psi\cdot d\boldsymbol{a} = \frac{i}{\hbar}\int (f_\lambda + m)\,\psi\, d^3x - \frac{i\hbar\eta_\lambda}{2m}\int \nabla^2\psi\, d^3x. \tag{5.11}$$

On equation (5.10) the second term on the right can be simplified using Green's first identity. This term will represent the kinetic term of the Lagrangian. The first term of equation (5.10) will give the DSR interaction. Equation (5.10) as it is, can't be written as a surface integral, because of the DSR constribution. To write this part in terms of a surface integral we have to assume that the wave function can be written in the following manner

$\psi = \nabla^2 \xi$.

$$\begin{aligned}\oint \hat{\mathbf{\Omega}}\psi \cdot d\boldsymbol{a} &= \frac{i}{\hbar}\int (f_\lambda + m)\, \nabla^2 \xi \, d^3x - \frac{i\hbar\eta_\lambda}{2m}\oint \boldsymbol{\nabla}\psi \cdot d\boldsymbol{a} \\ &= \frac{i}{\hbar}(f_\lambda + m)\oint \boldsymbol{\nabla}\xi \cdot d\boldsymbol{a} - \frac{i\hbar\eta_\lambda}{2m}\oint \boldsymbol{\nabla}\psi \cdot d\boldsymbol{a}.\end{aligned}$$

Where we have used Green's first identity again for the first term. Now we will write the last equation under one surface integral

$$\oint [\hat{\mathbf{\Omega}}\psi - \frac{i}{\hbar}(f_\lambda + m)\, \boldsymbol{\nabla}\xi + \frac{i\hbar\eta_\lambda}{2m}\boldsymbol{\nabla}\psi] \cdot d\boldsymbol{a} = 0. \tag{5.12}$$

Because the volumen is arbitrary we have

$$\begin{aligned}\hat{\mathbf{\Omega}}\psi &= \frac{i}{\hbar}(f_\lambda + m)\, \boldsymbol{\nabla}\xi - \frac{i\hbar\eta_\lambda}{2m}\boldsymbol{\nabla}\psi \\ \hat{\mathbf{\Omega}}\psi &= \frac{\eta_\lambda}{2m}\hat{\boldsymbol{p}}\psi - \hbar^{-2}(f_\lambda + m)\, \hat{\boldsymbol{p}}\xi\end{aligned} \tag{5.13}$$

One may observe that by taking the divergence of $\hat{\mathbf{\Omega}}$ one recovers what we had before,i.e

$$\begin{aligned}\boldsymbol{\nabla} \cdot \hat{\mathbf{\Omega}}\psi &= \frac{i}{\hbar}(f_\lambda + m)\, \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\xi - \frac{i\hbar\eta_\lambda}{2m}\nabla^2\psi \\ &= \frac{i}{\hbar}\left\{ f_\lambda + m + \eta_\lambda \frac{p^2}{2m} \right\}\psi\end{aligned} \tag{5.14}$$

Our Lagrangian in operator form becomes

$$\mathcal{L} = \frac{1}{2}\Big(\psi^* \hat{\mathcal{H}}_\lambda \psi - \psi \hat{\mathcal{H}}_\lambda \psi^*\Big) - \frac{f_\lambda}{\hbar^2}\,(\hat{\boldsymbol{p}}\psi^* \cdot \hat{\boldsymbol{p}}\xi + \hat{\boldsymbol{p}}\psi \cdot \hat{\boldsymbol{p}}\xi^*) + \frac{\eta_\lambda}{2m}\,\hat{\boldsymbol{p}}\psi^* \cdot \hat{\boldsymbol{p}}\psi + \psi^* \hat{V}\psi, \tag{5.15}$$

where we have eliminated the rest energy term as is done in perturbation theory. Using the Euler-Lagrange equation of motion and multiplying by $\eta_\lambda$ both side we will get the "deformed" Schrödinger equation

$$\left\{\frac{\hat{p}_\lambda^2}{2m} + \eta_\lambda f_\lambda + \hat{V}_\lambda(\boldsymbol{x})\right\}\psi(\boldsymbol{x},t) \;=\; i\hbar_\lambda \frac{\partial}{\partial t}\psi(\boldsymbol{x},t)\,.$$

We observe that the unknown function $\xi$, used to derive the results does not appear in the deformed Schrödinger equation. It is introduced as a mathematical tool to be able to perform the calculation. The nature of this function can be explored as a extension of this work.

# Chapter 6

# Gauge Transformation

In this chapter we investigate local gauge transformation. This gauge transformation will be performed first to the Schrödinger equation directly, as is done in many books[27,28], and later to the Lagrangian of the theory. After studying the gauge transformation of the Schrödinger equation we will review its application to the ordinary Schrödinger Lagrangian. Then we will apply this technique to the DSR Lagrangian. The study of local gauge transformations is important because when we demand that a Lagrangian has this type of symmetry we introduce the electromagnetic interaction. Then using Noether's theorem we can find the conserved current. Because the Lagrangian of a non-relativistic particle is very similar to the DSR Lagrangian, we will use the non-relativistic function to make an educated guess of how our DSR Lagrangian should look like. Later, using the Euler-Lagrange equation, we will find the equation of motion. We will compare these results by applying the gauge transformation to the DSR Schrödinger equation.

## 6.1 Gauge Transformation of the Deformed Schrödinger Equation

In this section we perform the gauge transformation of the wave function direcly to the equation of motion as is done in [27,28]. We start with the DSR Schrödinger equation of the free particle:

$$\left\{-\frac{\hbar_\lambda^2}{2m}\nabla^2+\eta_\lambda f_\lambda\right\}\psi=i\hbar_\lambda\frac{\partial\psi}{\partial t}.$$

If we demand that the equation satisfies a gauge symmetry of the form $\psi'\rightarrow e^{-iq\Lambda}\psi$ we find, as in the previous section, that there is an extra term that comes from the derivative. It arises from the Laplacian operator. The Laplacian operator acting on the gauge transformation is:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}e^{-iq\Lambda}\psi) \;=\; e^{-iq\Lambda}\left\{\boldsymbol{\nabla}-iq\boldsymbol{\nabla}\Lambda\right\}^2\psi$$

Our Schrödinger equation transforms to:

$$\left[-\frac{\hbar_\lambda^2}{2m}\left\{\boldsymbol{\nabla}-iq\boldsymbol{\nabla}\Lambda\right\}^2+\eta_\lambda f_\lambda\right]\psi=i\hbar_\lambda\left\{\frac{\partial\psi}{\partial t}-iq\frac{\partial\Lambda}{\partial t}\psi\right\}.$$

As found earlier, the equation for the free particle is not gauge invariant. Therefore we abandon the idea of a gauge symmetry for it. Instead we write the following Schrödinger equation:

$$\left[\frac{1}{2m}\left\{-i\hbar_\lambda \boldsymbol{\nabla} - q_\lambda \boldsymbol{A}\right\}^2 + \eta_\lambda f_\lambda + q_\lambda A_0\right]\psi = i\hbar_\lambda \frac{\partial \psi}{\partial t}. \tag{6.1}$$

We note that we have redefined the charge by $q_\lambda = q\eta_\lambda = q(1 + \lambda mc^2 + 2\alpha^2\lambda^2 m^2 c^4)$. This equation is the Schrödinger equation with an electromagnetic interaction and is gauge invariant if the new function $\boldsymbol{A}$ and $A_0$ transformed in the following way

$$\begin{aligned} \boldsymbol{A}' &\rightarrow \boldsymbol{A} + q^{-1}\boldsymbol{\nabla}\Lambda, \\ A_0' &\rightarrow A_0 - q^{-1}\frac{\partial \Lambda}{\partial t}. \end{aligned}$$

Viewed this way, the electromagnetic interaction arises if we demand the equation to be gauge invariant, as it is done in ordinary quantum mechanics. One of the results of this derivation is the fact that the charge of the particle becomes energy dependent due to the factor $\eta_\lambda$. This is what happens with $\hbar$, as shown earlier.

## 6.2 Gauge Transformation on the Schrödinger Lagrangian

In this review of the local gauge transformation of the Schrödinger Lagrangian we follow the article by Colussi and Wickramasekara[29]. First we write the Lagrangian of the free particle.

$$
\begin{aligned}
\mathcal{L} &= -\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla}\psi\right)\cdot\left(\boldsymbol{\nabla}\psi^*\right) + \frac{i\hbar}{2}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right), \\
&= \frac{\hat{\boldsymbol{p}}\psi\cdot\hat{\boldsymbol{p}}\psi^*}{2m} + \frac{1}{2}\left(\psi^*\hat{H}\psi - \psi\hat{H}\psi^*\right).
\end{aligned}
\tag{6.2}
$$

Now we perfom the following local transformation on the wave function.

$$
\psi'\left(\boldsymbol{x},t\right) = e^{-i\frac{e}{\hbar}\Lambda(\boldsymbol{x},t)}\psi\left(\boldsymbol{x},t\right), \qquad \psi'^*\left(\boldsymbol{x},t\right) = e^{i\frac{e}{\hbar}\Lambda(\boldsymbol{x},t)}\psi^*\left(\boldsymbol{x},t\right). \tag{6.3}
$$

These new wave functions will be entered in the Lagrangian. We notice that because our unknown function $\Lambda$ depends on the coordinate and time parameters we will have some extra terms in the Lagragian. The derivatives with respect to time and spatial coordiantes transform like:

$$
\begin{aligned}
\partial_0\psi' &= e^{-i\frac{e}{\hbar}\Lambda}\left[\partial_0\psi - i\frac{e}{\hbar}\psi\,\partial_0\Lambda\right], \\
\hat{H}\psi' &= e^{-i\frac{e}{\hbar}\Lambda}\left[\hat{H}\psi - i\frac{e}{\hbar}\psi\,\hat{H}\Lambda\right], \\
\boldsymbol{\nabla}\psi' &= e^{-i\frac{e}{\hbar}\Lambda}\left[\boldsymbol{\nabla}\psi - i\frac{e}{\hbar}\psi\boldsymbol{\nabla}\Lambda\right], \\
\hat{p}\psi' &= e^{-i\frac{e}{\hbar}\Lambda}\left[\hat{\boldsymbol{p}}\psi - i\frac{e}{\hbar}\psi\hat{\boldsymbol{p}}\Lambda\right].
\end{aligned}
\tag{6.4}
$$

Our Lagrangian function transforms in the following manner:

$$
\begin{aligned}
\mathcal{L}' &= -\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla} - i\frac{e}{\hbar}\boldsymbol{\nabla}\Lambda\right)\psi\cdot\left(\boldsymbol{\nabla} + i\frac{e}{\hbar}\boldsymbol{\nabla}\Lambda\right)\psi^*, \\
&\qquad +\frac{i\hbar}{2}\left[\psi^*\left(\partial_t - i\frac{e}{\hbar}\partial_t\Lambda\right)\psi - \psi\left(\partial_t + i\frac{e}{\hbar}\partial_t\Lambda\right)\psi^*\right] \\
&= \frac{1}{2m}\left(\hat{\boldsymbol{p}} - i\frac{e}{\hbar}\hat{\boldsymbol{p}}\Lambda\right)\psi\cdot\left(\hat{\boldsymbol{p}} + i\frac{e}{\hbar}\hat{\boldsymbol{p}}\Lambda\right)\psi^* \\
&\qquad +\frac{1}{2}\left[\psi^*\left(\hat{H} - i\frac{e}{\hbar}\hat{H}\Lambda\right)\psi - \psi\left(\hat{H} + i\frac{e}{\hbar}\hat{H}\Lambda\right)\psi^*\right].
\end{aligned}
$$

This Lagrangian is not invariant under the gauge transformation. Following what was done in section 6.1 we choose to write the Lagrangian as:

$$
\begin{aligned}
\mathcal{L} &= \frac{i\hbar}{2}\psi^*\left(\partial_t + i\frac{e}{\hbar}A_0\right)\psi - \frac{i\hbar}{2}\psi\left(\partial_t - i\frac{e}{\hbar}A_0\right)\psi^*, \\
&\qquad -\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla} + i\frac{e}{\hbar}\boldsymbol{A}\right)\psi\cdot\left(\boldsymbol{\nabla} - i\frac{e}{\hbar}\boldsymbol{A}\right)\psi^* \\
&= \frac{1}{2}\psi^*\left(\hat{H} - eA_0\right)\psi - \frac{1}{2}\psi\left(\hat{H} + eA_0\right)\psi^* + \frac{1}{2m}\left(\hat{\boldsymbol{p}} + e\boldsymbol{A}\right)\psi\cdot\left(\hat{\boldsymbol{p}} - e\boldsymbol{A}\right)\psi^*.
\end{aligned}
$$

The only thing that we demand is that the functions $\boldsymbol{A}$ y $A_0$ transfom in the following way:

$$
\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\Lambda, \qquad A_0' = A_0 + \partial_t\Lambda. \tag{6.5}
$$

We identify these functions to be the vector and scalar potentials respectively. After using the Euler-Lagrange equation the Schrödinger equation with electromagnetic interaction is

$$
\hat{H}\psi = \frac{1}{2m}\left(\hat{\boldsymbol{p}} + e\boldsymbol{A}\right)^2\psi + eA_0\psi. \tag{6.6}
$$

## 6.3 Gauge Transfomation on the DSR Lagrangian

The Lagrangian for a free particle in the DSR theory is:

$$\begin{aligned}
\mathcal{L} &= \frac{i\hbar}{2}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right) - \frac{\hbar^2\eta_\lambda}{2m}\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\psi^* + f_\lambda\left(\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\xi + \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\xi^*\right), \\
&= \frac{1}{2}\left(\psi^*\hat{\mathcal{H}}_\lambda\psi - \psi\hat{\mathcal{H}}_\lambda\psi^*\right) + \frac{\eta_\lambda}{2m}\hat{\boldsymbol{p}}\psi^*\cdot\hat{\boldsymbol{p}}\psi - \hbar^{-2}f_\lambda\ \left(\hat{\boldsymbol{p}}\psi^*\cdot\hat{\boldsymbol{p}}\xi + \hat{\boldsymbol{p}}\psi\cdot\hat{\boldsymbol{p}}\xi^*\right), \\
\mathcal{L} &= \mathcal{L}_{NR} + \mathcal{L}_{NT}. \qquad (6.7)
\end{aligned}$$

Where the first term($\mathcal{L}_{NR}$) in the last equation is the non-relativistic case that was written in the last section. The second term($\mathcal{L}_{NT}$) includes the new terms from the DSR interactions. The term $\mathcal{L}_{NR}$ consists of the temporal and kinetic parts of the Schrödinger equation and will transform like the previous section. The new part we have to investigate is the term $\mathcal{L}_{NT}$ which includes of the function $\xi$. Now we assume that the unknown function $\xi$ transforms in the following manner

$$\xi' = e^{-i\frac{e}{\hbar}\Xi(\boldsymbol{x},t)}\xi. \qquad (6.8)$$

The new transformation of the function $\xi$ is similar to the one performd to the wave function. Performing the gauge transformation of $\mathcal{L}_{NT}$ yields:

$$
\begin{aligned}
\mathcal{L}'_{NT} &= f_\lambda \left\{ \left(\boldsymbol{\nabla} + ie\hbar^{-1}\boldsymbol{\nabla}\Lambda\right)\psi^* \cdot \left(\boldsymbol{\nabla} - ie\hbar^{-1}\boldsymbol{\nabla}\Xi\right)\xi \right. \\
&\qquad\qquad \left. + \left(\boldsymbol{\nabla} - ie\hbar^{-1}\boldsymbol{\nabla}\Lambda\right)\psi \cdot \left(\boldsymbol{\nabla} + ie\hbar^{-1}\boldsymbol{\nabla}\Xi\right)\xi^* \right\}, \\
&= -\hbar^{-2} f_\lambda \left\{ \left(\hat{\boldsymbol{p}} + ie\hbar^{-1}\hat{\boldsymbol{p}}\Lambda\right)\psi^* \cdot \left(\hat{\boldsymbol{p}} - ie\hbar^{-1}\,\hat{\boldsymbol{p}}\Xi\right)\xi \right. \\
&\qquad\qquad \left. + \left(\hat{\boldsymbol{p}} - ie\hbar^{-1}\hat{\boldsymbol{p}}\Lambda\right)\psi \cdot \left(\hat{\boldsymbol{p}} + ie\hbar^{-1}\,\hat{\boldsymbol{p}}\Xi\right)\xi^* \right\}.
\end{aligned}
$$

We observe that, like in the earlier case, the free particle Lagrangian is not gauge invariant and therefore we modify the Lagrangian to include a new function

$$
\mathcal{L}_{NT} = -\hbar^{-2} f_\lambda \left[ \left(\hat{\boldsymbol{p}} - e\boldsymbol{A}\right)\psi^* \cdot \left(\hat{\boldsymbol{p}} + e\boldsymbol{B}\right)\xi + \left(\hat{\boldsymbol{p}} + e\boldsymbol{A}\right)\psi \cdot \left(\hat{\boldsymbol{p}} - e\boldsymbol{B}\right)\xi^* \right]. \quad (6.9)
$$

Where the function $\boldsymbol{B}$ transforms as $\boldsymbol{B}' = \boldsymbol{B} + \boldsymbol{\nabla}\Xi$. If we want to obtain the equation of motion we use the Euler-Lagrange equation and take the corresponding derivative. If we divide the Lagrangian in two parts[1] $\mathcal{L} = \mathcal{L}_{NR} + \mathcal{L}_{NT}$ where the term $\mathcal{L}_{NR}$ gives us the Schrödinger equation which includes the electromagnetic field interaction that we already know. Our Euler-Lagrange equation is

$$
\left(\hat{H} - eA_0\right)\psi - \frac{\eta_\lambda}{2m}\left(\boldsymbol{p} + e\boldsymbol{A}\right)^2\psi + \frac{\partial \mathcal{L}_{NT}}{\partial \psi^*} - \sum_{i=1}^{3} p_i \frac{\partial \mathcal{L}_{NT}}{\partial\left(p_i \psi^*\right)} = 0.
$$

We observe that the function $\mathcal{L}_{NT}$ does not depend in the temporal derivative so we have eliminated this term from the Euler-Lagrange equation. First we

[1] In this Lagrangian we have already done the transformed of momentum to the minimal coupling.

will take the corresponding derivatives individually and then we will evaluate the scalar product that appears in equation (6.9).

$$
\begin{aligned}
(\hat{\boldsymbol{p}} - e\boldsymbol{A})\psi^{*}\cdot(\hat{\boldsymbol{p}} + e\boldsymbol{B})\xi &= \hat{\boldsymbol{p}}\psi^{*}\cdot\hat{\boldsymbol{p}}\xi + e\xi\,\hat{\boldsymbol{p}}\psi^{*}\cdot\boldsymbol{B} - e\psi^{*}\hat{\boldsymbol{p}}\xi\cdot\boldsymbol{A} - e^{2}\psi^{*}\xi\boldsymbol{A}\cdot\boldsymbol{B} \\
\frac{\partial\mathcal{L}_{NT}}{\partial\psi^{*}} &= e\hbar^{-2}f_{\lambda}\left\{\hat{\boldsymbol{p}}\xi\cdot\boldsymbol{A} + e\xi\boldsymbol{A}\cdot\boldsymbol{B}\right\} \qquad (6.10) \\
\frac{\partial\mathcal{L}_{NT}}{\partial(p_i\psi^{*})} &= -\hbar^{-2}f_{\lambda}(p_i + e\xi B_i) \\
p_i\frac{\partial\mathcal{L}_{NT}}{\partial(p_i\psi^{*})} &= -\hbar^{-2}f_{\lambda}\left\{p_ip_i\xi + ep_i(\xi B_i)\right\} \\
&= -\hbar^{-2}f_{\lambda}\left\{p_ip_i\xi + e\xi p_iB_i + eB_ip_i\xi\right\} \\
\sum_{i=1}^{3}p_i\frac{\partial\mathcal{L}_{NT}}{\partial\left(p_i\psi^{*}\right)} &= -\hbar^{-2}f_{\lambda}\left\{p^{2}\xi + e\xi\hat{\boldsymbol{p}}\cdot\boldsymbol{B} + e\boldsymbol{B}\cdot\hat{\boldsymbol{p}}\xi\right\} \qquad (6.11)
\end{aligned}
$$

Using the results found above the equation of motion is

$$
\begin{aligned}
\left(\hat{H} - eA_0\right)\psi - \frac{\eta_\lambda}{2m}\left(\hat{\boldsymbol{p}} + e\boldsymbol{A}\right)^{2}\psi + e\hbar^{-2}f_{\lambda}(\hat{\boldsymbol{p}}\xi\cdot\boldsymbol{A} &+ e\xi\boldsymbol{A}\cdot\boldsymbol{B}) \\
&+ \hbar^{-2}f_{\lambda}(p^{2}\xi + e\xi\hat{\boldsymbol{p}}\cdot\boldsymbol{B} + e\boldsymbol{B}\cdot\hat{\boldsymbol{p}}\xi) = 0,
\end{aligned}
$$

Multiplying the whole equation by $\eta_\lambda$ we get,

$$
\begin{aligned}
\left(\eta_\lambda\hat{H} - e_\lambda A_0\right)\psi = \frac{1}{2m}\left(\hat{\boldsymbol{p}}_\lambda + e_\lambda\boldsymbol{A}\right)^{2}\psi + \eta_\lambda f_\lambda\psi - e_\lambda\hbar^{-2}f_{\lambda}(\hat{\boldsymbol{p}}\xi\cdot\boldsymbol{A} &+ e\xi\boldsymbol{A}\cdot\boldsymbol{B}) \qquad (6.12) \\
&- \hbar^{-2}e_\lambda f_{\lambda}(\xi\hat{\boldsymbol{p}}\cdot\boldsymbol{B} + \boldsymbol{B}\cdot\hat{\boldsymbol{p}}\xi).
\end{aligned}
$$

Although the function $\xi$ was use in the previous chapter as intermediate in the calculation of the DSR Lagrangian, it did not appear in the final equation. In this case, when the local gauge transformation is permormed one ends up with the function $\xi$ and a new vector function $\boldsymbol{B}$ that will be associated with a potential term like $\boldsymbol{A}$.

# Chapter 7

# Conclusion

In this work we have investigated the structure of a DSR theory in the context of non-relativistic mechanics. Our main goal was to write a Schrödinger -like equation using a relativistic deformed dispersion relation. To write this equation two sets of aproximations were made. The first one was to approximate the full deformed special relativity energy-momentum relation equation (3.1). The second aproximation was to pass to the non-relativistic limit, that is at low momentum (energy). After these aproximations were done we found what was called the DSR Schrödinger equation. For the time-independent DSR Schrödinger equation we noticed an energy shift of the form $E_\lambda = E - \eta_\lambda f_\lambda$. Where $E$ is the energy as defined in the "standard" Schrödinger equation. For the temporal part of the Schrödinger equation we have a change of scale in the Planck constant from $\hbar$ to $\hbar_\lambda = \eta_\lambda \hbar$, were $\hbar$ becomes energy dependent as found in [15].

We solved some "classical" problems like the harmonic oscillator, the free particle, finding the Green's function for the free particle and the hydrogen atom. For the case of the harmonic oscillator we redefined the angular frenquency as $\omega_\lambda^2 = \eta_\lambda \omega^2$. This was done to write the ladder operators in a symmetrical form and to use them to solve the problem. In this case we saw an energy shift as was predicted. If we take a look at the corrected energy eigenvalues for the harmonic oscillator we find $E_\lambda^n = \left(1 + \frac{3}{2}\lambda mc^2\right)(n + 1/2)\hbar\omega$. These corrected energy eigenvalues have the effect of a change of scale because of the term $\eta_\lambda^{3/2}$, this rescale of energy is of order of $10^{-23}$ to first order in $\lambda$. The leading order DSR correction to the energy eigenvalues of the hydrogen atom show no change of scale. The corrected energy eigenvalues in this case are:

$$E_\lambda^n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}.$$

These results do not depend on the constant $\eta_\lambda$ or $f_\lambda$ as does the harmonic oscillator.

In both problems will have an energy shift because the energy was defined as $E_\lambda = E - \eta_\lambda f_\lambda$ but the harmonic oscillator also got a change of scale on the angular frequency and the Planck constant. The eigenvalues of the hydrogen atom will get a energy shift of the order of $10^{-36}J$ in first order in $\lambda$. When investigating the wave function of the hydrogen atom we found that the Bohr's radius also got a correction from DSR. The correction was a change of scale, $a_\lambda = \eta_\lambda a_0 = a_0(1 + \lambda mc^2)$, where $a_0$ is the uncorrected Bohr's radius. For both cases, the free particle and the Green's function for the free particle, we saw a change on the Planck constant and a phase factor of the form $e^{-if_\lambda t/\hbar}$. Although this factor appear in the wave function and the Green's function the term are unobservable, because in such calculations we look at the square modulus of the wave and Green's function and this factor then disappears

The Lagrangian formalism is a powerful tool in physics. We took the task of writing a Lagrangian function from which we could derive the DSR Schrödinger equation. For this we had to introduce an unknown function $\xi$ that would not take part of the final equation. The resulting equation by the Lagrangian method is in complete agreement with the DSR Schrödinger equation derived in chapter two. $\xi$ was only used as an intermediate step for the calculation. Later on, when we made a local gauge transformation to the wave function that appeared in the Lagrangian, we saw that this function and its derivative did appeared. Compared with the gauge transformation done to the Schrödinger equation these two results seem different. But in the case of the Lagrangian method, were the function $\xi$ and its derivative appeared, we associated them with a potential term and because the potential is arbitrary we can redefine it as we like. All of our results in this work will reduce to the non-relativistic limit when $\lambda \to 0$. It was found that the charge of the particle becomes energy dependent due to the factor $\eta_\lambda$.

For future work one can investigate the role of the unknown function $\xi$ in the case for the interaction with the electromagnetic field were the function $\xi$ appears. Because of the restriction on the wave function $\psi = \nabla^2\xi$ one could try to solve this equation for $\xi$ for the one dimensional case. Then use the solution of $\xi$ and look if equation (6.12) simplifies or if it can be solve for this special case. Also a the function $\boldsymbol{B}$ is of interest. Future work can be done to investigate the nature of this unknown function. One can futher investigate if the DSR Lagrangian for the electromagnetic field exhibits other kinds of symmetries. The Noether current can be calculated yo study if any new conserved quantities can be found. Some preliminary work has been done to derive the DSR Schrödinger equation from the DSR Klein-Gordon equation. We propose as future work that this equation be written and study a way to consistently approximate this equation in the non-relativistic regime.

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