τ-MULTIPLICATIVE SETS

By

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Recently, an increasing interest has arisen in factorization with respect to unique representation of elements in an integral domain D, into elements distinct from irreducible elements.

Motivated by McAdams and Swan work, Anderson and Frazier developed a theory called *the theory of τ-factorizations*. It is a type of generalized factorization theory on integral domains. They used symmetric relations (denoted by τ) on the set of nonunit nonzero elements of an integral domain D (denoted by $D^\sharp$), in order to define what they called a τ-factorization of an element of $D^\sharp$. They called any factorization of an element x in $D^\sharp$ of the form $x = \lambda x_1 \cdot x_2 \cdots x_n$ where $x_i \tau x_j$ for all $i \neq j$ with $1 \leq i, j \leq n$ and $\lambda \in U(D)$, a *τ-factorization of x*. They classified the results based on three types of relations: divisive, multiplicative and associated-preserving relations. This theory has been studied by Hamon (2007), Ortiz (2008), Reinkoester (2010) and Juett (2012).

This work considered some preliminary definitions in order to study the theory of τ-factorizations with respect to an element, and further develops the most important types of relation of this theory. Several equivalences to the main types of definitions were obtained and then used to prove known results from other perspectives. In some cases, the results were more natural and their proofs were easier than the one provided in previous research works. It must be noted that this is the first attempt to try to understand what divisive, multiplicative and associated-preserving relations mean. All the work previously done considered such type of relations, but the authors did not try to understand the nature of them. They used them only to prove theorems, because these relations provided a good behavior when the τ-factorizations were studied. Furthermore, this investigation studied some sets with specific properties with respect to a symmetric relation on $D^\sharp$ and the connection with the τ-factorization theory. Finally, several examples were provided and some results were developed about the "*τ-sets*" on usual commutative ring properties.

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Recientemente, un creciente interés a surgido en factorización con respecto a representaciones únicas de elementos en un dominio integral D, especialmente en elementos distintos de elementos irreducibles.

Motivados por el trabajo de McAdams y Swan, Anderson y Frazier desarrollaron una teoría llamada *la teoría de τ-factorizaciones*. Esta es un tipo de teoría de factorizaciones generalizadas en dominios integrales. Ellos usaron relaciones simétricas (denotadas por τ) en el conjunto de elementos diferentes de cero y unidades de un dominio integral D (denotado por $D^\sharp$), para definir una *τ-factorización* de un elemento en $D^\sharp$. Ellos definieron una τ-factorización de un elemento x en $D^\sharp$ como una expresión de la forma $x = \lambda x_1 \cdot x_2 \cdots x_n$ donde $x_i \tau x_j$ para todo $i \neq j$ con $1 \leq i, j \leq n$ y $\lambda \in U(D)$. Anderson and Frazier (2006) clasificaron los resultados basados en tres tipos de relaciones; divisiva, multiplicativa y una relación que preserva asociados. Esta teoría ha sido estudiada por Hamon (2007), Ortiz (2008), Reinkoester (2010) and Juett (2012).

En este trabajo se consideraron algunas definiciones preliminares con el fin de estudiar la teoría de τ-factorizaciones con respecto a un elemento y desarrollar más a fondo las más importantes definiciones acerca de esta teoría. Fueron obtenidas algunas equivalencias de los principales tipos de relaciones y se utilizaron para probar resultados conocidos desde otra perspectiva. En algunos casos, los resultados fueron más naturales y sus pruebas más fáciles que las presentadas en trabajos previos. Este es el primer intento de tratar de entender el significado de las relaciones divisivas, multiplicativas, y las relaciones preserva-asociados. Toda las investigaciones previas consideraron esos tipos de relaciones, pero no estudiaron su naturaleza. Solo fueron usadas para probar teoremas, porque se necesitaban. Además, esta investigación estudió algunos conjuntos con propiedades específicas con respecto a una relación simétrica en $D^\sharp$ y la conexión con la teoría de τ-factorización. Finalmente, varios ejemplos fueron proporcionados y algunos resultados fueron desarrollados acerca de los "*τ-conjuntos*" en usuales propiedades de anillos conmutativos.

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Contents

Notations

R	Ring.
D	Integral domain.
$D^\sharp$	Set of the nonunit nonzero elements of D.
$U(D)$	Set of the units of D.
τ	Symmetric relation.
λ	Unit on D.
$x\tau y$	x is related with y.
$t \mid_\tau x$	t τ-divide to y.
$x \sim y$	x is associate to y.
$Z_\tau(x)$	τ centralizer of x.
$A_\tau(S)$	Set of points in S in which τ is associated-preserving.
$M_\tau(S)$	Set of points in S in which τ is multiplicative.
$D_\tau(S)$	Set of points in S in which τ is divisive.
$co_{D^\sharp}(S)$	Complement of S on $D^\sharp$.
$gcd(x,y)$	The greatest common divisor of x and y.
$\mathbb{Z}$	Integer numbers.
$\mathbb{Q}$	Rational numbers.

Chapter 1
INTRODUCTION

The definitions of divisibility and prime integers in $\mathbb{Z}$ were analogously defined over an integral domain D. Such concepts have been of special interest to algebraists, motivating them to consider integral domains with convenient structures in order to obtain similar results from the ring $\mathbb{Z}$. For example, the Fundamental Theorem of Arithmetic, every positive integer greater than 2 can be written uniquely (up to order) into a product of primes. It is the type of desire structures due its uniqueness. In an integral domain was defined with similar structure called a Unique Factorization Domain. Another example are the Euclidean domains, which are domains that satisfies the Euclidean Division Algorithm. See [7] for details.

In this thesis, D will be an integral domain with group of units $U(D)$ and $D^\sharp$ the set of nonzero nonunits elements of D. An element $x \in D^\sharp$ is called an *atom* or an irreducible element if it can not be factored as $x = yz$ where $y, z \in D^\sharp$ and it is a *prime* element if $x \mid yz$ implies $x \mid y$ or $x \mid z$. A lot of mathematicians that studied factorizations (Anderson, Zafrullah, Cohn, etc) have been focused in factorizations into atoms. Hence, an integral domain D is called *atomic* if each $x \in D^\sharp$ can be expressed as a finite product of atoms and is called a unique factorization domain (UFD) if in addition any atomic factorization of each element in $D^\sharp$ is unique up to associates and order of factors. Recently, algebraists have studied atomic domains with properties weaker than UFD. An atomic domain D in which for each $x \in D^\sharp$, there is a natural number N_x such that for any atomic factorization of x the number of factors is less than N_x, is called a *bounded factorization domain* (BFD). D *satisfies the ascending chain condition on principal ideals* (ACCP), if there does not exist an infinite strictly ascending chain of principal ideals of D. An atomic domain D is a *half-factorial domain* (HFD) if each atomic factorization of $x \in D^\sharp$ has the same length. The connections of these structures is summarized in Figure 1–1 (appears in [4] and [12] Figure 1.1), for more details or proofs see [4].

$$
\begin{tikzcd}
 & \text{HFD} \arrow[dr] & & & \\
\text{UFD} \arrow[ur] \arrow[r] \arrow[dr] & \text{FFD} \arrow[r] \arrow[d] & \text{BFD} \arrow[r] & \text{ACCP} \arrow[r] & \text{atomic} \\
 & \text{idf-domain} & & &
\end{tikzcd}
$$

Figure 1–1: Usual factorization properties on integral domains

It must be noted that has been of great interest to generalize several concepts, mostly to obtain unique representation of elements. Therefore the nature of factorization into rigid elements, primal elements, etc. See [14] and [3].

For example, Stephen McAdam and Richard Swan [11] developed the definition of comaximal factorization which is an example of a non-atomic factorization. For $x \in D^\sharp$, a comaximal factorization of x, is a factorization $x = x_1 \cdot x_2 \cdots x_n$ where for all $i \neq j$, x_i and x_j are comaximal, that is, $(x_i, x_j) = D$. They created similar definitions for atomic, unique factorization and atomic domain (pseudo-irreducible, unique comaximal factorization domain (UCFD) and comaximal factorization domain (CFD), respectively). Inspired by McAdam and Swan's work, in 2006, Anderson and Frazier [1] defined the theory of τ-*factorization*. They chose a symmetric relation on the nonzero nonunit elements of D, and only allowed two or more elements to be multiplied if and only if they were pairwise related under the symmetric relation. Formally, for a symmetric relation τ, a τ-factorization of an element x in $D^\sharp$ is an expression of the form $x = \lambda x_1 \cdot x_2 \cdots x_n$ where $x_i \tau x_j$ for all $i \neq j$ with $1 \leq i, j \leq n$ and $\lambda \in U(D)$ (the set of the units of D). For example, define $x\tau y \Leftrightarrow (x_i, x_j) = D$, then the comaximal product is obtained and if it is defined $\tau = D^\sharp \times D^\sharp$, the usual product is obtained. In 2007, Hamon [6] developed some topics about this theory using the relation τ defined by $x\tau y$ if and only if $n \mid x - y$; and in 2008, Ortiz [2] presented a generalization of the τ-factorization theory, called the Γ-$factorization$ theory. This last theory was studied more carefully by Juett in 2012 [8, 9] and extended to cancelative monoids.

Let τ be a symmetric relation on $D^\sharp$. The relation τ is *multiplicative*, if for any $x, y, z \in D^\sharp$, $x\tau y$ and $x\tau z$ imply $x\tau yz$; *divisive* if for any $x, y, x', y' \in D^\sharp$, $x\tau y$, $x' \mid x$ and $y' \mid y$ imply $x'\tau y'$; and *associated-preserving* if for any $x, y, t \in D^\sharp$ with $x \sim t$, then $x\tau y$ imply $t\tau y$. With the divisive relations any τ-factorization obtained by replacing a τ-factorization of a τ-factor is again a τ-factorization, and the unit can be omitted in front. On the other hand, with the multiplicative relations any τ-factorization can be expressed as the product of two τ-factors. Also the unit can be omitted in front with the associated-preserving relations, and in this way it is more general since any divisive relation is associated-preserving. Those are the principal reasons of the introduction of these definitions. In [1] and [5] the authors were able to establish the implication of usual weaker structures and the τ-structures defined analogously. Figure 1–2 (Figure 1.2 [12]) provide a good summary of their main results.

$$\begin{tikzcd}
\text{UFD} \arrow[r] \arrow[dd, "*"'] & \text{FFD} \arrow[r] \arrow[d, "*"'] & \text{BFD} \arrow[r] \arrow[dd, "*"'] & \text{ACCP} \arrow[r] \arrow[dd] & \text{atomic} \\
 & \tau\text{ -FFD} \arrow[dr] & & & \\
\tau\text{ -UFD} \arrow[ur] \arrow[dr] & & \tau\text{ -BFD} \arrow[r, "*"] & \tau\text{ -ACCP} \arrow[r, "*"] & \tau\text{ -atomic} \\
 & \tau\text{ -HFD} \arrow[ur] & & &
\end{tikzcd}$$

Figure 1–2: τ-factorization properties in integral domains, where $*$ means that τ is a divisive relation.

The author in [2] presented the same diagram for several different types of frameworks of the general theory of τ-factorization.

The theory of τ-factorization could be visualized as a restriction of the usual factorization theory, the difficulty in this theory arose in the fact of not knowing when two elements are related, that is, they can not be τ-multiplied. Some of the examples in [2], explain why this theory makes an important contribution, but others are very synthetic (because they just allow a finite number of elements to be related). Therefore, this investigation studied the set of elements related with a specific element. Let $x \in D^\sharp$, the set $Z_\tau(x) = \{y \in D^\sharp : x\tau y\}$ was called the τ-*centralizer of* x. Using $Z_\tau(x)$, equivalent statements were found to the definitions of multiplicative, divisive and associated-preserving relations. This is the first time this type of relations are deeply studied or thought from a different point of view. These equivalences will give a tool to understand why such relations are well behaved. Approach to some results will be presented, the connections between the presented definitions and the implications in the theory of τ-factorization in some known topics, specially when it is considered τ to be multiplicative or divisive. Similarly to a multiplicative set, a set $M \subseteq D^\sharp$ is defined to be τ-*multiplicative*, if for each $x, y \in M$ with $x\tau y$, $xy \in M$. Other τ-multiplicative were defined with specific properties similar to saturated sets and ideals. They are called the τ-*sets*. With the τ-sets considered on this work, some results and connections between them was established. Several results of properties of quasi-local rings and Kaplansky-like theorems will be presented. For future research, there are many sets to study and results with respect to the structure properties.

1.1 Objectives

Find sufficient or sufficient and necessary conditions, using the τ-centralizers of the elements of $D^\sharp$ and the τ-sets defined to connect the usual theory of the multiplicative sets, ideals and properties of domains, as well as the known τ-factorization theory.

Specifics objectives

- Get an equivalent definition to the definition of a τ-multiplicative relation, using the τ-centralizers of the elements in $D^\sharp$.
- Determine the connection and properties between the τ-sets.
- Determine sufficient and necessary conditions using the τ-sets to get results about a domain property.
- Study and characterize ideals with τ-sets properties.
- Define τ-multiplicative sets, with known properties in the usual theory in terms of the τ-factorization theory.

1.2 Chapters summary

In this work the theory of τ-factorization will be further developed and connections between this theory and usual topics of algebra will be found.

In the second chapter, the basic definitions of this theory and new additional definitions will be introduced. Examples will be presented so the reader can become more familiar with the notation. Also, there are several consequences and properties that come along with the definitions.

In the third chapter, several equivalences about the most known and used definitions of types of relations for the theory τ-factorization will be presented. The chapter introduces new approaches for known theorems, in order to obtain easier ways to prove them. Moreover, these equivalent statements will give a new point of view to study the theory of τ-factorizations.

The fourth chapter provides a connection to the usual theory of commutative rings. It will present consequences or results involving prime ideals, multiplicative sets and local rings. Furthermore, it will provide results about what kind of properties are obtained when considering special known sets in the usual theory that also have properties of τ-sets.

The last chapter will summarize the contributions obtained in the theory of τ-factorization, as well as in the commutative ring theory and introduce interesting topics to study in the future.

Chapter 2
NOTIONS OF THE THEORY OF τ-FACTORIZATION

In this chapter, the reader will find an introduction to the basic notions of the theory of τ-factorization as in [1]. Also some examples are given to illustrate this theory and make the reader more familiar with the notation and concepts.

2.1 Basic definitions of the theory of τ-factorization

Let D be an integral domain. The set of the nonzero nonunit elements of D is denoted by $D^\sharp$ and the set of the units of D by $U(D)$. Let τ be a symmetric relation on $D^\sharp$. The expression $x\tau y$ or $(x, y) \in \tau$, means x is related to y. The authors in [1] defined the following types of symmetric relations:

(1) The relation τ is *multiplicative*, if $x\tau y$ and $x\tau z$, then $x\tau yz$.

(2) The relation τ is *divisive*, if whenever $x\tau y$, for each x', y' in $D^\sharp$ such that $x' \mid x$ and $y' \mid y$ implies $x'\tau y'$.

(3) The relation τ to be *associated-preserving*, if $y \sim y'$ and $x\tau y$, then $x\tau y^{'}$.

Let x be in $D^\sharp$ and τ a symmetric relation on $D^\sharp$. Any expression of the form $x = \lambda x_1 \cdot x_2 \cdots x_n$ where $x_i \tau x_j$ for all $i \neq j$ in $\{1, \ldots, n\}$ and $\lambda \in U(D)$, is called a τ-$factorization$ of x. In this document, a τ-factorization of x is a τ-*product* of the x_i and x_i is a τ-$factor$ of x for each i. For simplicity, a τ-product will be denoted by ″.‟ and usual product just by concatenation. That is, $x \cdot y$ will means that $x\tau y$

and xy is a product that is not necessarily $x\tau y$.

For any $x, y \in D^\sharp$, x *τ-divides* y (denoted $x \mid_\tau y$), if there is a τ-factorization of y having x as a τ-factor, that is, $y = \lambda x \cdot x_1 \cdot x_2 \cdots x_n$ is a τ-*factorization* of y. The expressions $x = \lambda(\lambda^{-1}x)$ and $x = x$ are called trivial τ-factorizations of x, for any $\lambda \in U(D)$. An element in $D^\sharp$ that does not have a non-associated τ-factor is called *τ-atom* or *τ-irreducible*. For $x \in D^\sharp$, x is *τ-prime* if whenever $x \mid \lambda x_1 \cdot x_2 \cdots x_n$, where $\lambda x_1 \cdot x_2 \cdots x_n$ is a τ-factorization, then $x \mid x_i$ for some $i \in \{1, 2, \ldots, n\}$.

2.2 Examples

The following examples of symmetric relations will help the reader to familiarize with the notation and definitions of τ-factorizations. They have been studied in [5], [12] and [13]. Throughout the examples, D will be an integral domain.

(1) Consider $\tau = \emptyset$, it is divisive and multiplicative. Since no element $x \in D^\sharp$ has a non-trivial τ-factorization, then each x is a τ-atom. Therefore $x \mid_\tau y$ if and only if they are associates.

(2) Take $\tau = D^\sharp \times D^\sharp$. Note that τ is a multiplicative and divisive symmetric relation. Here, any element τ-prime and τ-irreducible is prime and irreducible respectively. Note that with this symmetric relation, the usual notions of factorization and divisivility on the nonzero nonunit elements, coincide with the notion of τ-factorizations and τ-divisivility.

(3) Suppose that M is any non-empty subset of $D^\sharp$ and take $\tau = M \times M$, then $x\tau y$ if and only if $x, y \in M$. Here, τ is multiplicative (respectively divisive) if and only M is a multiplicative set (respectively closed under nonunit factors). This

symmetric relation allows to obtain factorizations into special type of elements, by taking M as the set of such elements. Observe that a τ-factorization here is just a factorization of elements of M. For instance, take M the set of atoms in $D^{\sharp}$, then a τ-factorization is just the usual irreducible factorizations. In the same way, assume M is the set of prime elements, primary elements, rigid elements and other elements studied before 2004.

(4) Denote the greatest common divisor of x and y , by $gcd(x, y)$. Consider the symmetric relation τ in $D^{\sharp}$, defined by, $x\tau y$ if and only if $gcd(x, y) = 1$. Observe that if $t \mid x$ and $s \mid y$ with $d = gcd(t, s)$, d also divides x and y. If $x\tau y$, any factor of x in $D^{\sharp}$ must be relative prime to y, so is related to y under $\tau_{[\,]}$. Thus, $\tau_{[\,]}$ is a divisive relation and hence associated-preserving. Moreover, $\tau_{[\,]}$ is multiplicative only if for x, y, z in $D^{\sharp}$ such that $gcd(x, y) = 1$ and $gcd(x, z) = 1$, then $gcd(x, yz) = 1$. Unfortunately this does not happen in many structures. Therefore, τ is not necessarily multiplicative. The relation τ was investigated in [12] and [13].

(5) Define $f\partial g$ if and only if deg (f) = deg (g) on $D[x]$, where deg (f) is the degree of f as defined in [7]. Of course, ∂ is neither multiplicative nor divisive. Now, suppose that $f\partial g$ and let h be an associate of g, then $h = \lambda g$ for some unit λ. Observe, $deg\ (h) = deg\ (\lambda) + deg\ (g) = 0 + deg\ (g) = deg\ (g)$ therefore, $f\partial g$. It implies that the symmetric relation ∂ is associated-preserving. This relation was studied in [5].

(6) Consider $D = Z$ and define $x\tau_n y$ if and only if $x \equiv y\, mod\, n$. The only case τ_n is associated-preserving is when $n = 2$. Observe that this symmetric relation is never divisive. Suppose that τ is divisive, then τ must be associated-preserving. Let $t \in \mathbb{Z}$ be arbitrary, then $\bar{t} \in \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$, where $\bar{t} = \{s \in \mathbb{Z} \mid s\tau_n t\}$. Take any $s \in \bar{t}$,

then $s = t + pn$ for some $p \in \mathbb{Z}$ and $s\tau_n t$. Now, since τ is associated-preserving, $s\tau_n(-t)$, i.e., $s = -t + p'n$ for some $p' \in \mathbb{Z}$. As a consequence $t + pn = -t + p'n$, hence $2t = n(p' - p)$. In conclusion $2t \equiv 0 \ mod \ n$ for all $t \in \mathbb{Z}$, but this only happens if $n = 2$. If $n > 2$, τ_n is not divisive. For the case $n = 2$, note for example that $8\tau_2 10$, but 4 is not τ_2-related to 5. So, τ_n is never divisive.

On the other hand, suppose that $n \neq 2$. Observe that $2 \equiv 2 \ mod \ n$, but $2 \neq 4$ $mod \ n$. Hence τ_n is not multiplicative. Now, suppose $n = 2$. Assume that $x\tau_2 y$ and $x\tau_2 z$. Observe that $x\tau_2 y$ if and only if x, y are both even or x, y are both odd. If x is even, then this forces y and z to be even and of course yz is even. Analogously, if x is odd, then y, z and yz are odd. Hence, $x\tau_2 yz$ in either case. Therefore, τ_n is multiplicative only when $n = 2$. For more details on τ_n see in [1],[5], [6] and [12].

Some of the most important concepts created in [1] and [5] are the concepts of multiplicity and divisibility of symmetric relations. Most of the main results were obtained assuming τ multiplicative, divisive or both.

Theorem 2.1 (Theorem 2.2, [1])**.** *Let D be an integral domain, and let τ be a symmetric relation on $D^\sharp$.*

(1) *Suppose that τ is divisive and let $x \in D^\sharp$. Then $x = \lambda x_1 \cdot x_2 \cdots x_n$ is a τ-factorization of x if and only if $x = x_1 \cdots (\lambda x_i) \cdots x_n$ is a τ-factorization of x for each i.*

(2) *Suppose that τ is divisive. Let $x = x_1 \cdot x_2 \cdots x_n$ be a τ-factorization of a, and let $x_i = y_1 \cdot y_2 \cdots y_n$ be a τ-factorization of x_i for some i. Then $x = x_1 \cdots x_{i-1} \cdot y_1 \cdot y_2 \cdots y_n \cdot x_{i+1} \cdots x_n$ is again a τ- factorization of x.*

(3) *Suppose that τ is multiplicative. Let $x = \lambda x_1 \cdot x_2 \cdots x_n$ be a τ-factorization of x. Then $x = x_1 \cdots x_{i-1} \cdot (x_i x_{i+1}) \cdot x_{i+2} \cdots x_n$ is a τ-factorization of x, where $x_i x_{i+1}$ is a product of two elements.*

Observe that if τ is divisive, the replacement of a τ-factor on a τ-factorization with a reduced τ-factorization (see the definition a paragraph ahead) of the τ-factor gives again a τ-factorization. On the other hand, if τ is multiplicative, it is possible to combine τ-factors, doing this several times any τ-factorization can be express as the τ-product of two τ-factors. From (1) if $y \sim y'$ and τ is divisive, $x\tau y \Leftrightarrow x\tau y'$. If τ is divisive then τ is associated-preserving. In other words, the unit λ can be omitted.

In (2) the expression $x = x_1 \cdots x_{i-1} \cdot y_1 \cdot y_2 \cdots y_n \cdot x_{i+1} \cdots x_n$ is called a τ-*refinement* of x. Therefore, if τ is divisive then D accepts τ-refinements. Must note that this property of τ-refinements is the reason of why the authors in [1] and [5] obtained most of their theorems.

The first authors developed this theory using a unit λ as a factor in the τ-factorizations. However, the theory was developed without this assumption by letting a τ-factorization of an element $x \in D^\sharp$ to be $x = x_1 \cdot x_2 \cdots x_n$ where $x_i \tau x_j$ for all $i \neq j$. Such τ-factorizations are called *reduced τ-factorizations*, denoted ${}_r\tau$-factorization to distinguish from the usual τ-factorizations. Considering the theory from this perspective, an element $x \in D^\sharp$ is a τ-atom or a ${}_r\tau$-atom if x has only the trivial ${}_r\tau$-factorization $x = x$; and a ${}_r\tau$-prime, if for any ${}_r\tau$-factorization $x_1 \cdot x_2 \cdots x_n$ such that $x \mid x_1 \cdot x_2 \cdots x_n$, $x \mid x_i$ for some i. In [2] the authors show that without the associated-preserving property the ${}_r\tau$-factorizations, ${}_r\tau$-atoms, ${}_r\tau$ primes do not behave well. So the authors suggested to assume associated-preserving in order to avoid to deal with this type of inconvenient. On the other hand, the authors in

[2], [6] and [12] studied the ${}_r\tau$-factorizations and the τ_n-factorizations for $n > 2$ in which are relations that not preserve associates. So, it seems that there has been an interest in understanding the not well behaved cases of these ${}_r\tau$-factorizations.

2.3 Definitions created for our purpose

In this section, some sets are developed with specific properties that depend on a symmetric relation τ. A generalization of a multiplicative set and of a saturated set can be done using the definitions of τ-products or τ-factorizations as follows.

Definition 2.1. *Let τ be a symmetric relation on $D^\sharp$ and $M \subseteq D^\sharp$.*

(1) *M is τ-multiplicative if x,$y \in M$ such that $x\tau y$ implies $xy \in M$.*

(2) *M is a τ-ideal if for each $x \in M$ and $y \in D^\sharp$ such that $x\tau y$, $y \in M$.*

(3) *M is co-τ-saturated if M is τ-multiplicative and has the following property :* $(\forall x \in M)(\forall y \in D^\sharp)\ \big\{(x\tau y) \Longrightarrow \big\{(\forall t \in D^\sharp)(t \mid y) \Rightarrow (t \in M)\big\}\big\}$.

(4) *Let $x \in D^\sharp$ be arbitrary. The set $\{y \in D^\sharp \mid x\tau y\}$ is called the τ-centralizer of x and denoted $Z_\tau(x)$.*

(5) *M is τ-prime if for each τ-factorization $x = \lambda x_1 {\cdot} x_2 \cdots x_n \in M$, $x_i \in M$ for some i.*

Given that in the most important results obtained by Anderson and Frazier [1], they considered divisive and multiplicative symmetric relations, the work done in this thesis is interested in symmetric relations that maintain these kind of properties but from the point of view of just one element. Thus, the definition of sets that have the property of been divisive, multiplicative or associated-preserving with respect to a given element.

Definition 2.2. *Let τ be a symmetric relation on $D^\sharp$, $S \neq \emptyset$ such that $S \subseteq D^\sharp$ and $x \in D^\sharp$, the relation:*

(1) *τ is multiplicative with respect to x if whenever $x\tau y$ and $x\tau z$, $x\tau yz$. The set of elements $x \in S$ such that τ is multiplicative with respect to x is denoted by $M_\tau(S)$.*

(2) *τ is divisive with respect to x if for each $y \in D^\sharp$ with $x\tau y$, $x\tau t$, for each $t \in D^\sharp$ such that $t \mid y$. The set of elements $x \in S$ such that τ is divisive with respect to x is denoted $D_\tau(S)$.*

(3) *τ is associated-preserving with respect to x if for each $y \in D^\sharp$ such that $x\tau y$, $x\tau y'$ for all $y \sim y'$. The set of elements $x \in S$ such that τ is associated-preserving with respect to x is denoted $A_\tau(S)$.*

(4) *Let $S \subseteq D^\sharp$ be arbitrary. The set $\{x \in D^\sharp \mid x\tau y, \forall y \in S\}$ is called the τ-centralizer of S and denoted by $Z_\tau(S)$.*

The definitions of $M_\tau(\emptyset)$, $A_\tau(\emptyset)$ and $D_\tau(\emptyset)$ can be defined by $D^\sharp$ or $\emptyset$, but both cases gives some problems. Therefore, this work only use these definitions for non-empty sets. Clearly $Z_\tau(\emptyset) = \emptyset$ there is no problem considering it, but, it is kind of useless. Let $S \subseteq D^\sharp$ be arbitrary and assume that τ is a symmetric relation on $D^\sharp$. Consider S' the set of elements of S with empty τ-centralizer. By definition $S' \subseteq M_\tau(S)$, therefore, if $x \in S - M_\tau(S)$, then $Z_\tau(x) \neq \emptyset$ and there are $y, z \in D^\sharp$ with $x\tau y$ and $x\tau z$, but $yz \notin Z_\tau(x)$.
Let $S'' = \{x \in S : \textit{ whenever } x\tau y \;\, x\tau z, x\tau yz\}$, then $M_\tau(S) = S' \cup S''$. The same analysis applies to $D_\tau(S)$ and $A_\tau(S)$, obtaining similar results. In (4)., if $S = \{x\}$,

then is easy to see that $Z_\tau(S) = Z_\tau(x)$, which make sense completely.

The following proposition is a consequence of the previous definitions.

Proposition 1. *Let τ be a symmetric relation on an integral domain $D^\sharp$. Then*

(1) *τ is associated-preserving if and only if τ is associated-preserving with respect to x for all $x \in D^\sharp$,*

(2) *τ is divisive if and only if τ is divisive with respect to x for all $x \in D^\sharp$,*

(3) *τ is multiplicative if and only if τ is multiplicative with respect to x for all $x \in D^\sharp$.*

2.3.1 Examples

Let D be an integral domain. The following examples would be of help for the reader to understand the definitions and notation.

(1) Let $n \in \mathbb{N}$ be fixed and consider $D[x]$ as our domain.

Take $S = \{f \in D[x] : 1 \leq deg\ (f) \leq 2n\}$ and define $f\tau g$ if and only if $deg\ (f) = deg\ (g) \leq n$. Note that τ is symmetric, and if $(D[x])^\sharp = D[x] - \{U(D) \cup \{0\}\}$ then $S \subseteq (D[x])^\sharp$. Suppose that $f, g \in S$ such that $f\tau g$, then $deg\ (fg) = deg\ (f) + deg\ (g) \leq n + n = 2n$. Therefore, S is a τ-multiplicative set. Let $f \in S$ and $g \in (D[x])^\sharp$ be such that $f\tau g$, then $deg\ (f) = deg\ (g) \leq n$, and therefore $g \in S$. Hence, S is a τ-ideal and S is a co-τ-saturated set.

Take $f \in (D[x])^\sharp$ such that $deg\ (f) = 1$, then $f\tau f$. However, f is not related to f^2. Therefore, τ is not a multiplicative relation. Since $deg\ (d) = 0$ for all $d \in D^\sharp - \{0\}$, $D^\sharp \times D^\sharp \subseteq \tau$ and $D_\tau(S) = M_\tau(S) = A_\tau(S) = S$ for all $S \subseteq D^\sharp$.

Suppose $n \geq 2$ and $f = (x+a)^2$ where $a \in D^\sharp$, then $f\tau f$. Note that $x+a \mid (x+a)^2$ and $(x+a)^2 \mid (x+a)^2$, but $x+a$ is not related to $(x+a)^2$. Hence for $n \geq 2$, τ can not be divisive. Suppose that $n = 1$ and let $f, g \in D^\sharp$ be such that $f\tau g$, then $f, g \in D^\sharp$ or $deg\ (f) = deg\ (g) = 1$. If $f, g \in D^\sharp$, any factor of f is going to be related to any factor of g in $D^\sharp$. If $deg\ (f) = deg\ (g) = 1$, then $df\tau g$ for all $d \in D^\sharp$ but d is not related with g. In conclusion, τ is not divisive if $n = 1$. Finally, τ is multiplicative and divisive with respect to f if and only $deg\ (f) = 0$.

(2) Let $z \in D^\sharp$ be arbitrary. Consider $S_z = (z) - \{0\} \subseteq D^\sharp$ where (z) is the ideal generated by z. Define $x\tau y$ if and only if $x - y \in (z)$ and $x \neq y$. Then τ is symmetric and S_z is a τ-multiplicative set. Furthermore, τ is not necessarily divisive or multiplicative. If $x \in S_z$ and $y \in D^\sharp$ such that $x\tau y$, then $x - y \in (z)$ and $x - (x - y) \in (z)$, $y \in S_z$. In conclusion, S_z is a τ-ideal.

Let $x \in S_z$, then $Z_\tau(x) = S_z - \{x\}$. Now, take any $y \in co_{D^\sharp}(S_z)$, since $S_z \cup \{0\}$ is a τ-ideal, $xy \in S_z$. Hence $xy\tau t$ for all $t \in S_z$ with $t \neq xy$. Note that $y \mid xy$, but for $t \neq xy$ in S_z y is not related to t. In conclusion, τ is not divisive and S_z is not a co-τ-saturated set. In fact, τ is not divisive with respect to any element in S_z. Let $x, y \in S_z$ be arbitrary, then $xy \neq x$ and $xy \neq y$. Clearly $x\tau xy$ and $y\tau xy$, but x, y are not related to xy. In conclusion, τ is not multiplicative. On the other hand, suppose that there is $x \in S_z$ such that $x \neq yz$ for all $y, z \in S_z$, then τ is multiplicative with respect to x. In fact, $M_\tau(S) = \{x \in S_z : x \neq yz \ \ for\ \ all\ \ y, z \in S_z\}$ for all $S \subseteq S_z$.

If $x \in co_{D^\sharp}(S_z)$, then τ is not necessarily associated-preserving or multiplicative with respect to x. For instance, take $D = \mathbb{Z}$ and $S_8 = (8) - \{0\}$. Since $5 - (-3) \in S_8$, then $5\tau - 3$. However, $-3 \sim 3$ but $5 - 3 = 2 \notin S_8$. Hence, 5 is not related to 3 and τ is not an associated-preserving relation. Furthermore, observe that $5\tau - 3$, but 5 is not related to 9. Then τ is not a multiplicative relation.

Let $x \in S_z$ be such that $x\tau y$, then $y \in S_z$. Since $S_z \cup \{0\}$ is an ideal, $\lambda y \in S_z$ for all $\lambda \in U(D)$, and $x - \lambda y \in S_z$ for all $\lambda \in U(D)$. In conclusion, if $x \in S_z$, τ is associated-preserving with respect to x.

(3) Let $S \subseteq D^\sharp$ and take $\tau_S = S \times S$. Note that, S is a τ_S-multiplicative set if and only if S is a multiplicative set. Furthermore, in case S is a multiplicative set S is a τ_S-ideal. But S is not necessarily a co-τ_S-saturated set. In this relation $Z_{\tau_S}(x) = S$ for all $x \in S$ and the only case S is a co-τ_S-saturated set is when S is closed under proper factors.

(4) Let I be a proper ideal in D, and take $S = I - \{0\} \subseteq D^\sharp$. Define $a\tau b$ if and only if $a \sim b$. Note that τ is symmetric. Since I is an ideal, S is a τ-multiplicative set and a τ-ideal. Then S does not necessarily satisfy the saturation property.

(5) Let $S = \{2p : p \in \mathbb{Z}\} \subseteq \mathbb{Z}$ and consider the symmetric relation τ_2, $x\tau_2 y$ if and only if $x, y \in (2)$, the ideal generated by 2. Since the product of even numbers is even, then S is a τ-multiplicative set. If $x\tau_2 y$, then x, y are both even or both odd, so S is a τ-ideal. Now, $6\tau_2 10$ and $5 \mid 10$ but $5 \notin S$, so S is not co-τ-saturated.

(6) Let $D = \mathbb{Z}$ and take $S_n = (n) - \{0\}$ where $n \in \mathbb{N}$. Here $D^\sharp = D - \{\pm 1, 0\}$. Define $x\tau_n y$ if and only $x - y \in (n)$ and $x \neq y$. Let $x \in S_n$ and $y \in D^\sharp$ such that $x\tau_n y$, then $x - y \in (n)$ and $-y \in (n)$, so $y \in S_n$. In conclusion, S_n is a τ-ideal. On the other hand, if $n = 2$, then $4, 6 \in S_2$, $4\tau_2 6$, but $3 \in S_2$ and $3 \mid 6$. This shows that

S_2 is not a co-τ_2-saturated set.

In (3) define $\tau_{co} = co_{D^\sharp}(S) \times co_{D^\sharp}(S)$. Note that, τ_{co} is divisive if and only $co_{D^\sharp}(S)$ is closed under proper factors. Assume τ_{co} is a divisive relation. Let $x \in co_{D^\sharp}(S)$ be fixed but arbitrary, then $x\tau_{co}x$. Since τ_{co} is divisive, each pair of factors of x in $D^\sharp$ are related, and by definition they are in $co_{D^\sharp}(S)$, that is, $co_{D^\sharp}(S)$ is closed under proper factors. Assume $co_{D^\sharp}(S)$ is closed under proper factors and let $x, y \in D^\sharp$ be such that $x\tau_{co}y$, then $x, y \in co_{D^\sharp}(S)$. By hypothesis all the factors of x and y are in $co_{D^\sharp}(S)$, that is, any factor of x is going to be related to any factor of y. This prove that τ_{co} is divisive. It is easy to prove that if additionally S is a multiplicative set, then S is a τ_S-ideal.

2.4 Understanding the new definitions

In the examples shown in the previous section, there were symmetric relations that make a set $S \subseteq D^\sharp$ a τ-ideal and a co-τ-saturated set. However, there are examples of τ-ideal that are not necessarily co-τ-saturated sets. In any of these cases, however there is no co-τ-saturated set that is not a τ-ideal. This can be observed in the following theorem.

Theorem 2.2. *Let D be an integral domain, τ a symmetric relation on $D^\sharp$ and $M \subseteq D^\sharp$.*

(1) *If M is a co-τ-saturated set, then M is a τ-ideal.*

(2) *Assume τ is also divisive. Then M is a co-τ-saturated set if and only if M is a τ-ideal.*

Proof. For (1) Suppose that M is a co-τ-saturated set, then M is a τ-multiplicative. Let $x \in M$ be arbitrary and suppose that there is $y \in D^\sharp$ such that $x\tau y$. Since $y \mid y$, then $y \in M$ by hypothesis. Hence M is a τ-ideal. To show (2) suppose that τ is divisive. For the other direction, suppose that M is a τ-ideal, then τ is a multiplicative set. Let $x \in M$ be arbitrary and suppose that there is $y \in D^\sharp$ such that $x\tau y$. Since τ is divisive, any proper factor of y is going to be related to x, and since M is a τ-ideal each one of those factors is going to be in M. Hence M is a co-τ-saturated set.

□

It would be very interesting to find a non-empty symmetric relation in which the τ-centralizer of each element in $D^\sharp$ is also non-empty, because if there are other τ-sets, then they can be useful to obtain connections with known properties in abstract algebra. For instance, suppose that $S \subseteq D^\sharp$ such that S is a co-τ-saturated set and $Z_\tau(x) \neq \emptyset$ for all $x \in S$. Let $x \in S$ be arbitrary and let $y \in D^\sharp$ be such that $y \mid x$. Then $Z_\tau(x) \neq \emptyset$ and there is $t \in D^\sharp$ such that $x\tau t$. This happens if and only if $t\tau x$, so $y \in S$. In conclusion, for all $y \in D^\sharp$ such that $y \mid x$, $y \in S$. Hence, if τ is divisive and S is a τ-ideal, by Theorem 2.2 the same result is obtained.

A common question about these "absorption properties" in the previous paragraph, is how far is a co-τ-saturated set from being a saturated set. Since a saturated set contains all the units of a ring (in particular on a domain), it's not possible for a co-τ-saturated set to be saturated, even if the domain were in fact a field. But, the properties of a saturated set in $D^\sharp$ can be maintained, that is, a set that is closed under factors in $D^\sharp$.

Theorem 2.3. *Let S be a co-τ-saturated set and $Z_\tau(x) \cap Z_\tau(y) \neq \emptyset$ for all $x, y \in S$. Suppose that τ is multiplicative, then S is a saturated set in $D^\sharp$.*

Proof. Since $Z_\tau(x) \cap Z_\tau(y) \neq \emptyset$ for all $x, y \in S$, then $Z_\tau(x) \neq \emptyset$ for all $x \in S$. Hence the "absorption properties " holds. Let $x, y \in S$ be arbitrary, then by hypothesis $Z_\tau(x) \cap Z_\tau(y) \neq \emptyset$. Therefore, there is $p \in S$ with $p\tau x$ and $p\tau y$. Since τ

is multiplicative, then $p\tau xy$ and by hypothesis $xy \in S$. Hence S is a multiplicative set.

On the other hand, suppose that $S \subset D^\sharp$ is a co-τ-saturated set, τ is transitive and $Z_\tau(x) \cap Z_\tau(y) \neq \emptyset$ for all $x, y \in S$. Let $x, y \in S$, then $Z_\tau(x) \cap Z_\tau(y) \cap S \neq \emptyset$ and there is $t \in S$ such that $x\tau t$ and $t\tau y$. Since τ is transitive, $x\tau y$. But τ is multiplicative, then $xy \in S$. In conclusion, S is closed under proper factors and is a multiplicative set.

$\square$

The same result in Theorem 2.3 can be obtained replacing the condition of τ being a multiplicative relation with τ be a transitive relation and the existence of an element x in S with $S \subseteq Z_\tau(x)$. Another good question about the connection of a co-τ-saturated set S and its complement $co_{D^\sharp}(S)$. The following are some cases with their respective implications. If S is a τ-ideal set, the τ-centralizer of any element in S it is going to be contained in S. Then, it is important to know about how many elements are related to each element in S because it gives information about how big is S. In particular, if there is $x \in S$ with $Z_\tau(x) = D^\sharp$, each $y \in D^\sharp$ is related to x and $y \in S$, therefore $S = D^\sharp$. If S is a τ-ideal, then $S \supset \bigcup_{x \in S} Z_\tau(x)$. Furthermore, if $Z_\tau(x) \neq \emptyset$ for all $x \in S$, then $S = \bigcup_{x \in S} Z_\tau(x)$. Hence, if τ is a multiplicative symmetric relation, $Z_\tau(x)$ is a multiplicative set for each $x \in S$. Therefore S is the union of multiplicative sets (in the usual sense).

The properties about a τ-ideal seem to be interesting when $Z_\tau(x) \neq \emptyset$ for all $x \in S$ is assumed. On the other hand, note that S is a co-τ-saturated set where each pair of elements are related with respect to the relation τ if and only if $S \times S \subseteq \tau$. Hence, $S \cup U(D)$ is saturated in the usual sense. Moreover, a partial previous results can be obtained with a weaker hypothesis. For instance, if $Z_\tau(x) \neq \emptyset$ for all $x \in S$,

S is not necessarily a multiplicative set, but S is closed under factors on $D^\sharp$.

It is known that the τ-centralizer of each element in a τ-ideal set S is going to be contained in S. The set S can be expressed as the union of two sets by considering the set of elements of with empty τ-centralizer.

$$S = \left(\bigcup_{x\in S} Z_\tau(x)\right) \bigcup \{x \in S \mid Z_\tau(x) = \emptyset\}$$

.

For co-τ-saturated sets similar results holds. On the other hand, the same can be done with the complement of S with respect to $D^\sharp$. That is,

$$D^\sharp - S = \left(\bigcup_{x\in D^\sharp - S} Z_\tau(x)\right) \bigcup \{x \in D^\sharp - S \mid Z_\tau(x) = \emptyset\}$$

.

Furthermore, if $Z_\tau(x) \neq \emptyset$ for all $x \in D^\sharp - S$ then $D^\sharp - S = \bigcup_{x\in D^\sharp - S} Z_\tau(x)$. If S is a τ-ideal, then $D^\sharp = P \cup P'$, where $P = \left(\bigcup_{x\in S} Z_\tau(x)\right) \bigcup \{x \in S \mid Z_\tau(x) = \emptyset\}$ and $P' = \bigcup_{x\in D^\sharp - S} Z_\tau(x) \bigcup \{x \in D^\sharp - S \mid Z_\tau(x) = \emptyset\}$. If $Z_\tau(x) \neq \emptyset$ for all $x \in D^\sharp$, $D^\sharp = \left(\bigcup_{x\in S} Z_\tau(x)\right) \bigcup \left(\bigcup_{x\in D^\sharp - S} Z_\tau(x)\right)$.

Theorem 2.4. *Let D be an integral domain, $M \subseteq D$ and τ_1, τ_2 symmetric relations on $D^\sharp$. Assume $\tau_1 \leq \tau_2$, that is $\tau_1 \subseteq \tau_2$, then the following properties hold.*

(1) *$Z_{\tau_1}(x) \subseteq Z_{\tau_2}(x)$ for all $x \in D^\sharp$.*

(2) *$Z_{\tau_1}(x) = Z_{\tau_2}(x)$ for all $x \in D^\sharp$ if and only if $\tau_1 = \tau_2$.*

(3) *Let $x \in D^\sharp$. If $Z_{\tau_1}(x) = Z_{\tau_2}(x)$, then $(\forall y \in D^\sharp)(x \mid_{\tau_1} y \Longleftrightarrow x \mid_{\tau_2} y)$.*

(4) *If M is a τ_2-multiplicative set, then M is a τ_1-multiplicative set.*

(5) *If M is a co-τ_2-saturated set, then M is a co-τ_1-saturated set.*

(6) *If M is a τ_2-ideal, then M is a τ_1-ideal.*

(7) *If M is a τ_2-prime set, then M is a τ_1-prime set.*

Proof. For (1), let $x \in D^\sharp$ and $y \in Z_{\tau_1}(x)$ be arbitrary. Then $x\tau_1 y$ or $(x,y) \in \tau_1$. Since $\tau_1 \subseteq \tau_2$, therefore $x\tau_2 y$ and $y \in Z_{\tau_2}(x)$. In conclusion, $Z_{\tau_1}(x) \subseteq Z_{\tau_2}(x)$. To show (2), suppose that $Z_{\tau_1}(x) = Z_{\tau_2}(x)$ for all $x \in D^\sharp$. Let $(x,y) \in \tau_1$ be fixed but arbitrary, then $x\tau_1 y$ and $x \in Z_{\tau_1}(y) = Z_{\tau_2}(y)$. Hence $x\tau_2 y$ or equivalently $(x,y) \in \tau_2$. Then $\tau_1 \subseteq \tau_2$ and similarly $\tau_2 \subseteq \tau_1$. Therefore $\tau_1 = \tau_2$. The converse follows by part (1). To prove (3), suppose that $x \mid_{\tau_1} y$, then there is a τ_1-factorization $\lambda x \cdot x_2 \cdots x_n$ such that $y = \lambda x \cdot x_2 \cdots x_n$. Therefore $x_2, \dots, x_n \in Z_{\tau_1}(x) = Z_{\tau_2}(x)$, i.e., $x\tau_2 x_i$ for all $i \in \{2, \dots, n\}$. But τ is transitive, so that $x_i \tau_2 x_j$ where $i, j \in \{2, \dots, n\}$. In conclusion, $\lambda x x_2 \cdots x_n$ is a τ_2-factorization, i.e., $x \mid_{\tau_2} y$. In other words for any τ_1-factorization $\lambda x_1 \cdot x_2 \cdots x_n$ where x is one of $x_1, x_2, \dots, x_n$, $\lambda x_1 \cdot x_2 \cdots x_n$ is in fact a τ_2-factorization. Analogously if $x \mid_{\tau_2} y$ then $x \mid_{\tau_1} y$.

For (4), let $x, y \in M$ such that $x\tau_1 y$, then $x\tau_2 y$ and by hypothesis $xy \in M$. In consequence, M is a τ_1-multiplicative set. Let $x \in M$ such that $x\tau_1 y$ for some $y \in D^\sharp$. Assume that $t \in D^\sharp$ with $t \mid y$, must show that $t \in M$. Since $\tau_1 \leqslant \tau_2$, $x\tau_2 y$. Now, M is a co-τ_2-saturated set, hence $t \in M$. therefore M is a co-τ_1-saturated set. This shows (5). For (6), let $x \in M$ be such that $x\tau_1 y$ for some $y \in D^\sharp$. Since $\tau_1 \leqslant \tau_2$, $x\tau_2 y$. But M is a τ_2-ideal therefore $y \in M$. So M is a τ_1-ideal set. Finally, to show (7), let $x = \lambda x_1 \cdot x_2 \cdots x_n \in M$ be a τ_1-factorization, then $x_i \tau_1 x_j$ for all $i \neq j$. By hypothesis $x_i \tau_2 x_j$ for all $i \neq j$, i.e., $x = \lambda x_1 \cdot x_2 \cdots x_n \in M$ is a τ_2-factorization. Since M is a τ_2-prime set, $x_i \in M$ for some i. Hence M is a τ_1-prime set.

$\square$

The condition in (1) is also sufficient for $\tau_1 \leq \tau_2$. Suppose that $Z_{\tau_1(x)} \subseteq Z_{\tau_2}(x)$. Let $(x,y) \in \tau_1$, then $x\tau_1 y$, i.e., $y \in Z_{\tau_1}(x) \subseteq Z_{\tau_2}(x)$. Therefore, $(x,y) \in \tau_2$. So, $\tau_1 \leq \tau_2$. In conclusion, $\tau_1 \leq \tau_2$ if and only if $Z_{\tau_1}(x) \subseteq Z_{\tau_2}(x)$ for all $x \in D^\sharp$.

In the definition 2.1 of a co-τ-saturated set M, the property :

$$(\forall x \in M)(\forall y \in D^\sharp)\ \big\{(x\tau y) \Longrightarrow \big\{(\forall t \in D^\sharp)(t \mid y) \Rightarrow (t \in M)\big\}\big\}$$

is useful to obtain important information about M. For instance, see Example (3). This property is called "the co-τ-saturated property". However, there are sets that have this property but they are not τ-multiplicative sets. For example, let $n \in \mathbb{Z}$ with $n > 1$ fixed. Consider the set $S = \{m \in \mathbb{Z} : m \mid n\}$ and define $\tau_S = S \times S$. Since $n\tau_S n$ and n is not related to n^2, τ is not a multiplicative relation. But S is closed under proper factors, therefore S has the co-τ-saturated property. Observe that in example (3), the fact that S is closed under proper factors does not depend on S being a τ-multiplicative set, just on the fact that S has the co-τ-saturated property. Then, these properties can be useful to get information about a set that just has the co-τ-saturated property and also useful in obtaining properties of a symmetric relation. The following theorem will show it.

Theorem 2.5. *Let D be an integral domain, suppose that τ is a multiplicative and symmetric relation on $D^\sharp$ and let $x \in D^\sharp$ be arbitrary. The following properties holds.*

(1) *$Z_\tau(x)$ is a multiplicative set.*

(2) *If τ is transitive, then $Z_\tau(x)$ is a τ-ideal.*

(3) *$Z_\tau(x) \neq \emptyset$ implies $Z_\tau(x)$ is infinite.*

(4) *Let $x = x_1 \cdot x_2 \cdots x_n$ be a reduced τ-factorization. If there is $t \in \{1, 2, \ldots, n\}$ such that $x_t \tau x_t$, then $x_t \tau x$.*

(5) *Suppose that $Z_\tau(z)$ has the co-τ-saturated properties for all $z \in D^\sharp$, then τ is divisive.*

(6) *τ is a transitive and divisive relation if and only if $Z_\tau(z)$ has the co-τ-saturated properties for all $z \in D^\sharp$.*

(7) *Suppose that S is a τ-ideal and $Z_\tau(S) \cap S \neq \emptyset$. Then for each $x \in Z_\tau(S) \cap S$, $S = Z_\tau(x)$.*

Proof. For (1), assume that $y, z \in Z_\tau(x)$. Then $x\tau y$ and $x\tau z$, so $yz\tau x$, that is $yz \in Z_\tau(x)$. Hence $Z_\tau(x)$ is a multiplicative set. For (2), notice that by (1), $Z_\tau(x)$ is a multiplicative set. Let $y \in Z_\tau(x)$ and $z \in D^\sharp$ such that $y\tau z$. Then $x\tau y$ and $y\tau z$. Since τ is a transitive relation, $x\tau z$ and $z \in Z_\tau(x)$. Hence, $Z_\tau(x)$ is a τ-ideal and (2) holds.

To prove (3), assume that $Z_\tau(x) \neq \emptyset$. Then there is $y \in D^\sharp$ such that $x\tau y$. Let's prove by induction that $y^n\tau x$ for all $n \in \mathbb{N}$. For $k = 1$, $x\tau y$. Suppose that it is true for $k = n-1$, then $x\tau y^{n-1}$. Since $x\tau y$ and τ is a multiplicative relation, $x\tau yy^{n-1}$ and hence $x\tau y^n$. Note that if $n < m$ and $y^n = y^m$, $y^n(1 - y^{m-n}) = 0$, hence $1 = y^{m-n}$ (D is an integral domain and $y \in D^\sharp$). In conclusion, the elements $y, y^2, \ldots y^n, \ldots$ are all distinct. This implies that $Z_\tau(x)$ is infinite. To show (4), suppose that $x \in D^\sharp$ and let $x_1 \cdot x_2 \cdots x_n$ be a reduced τ-factorization of x. Then $Z_\tau(x) = Z_\tau(x_1 \cdot x_2 \cdots x_n) \supseteq \bigcap_i^n Z_\tau(x_i)$ (τ is multiplicative). Let $t \in \{1, \ldots, n\}$ be such that $x_t\tau x_t$, then $x_t \in \bigcap_i^n Z_\tau(x_i)$. In conclusion $x_t \in Z_\tau(x)$.

For (5), let $x, y, x', y' \in D^\sharp$ such that $x' \mid x$, $y' \mid y$ and $x\tau y$, then $y \in Z_\tau(x)$. Since $Z_\tau(x)$ is a co-τ-saturated set, $y' \in Z_\tau(x)$, so $x \in Z_\tau(y')$. But, $Z_\tau(y')$ is also a co-τ-saturated set, then $x' \in Z_\tau(y')$, that is, $x' \sim y'$. Therefore τ is a divisive relation.

To prove (6), suppose τ is a transitive and divisive relation. Let $z \in D^\sharp$, $x \in Z_\tau(z)$ arbitrary and $y \in D^\sharp$ such that $x\tau y$. Let $t \mid y$ for some $t \in D^\sharp$. Since τ is divisive, $x\tau t$ and using the fact τ is transitive, $t\tau z$, i.e., $t \in Z_\tau(z)$. Therefore $Z_\tau(z)$ has the co-τ-saturated properties for all $z \in D^\sharp$. For the converse, suppose that $Z_\tau(z)$ has the co-τ-saturated properties for all $z \in D^\sharp$. By (5), τ is divisive. Let $x, y, z \in D^\sharp$ such that $x\tau y$ and $y\tau z$, then $y \in Z_\tau(x)$ and $z \in Z_\tau(x)$ ($Z_\tau(x)$ has the co-τ-saturated properties). So, τ is a transitive relation. Finally, to prove (7), let $x \in Z_\tau(S) \cap S$ be arbitrary, then $x \in Z_\tau(S)$ and $x \in S$. By the first case, $S \subseteq Z_\tau(x)$ and by the second case, $Z_\tau(x) \subseteq S$. Therefore $S = Z_\tau(x)$.

□

Note that in (7), if τ is a multiplicative relation, S is a multiplicative set.

Theorem 2.1 shows that if τ is a multiplicative relation is possible to express any τ-factorization as the τ-product of two τ-factors. However, the same result can be obtain if for each τ-factorization τ is multiplicative with respect to one of the τ-factors. Let $\lambda x_1 \cdot x_2 \cdots x_n$ be a τ-factorization of y. If there is a τ-factor x_i in which τ is multiplicative with respect to x_i, then there is a τ-factorization for y of the form $y = \lambda x_i s$ for some $s \in D^\sharp$. Assume there is a τ-factor x_i in which τ is multiplicative with respect to x_i, say x_1. Since $\lambda x_1 \cdot x_2 \cdots x_n$ is a τ-factorization of y, then $x_1 \tau x_j$ for all $j \in \{2, 3, \ldots, n\}$. Furthermore using the fact that τ is multiplicative, $x_1 \tau (x_2 \cdots x_n)$. Let $s = x_2 \cdots x_n$, then $y = \lambda x_1 s$ where $s \in D^\sharp$ and $x_1 \tau s$.

Consider a non-empty τ-centralizer of a subset S of $D^\sharp$, then there is $x \in D^\sharp$ with non-empty τ-centralizer that contains all the elements of S. However, it does not give any information about the equality of S and $Z_\tau(S)$, even if x is in S or its complement. Unless S has some additional property like being a τ-ideal or a co-τ-saturated set. On the other hand, if the additional properties are assumed with respect to τ, the following theorem gives the connections between S, $Z_\tau(S)$ and $Z_\tau(x)$ for all $x \in Z_\tau(S)$.

Theorem 2.6. *Let D be an integral domain, suppose that τ is a symmetric relation on $D^\sharp$ and $S \subseteq D^\sharp$. Suppose $x \in D^\sharp$, then:*

(1) *$S \subseteq Z_\tau(x)$ if and only if $x \in Z_\tau(S)$.*

(2) *If τ is a transitive relation and $Z_\tau(x) \neq \emptyset$, the following statements are equivalents:*

(a) $S \subseteq Z_\tau(x)$,

(b) $x \in Z_\tau(S)$,

(c) $Z_\tau(x) \subseteq Z_\tau(S)$.

(3) *If τ is a transitive relation and S is a τ-ideal, then $S = Z_\tau(x) = Z_\tau(S)$ for all $x \in Z_\tau(S)$.*

(4) *$S \subseteq Z_\tau(x)$ for all $x \in D^\sharp$ if and only $D^\sharp = Z_\tau(S)$.*

Proof. (1) ($\Longrightarrow$) Suppose $S \subseteq Z_\tau(x)$. Given $y \in S$ fixed but arbitrary, $y \in Z_\tau(x)$, hence $x\tau y$. Then $x\tau y$ for all $y \in S$ and $x \in Z_\tau(S)$.

($\Longleftarrow$) Suppose $x \in Z_\tau(S)$. Let $y \in S$ be fixed but arbitrary. Since $x\tau t$ for all $t \in S$, in particular for $t = y$, $y \in Z_\tau(x)$. In conclusion $S \subseteq Z_\tau(x)$.

(2) By (1) parts (a) and (b) are equivalents. Suppose that $Z_\tau(x) \subseteq Z_\tau(S)$. Let $t \in S$ be fixed but arbitrary. Since $Z_\tau(x) \neq \emptyset$ there is $p \in Z_\tau(x)$. By hypothesis $p \in Z_\tau(S)$ and by definition $p\tau y$ for all $y \in S$. In particular for $y = t$, $t\tau p$. Since $p\tau x$ and the transitivity of τ, $t \in Z_\tau(x)$. Therefore $S \subseteq Z_\tau(x)$.

(3) Suppose τ is a transitive relation and S is a τ-ideal. Let $x \in Z_\tau(S)$ be fixed but arbitrary. By (2a) and (2c), $S \subseteq Z_\tau(x) \subseteq Z_\tau(S)$. If $y \in Z_\tau(S)$, then $y\tau t$ for all $t \in S$. But since S is a τ-ideal, $y \in S$, therefore $S \supseteq Z_\tau(S)$. This implies $S = Z_\tau(x) = Z_\tau(S)$ for all $x \in Z_\tau(S)$.

(4) ($\Longrightarrow$) Suppose $S \subseteq Z_\tau(x)$ for all $x \in D^\sharp$. Let $y \in D^\sharp$ be fixed but arbitrary. By hypothesis $S \subseteq Z_\tau(y)$, that is, $y\tau t$ for all $t \in S$. Hence $D^\sharp \subseteq Z_\tau(S)$.

($\Longleftarrow$) Suppose $D^\sharp = Z_\tau(S)$. Let $x \in D^\sharp$ be fixed but arbitrary, then $x \in Z_\tau(S)$. This implies that $t \in Z_\tau(x)$ for all $t \in S$. Therefore $S \subseteq Z_\tau(x)$ for all $x \in D^\sharp$.

$\square$

Example 2.4.1. *Let D be an integral domain and $a \in D^\sharp$ arbitrary but fixed. Let $S = \{a\}$ and define $\tau_a = \{(\lambda a, \mu a) \mid \lambda, \mu \in U(D)\}$. Since $a\tau_a\lambda a$ for all $\lambda \in U(D)$, then $Z_{\tau_a}(a) \neq \emptyset$. If $\lambda a\tau_a\mu a$ and $\mu a\tau_a\eta a$, then $\lambda a\tau_a\eta a$. Hence, τ is a transitive relation. By previous theorem $S \subseteq Z_{\tau_a}(a) = \{\lambda a \mid \lambda \in U(D)\}$, $a \in Z_{\tau_a}(S)$ and $Z_{\tau_a}(a) \subseteq Z_{\tau_a}(S)$. In fact, $Z_{\tau_a}(a) = Z_{\tau_a}(S) = \{\lambda a \mid \lambda \in U(D)\}$.*

Example 2.4.2. *Let D be an integral domain and $a \in D^\sharp$ arbitrary but fixed. Let $S = \{a\}$ and define $\tau_a = \{(\lambda a, \mu a) \mid \lambda, \mu \in U(D)\}$. Since $a\tau_a\lambda a$ for all $\lambda \in U(D)$, then $Z_{\tau_a}(a) \neq \emptyset$. If $\lambda a\tau_a\mu a$ and $\mu a\tau_a\eta a$, then $\lambda a\tau_a\eta a$. Hence, τ is transitive. By Theorem 2.6, $S \subseteq Z_{\tau_a}(a) = \{\lambda a \mid \lambda \in U(D)\}$, $a \in Z_{\tau_a}(S)$ and $Z_{\tau_a}(a) \subseteq Z_{\tau_a}(S)$. In fact, $Z_{\tau_a}(a) = Z_{\tau_a}(S) = \{\lambda a \mid \lambda \in U(D)\}$.*

The following example illustrate why the hypothesis of τ being transitive in Theorem 2.6 is necessary for theorem to holds.

Example 2.4.3. *Let D be an integral domain and $a, b \in D^\sharp$ such that a is not associated to b. Consider τ_a and τ_b as defined in Example 2.4.2. Define $\tau = \tau_a \cup \tau_b \cup \{(a, b)\}$. For any $\lambda \in U(D)$, $\lambda a\tau a$ and $a\tau b$, but $(\lambda a, b) \notin \tau$. Then τ is not a transitive relation. Let $S = \{a, b\}$, then S is a τ-ideal. But, $Z_\tau(a) = \{\lambda a \mid \lambda \in U(D)\} \nsubseteq Z_\tau(S)$.*

In the following theorem the are some results that try to understand the behavior of $D_\tau(S)$, $M_\tau(S)$, $Z_\tau(S)$ and $A_\tau(S)$ for a non-empty subset S.

Theorem 2.7. *Let D be an integral domain, τ be a symmetric relation on $D^\sharp$ and S and E non-empty subsets of $D^\sharp$ then:*

(1) *If $S \subseteq E$ then $D_\tau(S) \subseteq D_\tau(E)$, $M_\tau(S) \subseteq M_\tau(E)$ and $A_\tau(S) \subseteq A_\tau(E)$.*

(2) $D_\tau(D_\tau(S)) = D_\tau(S)$.

(3) $M_\tau(M_\tau(S)) = M_\tau(S)$.

(4) $A_\tau(A_\tau(S)) = A_\tau(S)$.

(5) *$Z_\tau(Z_\tau(F)) = Z_\tau(F)$ for all $F \subseteq D^\sharp$ if and only if τ is a transitive relation.*

(6) *Let $\wedge$ be an index set and S_λ a family of non-empty subsets of $D^\sharp$ with non-empty intersection, then :*

   (a) $D_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right) = \bigcup_{\lambda\in\wedge} D_\lambda\left(S_\lambda\right)$, $M_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right) = \bigcup_{\lambda\in\wedge} M_\lambda\left(S_\lambda\right)$ *and* $A_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right) = \bigcup_{\lambda\in\wedge} A_\lambda\left(S_\lambda\right)$.

   (b) $D_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right) = \bigcap_{\lambda\in\wedge} D_\lambda\left(S_\lambda\right)$, $M_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right) = \bigcap_{\lambda\in\wedge} M_\lambda\left(S_\lambda\right)$ *and* $A_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right) = \bigcap_{\lambda\in\wedge} A_\lambda\left(S_\lambda\right)$.

(7) $D_\tau\left(E - S\right) = D_\tau\left(E\right) - D_\tau\left(S\right)$, *for any* $S \subsetneq E$

Proof. (1) Suppose that $S \subseteq E$. Let $x \in D_\tau(S)$ be arbitrary, then $x \in S$ and τ is divisive with respect to x. By hypothesis $x \in E$ and τ is divisive with respect to x. Therefore $x \in D_\tau(E)$ and $D_\tau(S) \subseteq D_\tau(E)$. Similarly $M_\tau(S) \subseteq M_\tau(E)$ and $A_\tau(S) \subseteq A_\tau(E)$.

(2) By definitions $D_\tau(D_\tau(S))$ is the set of elements in $D_\tau(S)$ for which τ is divisive with respect to such elements, then $D_\tau(D_\tau(S)) \subseteq D_\tau(S)$. Let $x \in D_\tau(S)$ be fixed but arbitrary, then τ is divisive with respect to x. By definition of $D_\tau(D_\tau(S))$, $x \in D_\tau(D_\tau(S))$. Hence $D_\tau(S) \subseteq D_\tau(D_\tau(S))$.

(3) and (4) are proved similarly.

(5) ($\Longrightarrow$) Suppose that $Z_\tau(Z_\tau(F)) = Z_\tau(F)$ for all $F \subseteq D^\sharp$. Let $x, y, z \in D^\sharp$ be such that $x\tau y$ and $y\tau z$, then $x \in Z_\tau(y)$ and $z \in Z_\tau(y)$. But by hypothesis, $x \in Z_\tau(Z_\tau(y))$, hence $x\tau t$ for all $t \in Z_\tau(y)$. In particular, $x\tau z$. Therefore, τ is a transitive relation.

($\Longleftarrow$) Suppose τ is a transitive relation. Let $x, y \in Z_\tau(F)$ be fixed but arbitraries, then $x\tau t$ and $y\tau t$ for all $t \in F$. For any element $s \in F$, since τ is transitive, $x\tau s$ and $s\tau y$, implies $x\tau y$. In conclusion, $x\tau y$ for all $y \in Z_\tau(F)$, and so $Z_\tau(F) \subseteq Z_\tau(Z_\tau(F))$. Let $x \in Z_\tau(Z_\tau(F))$ be arbitrary, then $x\tau t$ for all $t \in Z_\tau(F)$. Take any $y \in Z_\tau(F)$, then $x\tau y$ and $y\tau s$ for all $s \in F$. Since τ is a transitive relation, $x\tau s$, for all $s \in F$. In conclusion $x \in Z_\tau(F)$, so $Z_\tau(Z_\tau(F)) \subseteq Z_\tau(F)$.

($6a$) Let $x \in \bigcup_{\lambda\in\wedge} D_\lambda(S_\lambda)$ be a fixed but arbitrary element, then $x \in D_\lambda(S_\lambda)$ for some $\lambda \in \wedge$, τ is divisive with respect to x and $x \in S_\lambda \subseteq \left(\bigcup_{\lambda\in\wedge} S_\lambda\right)$. Therefore $x \in D_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right)$. In conclusion $\bigcup_{\lambda\in\wedge} D_\lambda(S_\lambda) \subseteq D_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right)$.
Let $x \in D_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right)$ be arbitrary, then τ is divisive with respect to x and $x \in \bigcup_{\lambda\in\wedge} S_\lambda$, then $x \in S_\lambda$ for some $\lambda \in \wedge$. In consequence $x \in D_\lambda(S_\lambda)$ for some $\lambda \in \wedge$. It implies that $D_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right) \subseteq \bigcup_{\lambda\in\wedge} D_\lambda(S_\lambda)$.

($6b$) Let $x \in D_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right)$ be a fixed but arbitrary element, then τ is divisive with respect to x and $x \in S_\lambda$ for all $\lambda \in \wedge$. In consequence $x \in D_\lambda(S_\lambda)$ for all $\lambda \in \wedge$. It implies that $D_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right) \subseteq \bigcap_{\lambda\in\wedge} D_\lambda(S_\lambda)$.
Let $x \in \bigcap_{\lambda\in\wedge} D_\lambda(S_\lambda)$ be arbitrary, then $x \in D_\lambda(S_\lambda)$ for all $\lambda \in \wedge$, τ is divisive with respect to x and $x \in S_\lambda \supseteq \left(\bigcap_{\lambda\in\wedge} S_\lambda\right)$. Therefore $x \in D_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right)$. In conclusion $\bigcap_{\lambda\in\wedge} D_\lambda(S_\lambda) \subseteq D_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right)$.

(7) Let $x \in D_\tau(E - S)$ be a fixed but arbitrary element, then $x \in E - S$ and τ is divisive with respect to x. Hence $x \in E$ and $x \notin S$. Note that $x \notin S$ implies by definition that $x \notin D_\tau(S)$, and the fact that $x \in S$ and τ is divisive with respect to x implies $x \in D_\tau(E)$. Therefore $x \in D_\tau(E) - D_\tau(S)$.

Let $x \in D_\tau(E) - D_\tau(S)$ be arbitrary, then $x \in D_\tau(E)$ and $x \notin D_\tau(S)$, i.e., $x \in E$, $x \notin S$ and τ is divisive with respect to x. In summary, $x \in E - S$ and τ is divisive with respect to x. In consequence $x \in D_\tau(E - S)$ and therefore $D_\tau(E) - D_\tau(S) \subseteq D_\tau(E - S)$.

□

The results of the previous theorem can be generalized as follows. Assume P to be a τ-property with respect to an element x in $D^\sharp$. Let $S \subseteq D^\sharp$ a non-empty set. The set of elements $x \in S$ such that τ satisfies the property p with respect to x is denoted as $P_\tau(S)$. This generalizes the idea of $D_\tau(S)$, $M_\tau(S)$ and $A_\tau(S)$, where in these cases the τ-property P is the divisive (respectively multiplicative and associated-preserving) property with respect to given element in $D^\sharp$.

Theorem 2.8. *Let D be an integral domain, τ be a symmetric relation on $D^\sharp$ and S and E non-empty subsets of $D^\sharp$, then:*

(1) *If $S \subseteq E$ then $P_\tau(S) \subseteq P_\tau(E)$.*

(2) $P_\tau(P_\tau(S)) = P_\tau(S)$.

(3) *Let $\wedge$ be an index set and S_λ a family of non-empty subsets of $D^\sharp$ with non-empty intersection, then*

  (a) $P_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right) = \bigcup_{\lambda\in\wedge} P_\tau(S_\lambda)$.

  (b) $P_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right) = \bigcap_{\lambda\in\wedge} P_\tau(S_\lambda)$.

  (c) *$P_\tau(E - S) = P_\tau(E) - P_\tau(S)$, for any $S \subsetneq E$.*

Proof. (1) Suppose $S \subseteq E$. Let $x \in P_\tau(S)$ be arbitrary, then $x \in S$ and τ satisfies the τ-property P with respect to x. But by hypothesis $x \in E$, then $x \in P_\tau(E)$. Therefore $P_\tau(S) \subseteq P_\tau(E)$.

(2) By definition $P_\tau(P_\tau(S)) \subseteq P_\tau(S)$. Let $x \in P_\tau(S)$ be arbitrary, then τ-satisfies the property P with respect to x, then by definition $x \in P_\tau(P_\tau(S))$. In conclusion $P_\tau(S) = P_\tau(P_\tau(S))$.

(3) Let $\wedge$ be an index set and S_λ a family of non-empty subsets of $D^\sharp$ with non-empty intersection.

($3a$) Let $x \in P_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right)$ be arbitrary, then $x \in \bigcup_{\lambda\in\wedge} S_\lambda$ and τ satisfies the property P with respect to x. Hence $x \in S_\lambda$ for some $\lambda \in \wedge$ and $x \in P_\tau(S_\lambda)$ for some $\lambda \in \wedge$, i.e., $x \in \bigcup_{\lambda\in\wedge} P_\tau\left(S_\lambda\right)$. Then $P_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right) \subseteq \bigcup_{\lambda\in\wedge} P_\tau\left(S_\lambda\right)$. Let $x \in \bigcup_{\lambda\in\wedge} P_\tau\left(S_\lambda\right)$ be arbitrary, then $x \in P_\tau\left(S_\lambda\right) \subseteq S_\lambda$ for some $\lambda \in \wedge$. Hence $x \in \bigcup_{\lambda\in\wedge} S_\lambda$ and τ satisfies the property P with respect to x, i.e., $x \in P_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right)$, therefore $P_\tau\left(\bigcup_{\lambda\in\wedge} S_\lambda\right) = \bigcup_{\lambda\in\wedge} P_\tau\left(S_\lambda\right)$.

($3b$) Let $x \in P_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right)$ be arbitrary, then $x \in \bigcap_{\lambda\in\wedge} S_\lambda$ and τ satisfies the property P with respect to x, hence $x \in S_\lambda$ for all $\lambda \in \wedge$. Then $x \in P_\tau\left(S_\lambda\right)$ for all $\lambda \in \wedge$, i.e., $x \in \bigcap_{\lambda\in\wedge} P_\tau\left(S_\lambda\right)$ and $P_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right) \subseteq \bigcap_{\lambda\in\wedge} P_\tau\left(S_\lambda\right)$. Let $x \in \bigcap_{\lambda\in\wedge} P_\tau\left(S_\lambda\right)$ be arbitrary, then $x \in P_\tau\left(S_\lambda\right)$ for all $\lambda \in \wedge$, then $x \in S_\lambda$ for all $\lambda \in \wedge$ and τ satisfies the property P with respect to x. Therefore $x \in \bigcap_{\lambda\in\wedge} S_\lambda$ and $x \in P_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right)$. In conclusion $P_\tau\left(\bigcap_{\lambda\in\wedge} S_\lambda\right) = \bigcap_{\lambda\in\wedge} P_\tau\left(S_\lambda\right)$.

($3c$) Assume $S \subsetneq E$ where S and E are non-empty subsets of $D^\sharp$. Let $x \in P_\tau(E-S)$, then $x \in E-S$ and τ satisfies the property P with respect to x, so $x \in E$ and $x \notin S$. Therefore $x \in P_\tau(E)$ and $x \notin P_\tau(S)$. In resume $x \in P_\tau(E) - P_\tau(S)$.

Hence $P_\tau(E-S) \subseteq P_\tau(E) - P_\tau(S)$. Let $x \in P_\tau(E) - P_\tau(S)$, then $x \in P_\tau(E)$ and $x \notin P_\tau(S)$. Hence $x \in E$ and $x \notin P_\tau(S)$, so $x \in E - S$ and τ satisfies the property P with respec to x. Therefore $x \in P_\tau(E-S)$, i.e., $P_\tau(E-S) = P_\tau(E) - P_\tau(S)$.

$\square$

In this chapter, some properties of the theory of τ-factorizations depend on symmetric relations, and on the divisive, multiplicative and associated-preserving relations. If τ is a divisive relation, any τ-factorization after a τ-refinement is again a τ-factorization. If τ is a multiplicative relation, any τ-factorization can be expressed as the τ-product of lenght two, and if τ is an associated-preserving relation, the unit can be omitted. Also, a divisive relation is an associated-preserving relation.

The set $A_\tau(S)$ (respectively $D_\tau(S)$ and $M_\tau(S)$) where $S \subseteq D^\sharp$, is formed by the elements with empty τ-centralizer and the elements in which τ is associated-preserving (respectively divisive and multiplicative) with respect to such element. Any co-τ-saturated set is a τ-ideal. The converse is true when τ is a divisive relation. The properties obtained in the τ-sets were more interesting when considered with non-empty τ-centralizers for each one of their elements, and τ to be a multiplicative or a divisive relation. The τ-set property can be inheritable, that means whenever $\tau_1 \leq \tau_2$, a τ_2-set is a τ_1-set.

Chapter 3
EQUIVALENCES

During the development of this work, equivalences with respect to the types of relations considered in the theory of τ-factorization were found, using the definitions of τ-centralizer, $D_\tau(S)$, $M_\tau(S)$ and $A_\tau(S)$ for some non-empty subset S of $D^\sharp$.

3.1 Equivalences for a multiplicative relation

In this section, the equivalences to the definition of a multiplicative relation are given in terms of $Z_\tau(x)$ and $M_\tau(S)$, where $x \in D^\sharp$ and $\emptyset \neq S \subseteq D^\sharp$.

Theorem 3.1. *Let τ be a symmetric relation on $D^\sharp$. Then τ is a multiplicative relation if and only if $Z_\tau(x) \cap Z_\tau(y) \subseteq Z_\tau(xy)$ for all $x, y \in D^\sharp$.*

Proof. ($\Longrightarrow$) Suppose that τ is a multiplicative relation. Let $x, y \in D^\sharp$ be fixed but arbitrary elements and $t \in Z_\tau(x) \cap Z_\tau(y)$, then $t\tau x$ and $t\tau y$. Since τ is a multiplicative relation, $t\tau xy$. In consequence, $Z_\tau(x) \cap Z_\tau(y) \subseteq Z_\tau(xy)$ for all $x, y \in D^\sharp$.

($\Longleftarrow$) Suppose that $Z_\tau(x) \cap Z_\tau(y) \subseteq Z_\tau(xy)$ for all $x, y \in D^\sharp$. Let x, y and z be elements in $D^\sharp$ such that $x\tau y$ and $x\tau z$. By hypothesis, $x \in Z_\tau(y) \cap Z_\tau(z) \subseteq Z_\tau(yz)$ and $x\tau yz$. Thus, τ is a multiplicative relation.

$\square$

In Theorem 3.1, if $x = \lambda x_1 \cdot x_2 \cdots x_n$ is a τ-factorization, this implies that for any $i \neq j$, $x_i \tau x_j$. If x_1 is fixed, then $x_1 \in Z_\tau(x_i)$ for all $i \in \{2, \ldots, n\}$. Therefore $x_1 \tau (x_2 \cdots x_n)$ and x has a τ-factorization of length 2, when τ is a multiplicative relation. This is a consequence of Proposition 2.2 in [1].

Theorem 3.2. *A symmetric relation τ is multiplicative if and only if τ is contained in $M_\tau(S) \times M_\tau(S)$ for some set $S \subseteq D^\sharp$.*

Proof. If $\tau = \emptyset$, then $\tau \subseteq M_\tau(S) \times M_\tau(S)$ for any $S \subseteq D^\sharp$. So assume $\tau \neq \emptyset$.
($\Longrightarrow$) Assume τ to be a multiplicative relation. Take $S = \bigcup_{x \in D^\sharp} Z_\tau(x)$. Since τ is a multiplicative relation, $S \subseteq M_\tau(S)$. But $M_\tau(S) \subseteq S$, so $M_\tau(S) = S$. If $(x, y) \in \tau$, then $x\tau y$ and $x, y \in S$. Therefore $(x, y) \in S \times S \subseteq M_\tau(S) \times M_\tau(S)$. Consequently $\tau \subseteq M_\tau(S) \times M_\tau(S)$.

($\Longleftarrow$) Assume $\tau \subseteq M_\tau(S) \times M_\tau(S)$ for some $S \subseteq D^\sharp$. Let $x \in D^\sharp$ be an element with $x\tau y$ and $x\tau z$. In particular, $(x, y) \in \tau \subseteq M_\tau(S) \times M_\tau(S)$ for some $S \subseteq D^\sharp$. Hence $x \in M_\tau(S)$ and $x\tau yz$ (by hypothesis). Thus, τ is a multiplicative relation.

□

Theorem 3.3. *A symmetric relation τ is a multiplicative relation if and only if for all $x, y \in D^\sharp$ there is $S \subseteq D^\sharp$ such that $Z_\tau(x) \cap Z_\tau(y) \subseteq M_\tau(S)$.*

Proof. If $\tau = \emptyset$, then $Z_\tau(x) \cap Z_\tau(y) = \emptyset \subseteq M_\tau(S)$ for any subset S of $D^\sharp$. Hence, assume $\tau \neq \emptyset$.

($\Longrightarrow$) Suppose that τ is a multiplicative relation. Take $S = \bigcup_{x \in D^\sharp} Z_\tau(x)$, then by Theorem 2.7

$$\begin{aligned} M_\tau(S) &= M_\tau\{\bigcup_{x \in D^\sharp} Z_\tau(x))\} \\ &= \bigcup_{x \in D^\sharp} M_\tau\{Z_\tau(x)\} \end{aligned} \tag{3.1}$$

Since τ is a multiplicative relation $M_\tau\{Z_\tau(x)\} = Z_\tau(x)$ for all $x \in D^\sharp$. In conclusion, $M_\tau(S) = S$. Let $x, y \in D^\sharp$ and $t \in Z_\tau(x) \cap Z_\tau(y)$ be fixed but arbitrary. In particular, $t \in Z_\tau(x) \subseteq S = M_\tau(S)$. Consequently $Z_\tau(x) \cap Z_\tau(y) \subseteq M_\tau(S)$.

($\Longleftarrow$) Suppose that for all $x, y \in D^\sharp$, there is $S \subseteq D^\sharp$ such that $Z_\tau(x) \cap Z_\tau(y) \subseteq M_\tau(S)$. Let $x, y, z \in D^\sharp$ such that $x\tau y$ and $x\tau z$. Then $x \in Z_\tau(x) \cap Z_\tau(z) \subseteq M_\tau(S)$. Therefore $x \in M_\tau(S)$, and by the definition of $M_\tau(S)$, $x\tau yz$. Therefore, τ is a multiplicative relation.

□

Corollary 3.1. *τ is a multiplicative relation if and only if for all $x \in D^\sharp$ there is $S \subseteq D^\sharp$ such that $Z_\tau(x) \subseteq M_\tau(S)$.*

Proof. If $\tau = \emptyset$. then $Z_\tau(x) = \emptyset$ for all $x \in D^\sharp$ and $Z_\tau(x) \subseteq M_\tau(S)$ for all $S \subseteq D^\sharp$. Hence assume $\tau \neq \emptyset$.

($\Longrightarrow$) Suppose that τ is a multiplicative relation. Let $x \in D^\sharp$. By Theorem 2.7, there is $S \subseteq D^\sharp$ such that $Z_\tau(x) = Z_\tau(x) \cap Z_\tau(x) \subseteq M_\tau(S)$.

($\Longleftarrow$) Suppose that for all $x \in D^\sharp$ there is $S \subseteq D^\sharp$ such that $Z_\tau(x) \subseteq M_\tau(S)$. For all $x, y \in D^\sharp$ arbitrary, $Z_\tau(x) \cap Z_\tau(y) \subseteq Z_\tau(x) \subseteq S$ for some $S \subseteq D^\sharp$. Then it follows from Theorem 3.3.

□

In the next equivalence, other condition between the concepts of the set $M_\tau(S)$ and a multiplicative relation was obtained.

Theorem 3.4. *A symmetric relation $\tau \neq \emptyset$ is multiplicative if and only if $M_\tau(S) = S$ for all $S \subseteq D^\sharp$.*

Proof. ($\Longrightarrow$) Assume τ to be a multiplicative relation. By definition $M_\tau(S) \subseteq S$. Let $x \in S$ be fixed but arbitrary. If $Z_\tau(x) = \emptyset$, τ is multiplicative with respect to x vacuously, hence $x \in M_\tau(S)$. So assume $Z_\tau(x) \neq \emptyset$. For any $y, z \in D^\sharp$, if $x\tau y$ and $x\tau z$, then $x\tau yz$ and $x \in M_\tau(S)$.

($\Longleftarrow$) Suppose that $M_\tau(S) = S$ for all $S \subseteq D^\sharp$. Note that for all $x \in D^\sharp$, $M_\tau(\{x\}) = \{x\}$. Then the proof is concluded, because τ is multiplicative with respect to x for each $x \in D^\sharp$.

$\square$

3.2 Equivalences for a divisive relation

In this section, the equivalences to the definitions of a divisive relation is given in terms of $Z_\tau(x)$ for $x \in D^\sharp$ and $D_\tau(S)$ where $\emptyset \neq S \subseteq D^\sharp$. Must note that the statements are very similar to those in the previous section.

Theorem 3.5. *A symmetric relation τ is divisive if and only if $Z_\tau(xy) \subseteq Z_\tau(x) \cap Z_\tau(y)$ for all $x, y \in D^\sharp$ and τ is associated-preserving.*

Proof. ($\Longrightarrow$) Suppose τ is a divisive relation. Let $x, y \in D^\sharp$ and $t \in Z_\tau(xy)$ be fixed but arbitrary. By definition, $t\tau xy$. Since τ is a divisive relation, $t\tau x$ and $t\tau y$. Therefore $Z_\tau(xy) \subseteq Z_\tau(x) \cap Z_\tau(y)$.

($\Longleftarrow$) Assume $Z_\tau(xy) \subseteq Z_\tau(x) \cap Z_\tau(y)$ for all $x, y \in D^\sharp$. Let $x, y \in D^\sharp$ arbitrary but fixed such that $x\tau y$. Suppose $x^{'}, y^{'} \in D^\sharp$ such that $x^{'} \mid x$ and $y^{'} \mid y$. Hence, $x = x^{'}t$ and $y = y^{'}s$ for some $t, s \in D$. Therefore there are 3 cases; when $t, s \in D^\sharp$, $t, s \in U(D)$ and, $t \in D^\sharp$ and $s \in U(D)$. If $t, s \in D^\sharp$, then $Z_\tau(x't) \subseteq Z_\tau(x') \cap Z_\tau(t)$. Since $x\tau y$, $y \in Z_\tau(x't)$ and $y \in Z_\tau(x')$. Thus, $x' \in Z_\tau(y) \subseteq Z_\tau(y's) \subseteq Z_\tau(y') \cap Z_\tau(s)$ and $x'\tau y'$. In the second case since τ is associate-preserving, $x\tau x'$ and $y \sim y'$, $x'\tau y'$. For the last case, note that $y'\tau x$, because τ is associate-preserving. Now,

$y' \in Z_\tau(x't) \subseteq Z_\tau(x') \cap Z_\tau(t)$ and $x'\tau y'$. Therefore, in any case $x'\tau y'$ and τ is divisive

□

The last theorem gives a similar result as when τ is a multiplicative relation, this time with the assumption that the τ-centralizer of the product of any two elements in $D^\sharp$ is contained in the intersection of their respective τ-centralizers. However, the hypothesis of τ being an associated-preserving relation was needed. The following theorem shows the connections between the definition of $D_\tau(S)$ of a subset $S \subseteq D^\sharp$ and a divisive relation.

Theorem 3.6. *A symmetric relation τ is divisive if and only if $\tau \subseteq D_\tau(S) \times D_\tau(S)$ for some $S \subseteq D^\sharp$.*

Proof. If $\tau = \emptyset$, then $\tau \subseteq D_\tau(S) \times D_\tau(S)$ for any $S \subseteq D^\sharp$. So assume $\tau \neq \emptyset$. $(\Longrightarrow)$ Suppose τ is a divisive relation and consider $S = \bigcup_{x \in D^\sharp} Z_\tau(x)$. Since τ is a divisive relation, $S \subseteq D_\tau(S)$. But, $D_\tau(S) \subseteq S$, so $D_\tau(S) = S$. Moreover, if $x\tau y$, $x, y \in S$. Therefore $(x, y) \in S \times S = D_\tau(S) \times D_\tau(S)$, and consequently $\tau \subseteq D_\tau(S) \times D_\tau(S)$.

$(\Longleftarrow)$ Suppose that $\tau \subseteq D_\tau(S) \times D_\tau(S)$ for some $S \subseteq D^\sharp$. Let $x, y \in D^\sharp$ such that $x \in Z_\tau(y)$ and $t \mid y$. For some $t \in D^\sharp$, $(x, y) \in \tau \subseteq D_\tau(S) \times D_\tau(S)$ and $x \in D_\tau(S)$. Therefore $t \in Z_\tau(x)$, and hence τ is divisive.

□

Theorem 3.7. *A symmetric relation τ is divisive if and only if the following condition holds:*

$$(\forall x \in D^\sharp)(\forall y \in D^\sharp)\{(y \in Z_\tau(x)) \Longrightarrow \{(\forall t \in D^\sharp)(t \mid y) \Longrightarrow (t \in Z_\tau(x)\}\}.$$

Proof. Suppose that τ is a divisive relation. Let $x, y \in D^\sharp$ such that $y \in Z_\tau(x)$. Suppose $t \in D^\sharp$ with $t \mid y$. By divisivility, since $x \mid x$ and $t \mid y$, $t\tau x$. Therefore

$t \in Z_\tau(x)$. For the converse, suppose the above property holds. Let $x, y \in D^\sharp$ such that $y \in Z_\tau(x)$. Let $x', y' \in D^\sharp$ with $x' \mid x$ and $y' \mid y$. By hypothesis, $y' \in Z_\tau(x)$, that is, $x \in Z_\tau(y')$. Applying the hypothesis again, $x' \in Z_\tau(y')$, that is, $x' \tau y'$. Therefore, τ is a divisive relation.

$\square$

Theorem 3.8. *A symmetric relation τ is a divisive relation if and only if for all $x, y \in D^\sharp$ there is $S \subseteq D^\sharp$ such that $Z_\tau(xy) \subseteq D_\tau(S)$ and τ is an associated-preserving relation.*

Proof. If $\tau = \emptyset$ then $Z_\tau(xy) = \emptyset$ for all $x, y \in D^\sharp$, hence $Z_\tau(xy) \subseteq D_\tau(S)$ for any $S \subseteq D^\sharp$ and τ is an associated-preserving relation.

($\Longrightarrow$) Suppose that τ is a divisive relation. By Theorem 3.6, there is $S \subseteq D^\sharp$ such that $\tau \subseteq D_\tau(S) \times D_\tau(S)$. Let $p \in Z_\tau(xy)$ be fixed but arbitrary, then $(p, xy) \in \tau$. Hence $p \in D_\tau(S)$, therefore $Z_\tau(xy) \subseteq D_\tau(S)$.

($\Longleftarrow$) Assume τ to be an associated-preserving relation and for all $x, y \in D^\sharp$ there is a subset $S \subseteq D^\sharp$ such that $Z_\tau(xy) \subseteq D_\tau(S)$. Let $x, y \in D^\sharp$ be fixed but arbitrary and $p \in Z_\tau(xy)$. By hypothesis, $p \in D_\tau(S)$ for some $S \subseteq D^\sharp$ and τ is divisive with respect to p. Since $x, y \mid xy$, $x\tau p$ and $y \tau p$. Therefore $p \in Z_\tau(x) \cap Z_\tau(y)$ and $Z_\tau(xy) \subseteq Z_\tau(x) \cap Z_\tau(y)$ for all $x, y \in D^\sharp$. By Theorem 3.5 τ is a divisive relation.

$\square$

Theorem 3.9. *A symmetric relation τ is divisive if and only if for all $x \in D^\sharp$ there is a subset $S \subseteq D^\sharp$ such that $Z_\tau(x) \subseteq D_\tau(S)$.*

Proof. If $\tau = \emptyset$, then $Z_\tau(x) = \emptyset$ for all $x \in D^\sharp$. Hence $Z_\tau(x) \subseteq S$ for any $S \subseteq D^\sharp$. So, assume $\tau \neq \emptyset$.

($\Longrightarrow$) Suppose that τ is divisive. By Theorem 3.6 there is an $S \subseteq D^\sharp$ such that $\tau \subseteq D_\tau(S) \times D_\tau(S)$. Let $p \in Z_\tau(x)$ be fixed but arbitrary, then $p \tau x$. Hence,

$(p, x) \in \tau \subseteq D_\tau(S) \times D_\tau(S)$ and $p \in D_\tau(S)$. Therefore $Z_\tau(x) \subseteq D_\tau(S)$.

($\Longleftarrow$) Suppose that for all $x \in D^\sharp$ there is a subset $S \subseteq D^\sharp$ such that $Z_\tau(x) \subseteq D_\tau(S)$. Let $y, z \in D^\sharp$ be fixed but arbitrary. Take $x = zy$, by hypothesis there is a subset $S \subseteq D^\sharp$ such that $Z_\tau(zy) \subseteq D_\tau(S)$. That is, for all $z, y \in D^\sharp$ there is $S \subseteq D^\sharp$ such that $Z_\tau(zy) \subseteq D_\tau(S)$. On the other hand, let x, y be two elements $D^\sharp$ such that $x\tau y$ and $t \sim y$. Now, $x \in Z_\tau(y) \subseteq D_\tau(S)$ for some $S \subseteq D^\sharp$, and τ is divisive with respect to x, so $x\tau t$. Therefore τ is an associated-preserving relation. By previous theorem τ is a divisive relation.

□

Theorem 3.10. *A symmetric relation $\tau \neq \emptyset$ is divisive if and only if $D_\tau(S) = S$ for all $S \subseteq D^\sharp$ with $S \neq \emptyset$.*

Proof. ($\Longrightarrow$) Suppose that τ is a divisive relation. Let $S \subseteq D^\sharp$ be arbitrary. By definition, $D_\tau(S) \subseteq S$. To prove the other direction let $x \in S$ be arbitrary. If $Z_\tau(x) = \emptyset$, then $x \in D_\tau(S)$. So assume that $Z_\tau(x) \neq \emptyset$ and let $y \in D^\sharp$ such that $x\tau y$. Since τ is a divisive relation, any factor of y on $D^\sharp$ is related to x. Hence $D_\tau(S) = S$.

($\Longleftarrow$) Suppose $D_\tau(S) = S$ for all $S \subseteq D^\sharp$. Let $x \in D^\sharp$ be fixed but arbitrary. By hypothesis, $D_\tau(Z_\tau(x)) = Z_\tau(x)$. In conclusion, for all $x \in D^\sharp$, there is $S \subseteq D^\sharp$ such that $Z_\tau(x) \subseteq D_\tau(S)$. By Theorem 3.9, τ is a divisive relation.

□

Corollary 3.2. *Let τ be a symmetric relation on $D^\sharp$. Assume that τ is associated-preserving, then the following statements are equivalents.*

(1) *The relation τ is a multiplicative and divisive relation.*

(2) *For all $x, y \in D^\sharp$, $Z_\tau(xy) = Z_\tau(x) \cap Z_\tau(y)$.*

(3) *For all* $\emptyset \neq S \subseteq D^\sharp$ *with* $D_\tau(S) \neq \emptyset$ *and* $M_\tau(S) \neq \emptyset$, $M_\tau(D_\tau(S)) = S$ *or* $D_\tau(M_\tau(S)) = S$.

(4) *For all* $\emptyset \neq S \subseteq D^\sharp$, $D_\tau(S) = M_\tau(S) = S$.

Proof. It follows immediately from the previous theorems.

□

3.3 Equivalences for an associated-preserving relation

If τ is an associated-preserving symmetric relation, each τ-factorization can be written as a τ-reduced factorization and it is a way to dispense of the unit in front of a τ-factorization. In this section several equivalences will be presented with respect to associated-preserving relations.

Theorem 3.11. *A symmetric relation* τ *is associated-preserving if and only if* $A_\tau(S) = S$ *for all* $S \subseteq D^\sharp$.

Proof. If $\tau = \emptyset$, then $A_\tau(S) = S$ vacuously. So assume $\tau \neq \emptyset$.
($\Longrightarrow$) Suppose that τ is an associated-preserving relation. Let $S \subseteq D^\sharp$ be fixed but arbitrary. By definition, $A_\tau(S)$ is the set of elements x in S such that τ is associated-preserving with respect to x, then $A_\tau(S) \subseteq S$. For $x \in S$, τ is associated-preserving with respect to x, because τ is an associated-preserving relation. Therefore $x \in A_\tau(S)$. In conclusion $A_\tau(S) = S$

($\Longleftarrow$) Assume $A_\tau(S) = S$ for all $S \subseteq D^\sharp$. Let $x, y \in D^\sharp$ such that $x\tau y$. Then $x \in Z_\tau(y) = A_\tau(Z_\tau(y))$, that is, $x\tau t$ for any $t \sim y$. Therefore, τ is an associated-preserving relation.

□

Observe that any divisive relation is an associated-preserving relation, therefore a symmetric relation is divisive if and only if $A_\tau(S) = D_\tau(S)$ for any $S \subseteq D^\sharp$. But in general, $D_\tau(S) \subseteq A_\tau(S)$ for any $S \subseteq D^\sharp$.

Theorem 3.12. *A symmetric relation τ is associated-preserving if and only if there is a subset $\emptyset \neq S \subseteq D^\sharp$ such that $\tau \subseteq A_\tau(S) \times A_\tau(S)$.*

Proof. If $\tau = \emptyset$, then $\tau \subseteq A_\tau(S) \times A_\tau(S)$ for any $S \subseteq D^\sharp$. So assume $\tau \neq \emptyset$.
($\Longrightarrow$) Suppose τ is an associated-preserving relation. Take $S = \bigcup_{x\in D^\sharp} Z_\tau(x)$, then by Theorem 3.11 $A_\tau(S) = S$. If $(x, y) \in \tau$, then $x \in Z_\tau(y)$ and $y \in Z_\tau(x)$. But $Z_\tau(x), Z_\tau(y) \subseteq S$, hence $(x, y) \in S \times S = A_\tau(S) \times A_\tau(S)$. In conclusion, $\tau \subseteq A_\tau(S) \times A_\tau(S)$.

($\Longleftarrow$) Suppose there is a subset $S \subseteq D^\sharp$ such that $\tau \subseteq A_\tau(S) \times A_\tau(S)$. Let $x, y \in D^\sharp$ such that $x\tau y$. By hypothesis $(x, y) \in A_\tau(S) \times A_\tau(S)$. In particular, $x \in A_\tau(S)$. Hence $x \in S$ and τ is associated-preserving with respect to x. So, if $y \sim t$ then $x\tau t$. Thus, τ is associated-preserving.

□

Theorem 3.13. *A symmetric relation τ is associated-preserving if and only if for all $x, y \in D^\sharp$ there is a subset $S \subseteq D^\sharp$ such that $Z_\tau(x) \cap Z_\tau(y) \subseteq A_\tau(S)$.*

Proof. If $\tau = \emptyset$, then $Z_\tau(x) \cap Z_\tau(y) = \emptyset \subseteq A_\tau(S)$ for any set. Hence, assume $\tau \neq \emptyset$.
($\Longrightarrow$) Suppose τ is an associated-preserving relation. Let $x, y \in D^\sharp$ be fixed but arbitrary. Take $S = Z_\tau(x)$, then $Z_\tau(x) \cap Z_\tau(y) \subseteq Z_\tau(x)$. By Theorem 3.11 $A_\tau(Z_\tau(x)) = Z_\tau(x)$, therefore $Z_\tau(x) \cap Z_\tau(y) \subseteq A_\tau(S)$. In conclusion, for all $x, y \in D^\sharp$ there is $S = Z_\tau(x) \subseteq D^\sharp$ such that $Z_\tau(x) \cap Z_\tau(y) \subseteq A_\tau(S)$.

($\Longleftarrow$) Suppose for all $x, y \in D^\sharp$ there is $S \subseteq D^\sharp$ such that $Z_\tau(x) \cap Z_\tau(y)$ is a subset of $A_\tau(S)$. Let $x, y \in D^\sharp$ such that $x\tau y$. By hypothesis, there is $S \subseteq D^\sharp$ such that $Z_\tau(y) \cap Z_\tau(y) = Z_\tau(y) \subseteq A_\tau(S)$. Since $x \in Z_\tau(y)$, τ is associated-preserving with respect to x. Thus, $x\tau t$ for all $t \sim y$.

$\square$

Theorem 3.14. *A symmetric relation τ is an associated-preserving relation if and only if for all $x \in D^\sharp$ there is $\emptyset \neq S \subseteq D^\sharp$ such that $Z_\tau(x) \subseteq A_\tau(S)$.*

Proof. If $\tau = \emptyset$, $Z_\tau(x) = \emptyset \subseteq A_\tau(S)$. So assume $\tau \neq \emptyset$.
($\Longrightarrow$) Suppose τ is an associated-preserving relation. Let $x \in D^\sharp$ be fixed but arbitrary. By Theorem 3.11 $Z_\tau(x) = A_\tau(Z_\tau(x))$. Then $S = Z_\tau(x)$ does the work.

($\Longleftarrow$) Let $x, y \in D^\sharp$ be such that $x\tau y$, then there is a $S \subseteq D^\sharp$ such that $Z_\tau(y) \subseteq A_\tau(S)$. Note that $x \in Z_\tau(y)$ and τ is associated-preserving with respect to x. In conclusion, $x\tau t$ for all $t \in D^\sharp$. Thus, τ is an associated-preserving relation.

$\square$

Theorem 3.15. *A symmetric relation τ is an associated-preserving relation if and only if $Z_\tau(x) = A_\tau(x)$ for all $x \in D^\sharp$.*

Proof. ($\Longrightarrow$) Suppose τ is an associated-preserving relation. Let $p \in Z_\tau(x)$ be fixed but arbitrary, then $p\tau x$. Since τ is an associated-preserving relation, $x\tau p'$ for all $p' \sim p$. Thus $p \in A_\tau(x)$, so $Z_\tau(x) \subseteq A_\tau(x)$.

($\Longleftarrow$) Let $x, y \in D^\sharp$ such that $x\tau y$, then $y \in Z_\tau(x) = A_\tau(x)$. By definition of $A_\tau(x)$, $x\tau y$ for all $y \sim y'$. Therefore, τ is an associated-preserving relation.

$\square$

Theorem 3.16. *A symmetric relation τ is associated-preserving if and only if $Z_\tau(x) = Z_\tau(t)$ for all $x \sim t$.*

Proof. Suppose τ is an associated-preserving relation. Let $x, t \in D^\sharp$ such that $x \sim t$. Let $p \in Z_\tau(x)$, then $p\tau x$ and by hypothesis $p\tau t$. Thus, $p \in Z_\tau(t)$ and $Z_\tau(x) \subseteq Z_\tau(t)$. The other containment is similar. Assume that $x\tau y$ and $t \sim y$. By hypothesis

$Z_\tau(x) = Z_\tau(t)$. Now, $x \in Z_\tau(y)$ and $x \in Z_\tau(t)$, implies τ is an associated-preserving relation.

□

The following theorem generalizes the result about some of the equivalences found.

Theorem 3.17. *Let D be an integral domain, τ a symmetric relation on $D^\sharp$ and P a property with respect to τ, then the following statements are equivalents:*

(1) τ satisfy the property P for each $x \in D^\sharp$.

(2) $\tau \subseteq P_\tau(S) \times P_\tau(S)$ for some set $S \subseteq D^\sharp$,

(3) for all $x \in D^\sharp$, there is $S \subseteq D^\sharp$ such that $Z_\tau(x) \subseteq P_\tau(S)$,

(4) $P_\tau(S) = S$ for all $S \subseteq D^\sharp$.

Proof. ((1)$\Longrightarrow$ (2)) If $\tau = \emptyset$, then $\tau \subseteq P_\tau(S) \times P_\tau(S)$ for any $S \subseteq D^\sharp$. So assume $\tau \neq \emptyset$. Suppose τ is a divisive relation and consider $S = \bigcup_{x\in D^\sharp} Z_\tau(x)$. Since τ satisfies the property P, $S \subseteq P_\tau(S)$. But by definition $P_\tau(S) \subseteq S$, so $P_\tau(S) = S$. Moreover, if $x\tau y$, $x, y \in S$. Therefore $(x, y) \in S \times S = P_\tau(S) \times P_\tau(S)$, and consequently $\tau \subseteq P_\tau(S) \times P_\tau(S)$.

((2)$\Longrightarrow$ (3)) Let $x \in D^\sharp$ arbitrary and $y \in Z_\tau(x)$, then by hypothesis $(y, x) \in \tau \subseteq P_\tau(S) \times P_\tau(S)$ for some $S \subseteq P_\tau(S)$ and $y \in P_\tau(S)$. Therefore $Z_\tau(x) \subseteq P_\tau(S)$ for all $x \in D^\sharp$.

((3)$\Longrightarrow$ (4)) Let $S \subseteq D^\sharp$ arbitrary. By definition $P_\tau(S) \subseteq S$. Let $x \in S$ arbitrary. If $Z_\tau(x) = \emptyset$ then $x \in P_\tau(S)$. Assume $Z_\tau(x) \neq \emptyset$ then exist $y \in D^\sharp$ such that $x\tau y$. But by hypothesis there is $S' \subseteq D^\sharp$ such that $Z_\tau(y) \subseteq P_\tau(S')$, hence $x \in P_\tau(S')$. Therefore τ satisfies the property P with respect to x and $x \in S$, then

$x \in P_\tau(S)$. In conclusion $P_\tau(S) = S$ for all $S \subseteq D^\sharp$.

((4)$\Longrightarrow$ (1)) Note that for all $x \in D^\sharp$, $P_\tau(\{x\}) = \{x\}$. Then the proof is concluded, because τ satisfies the property P with respect to x for each $x \in D^\sharp$.

□

3.4 New approach for known theorems

The most important results of the theory of τ-factorization were obtained when multiplicative, associated-preserving and divisive relations were considered. The equivalences can be used to obtain alternative proofs of the theorems in [1], but to try to understand the nature of these types of relations. In some cases it is easier to prove them using the equivalences.

Theorem 3.18. *Let D be an integral domain and τ a symmetric relation on $D^\sharp$. Suppose τ is a divisive relation.*

(1) *Let $x = x_1 \cdot x_2 \cdots x_n$ be a τ-factorization of x, and let $x_i = y_1 \cdot y_2 \cdots y_n$ be a τ-factorization of x_i for some i. Then $x = x_1 \cdots x_{i-1} \cdot y_1 \cdot y_2 \cdots y_n \cdot x_{i+1} \cdots x_n$ is again a τ-factorization of x.*

(2) *Suppose τ is a multiplicative relation. Let $x = \lambda x_1 \cdot x_2 \cdots x_n$ be a τ-factorization of x. Then $x = \lambda x_1 \cdots x_{i-1} \cdot (x_i x_{i+1}) \cdot x_{i+2} \cdots x_n$ is a τ-factorization of x.*

Proof. Let $x = x_1 \cdots x_n$ be a τ-factorization of x and $x_i = y_1 \cdots y_m$ be a τ-factorization of x_i for some $i \in \{1, 2, \cdots n\}$, then $x_j \in Z_\tau(x_i) = Z_\tau(y_1 \cdots y_m)$ for all $i \neq j$. Since τ is divisive $Z_\tau(y_1 \cdots y_m) \subseteq \bigcap_{k=1}^m Z_\tau(y_k)$, hence $x_j \in \bigcap_{k=1}^m Z_\tau(y_k)$, i.e., $x_j \tau y_k$ for all $j \neq i$. So the first statement holds. For the second part, let $x = \lambda x_1 \cdot x_2 \cdots x_n$ be a τ-factorization of x. Then $x_k \in Z_\tau(x_i) \cap Z_\tau(x_{i+1})$ for all $k \neq i, i+1$. But by hypothesis $Z_\tau(x_i) \cap Z_\tau(x_{i+1}) \subseteq Z_\tau(x_i x_{i+1})$ for all $k \neq i, i+1$. Hence, $x_k \in Z_\tau(x_i x_{i+1})$ for all $k \neq i, j$. Therefore, $x = x_1 \cdots x_{i-1} \cdot (x_i x_{i+1}) \cdot x_{i+2} \cdots x_n$

is a τ-factorization of x.

□

Theorem 3.19. *Let τ be a divisive symmetric relation on the integral domain D. Let $x \in D^\sharp$ be a τ-prime element and $y \in D^\sharp$ a τ-atom. Then either $a\tau b$ or ab is a τ-atom.*

Proof. Suppose that is not a τ-atom. Let $ab = x_1 \cdot x_2 \cdots x_n$ be a τ-factorization, then $a \mid x_i$ for some $i \in \{1, 2, \dots, n\}$, say $a \mid x_1$. Write $x_1 = ra$, then $b = r \cdot x_2 \cdots x_n$. If r is an unit, $b = rx_2$. Note that $x_1 = ra$ and $b = rx_2$, and since τ is associated-preserving, $a\tau b$. So, the reader may assume that r is not unit. Note that $x_2, \dots, x_n \in Z_\tau(x_1) = Z_\tau(ra) \subseteq Z_\tau(r) \cap Z_\tau(a)$. In consequence, $x_2, \dots, x_n \in Z_\tau(r)$, i.e., $r \cdot x_2 \cdots x_n$ is a τ-factorization. A contradiction because is a τ-atom.

□

In this chapter equivalences for divisive, multiplicative and associated-preserving relations were found, using Definition 2.2. These results are connections between the theory of τ-factorization and our work, and it is possible to use them to analyse the main type of relations known in the theory of τ-factorization from other perspective.

Chapter 4
OTHER RESULTS

The idea of prime ideal arose from the natural generalization of the notion of a prime in the integers $\mathbb{Z}$, and plays an important role in the theory of commutative rings. The definition of a prime ideal can be recast in the follow way : I is prime if and only if M is multiplicative, where M is the complement of I, see [10]. From the definition of a τ-multiplicative set, the definition of a τ-prime set was considered based on such statement . Unit and zero elements were avoided in the theory of τ-factorization. For this definition would like to keep the property of a τ-prime ideal as defined in [11], without most of the ideal or subring properties.

Theorem 4.1. *Let D be an integral domain, τ a multiplicative symmetric relation on $D^\sharp$ and $M \subseteq D^\sharp$. Then M is a τ-prime set with respect to reduced τ-factorizations if and only if $co_{D^\sharp}(M)$ is a τ-multiplicative set.*

Proof. ($\Longrightarrow$) Suppose that M is a τ-prime set and let $x, y \in co_{D^\sharp}(M)$ be such that $x\tau y$. Since M is a τ-prime set $xy \in M$ implies $x \in M$ or $y \in M$, which it is a contradiction. Hence $xy \in co_{D^\sharp}(M)$ and $co_{D^\sharp}(M)$ is a τ-multiplicative set.

($\Longleftarrow$) Suppose that $co_{D^\sharp}(M)$ is a τ-multiplicative set and let $x_1 \cdot x_2 \cdots x_n \in M$. Suppose that $x_i \in co_{D^\sharp}(M)$ for all $1 \leq i \leq n$, let's prove by using induction that $x_1 \cdot x_2 \cdots x_n \in co_{D^\sharp}(M)$. If $n = 1$, there is nothing to prove. Consider $n \geq 2$. If $n = 2$, then $x_1 \tau x_2$ and $x_1 \cdot x_2$ is a τ-factorization. So that $x_1 x_2 \in co_{D^\sharp}(M)$ by hypothesis. If $n = 3$, using the fact τ is a multiplicative relation and $x_1 \cdot x_2 \cdots x_n$ is a τ-factorization, $x_1 \tau x_2 x_3$ and $x_1 x_2 x_3 \in co_{D^\sharp}(M)$. Now, $x_1 \tau (x_2 \cdots x_{n-1})$ follows for the fact that τ is a multiplicative relation. But $x_1 \tau x_n$, therefore $x_1 \tau (x_2 \cdot x_3 \cdots x_n)$ and $x_1 \cdot x_2 \cdot x_3 \cdots x_n \in co_{D^\sharp}(M)$(as a product, and later as a τ-product because they were originally related), which is a contradiction. In conclusion, M is a τ-prime set.

□

The above theorem focused only on reduced τ-factorizations because on it $\lambda = 1$ is assumed . But if τ is divisive or associated-preserving, such τ-factorizations can be considered. The same result can be obtained instead considering τ to be a divisive relation to consider $M - \{0\}$ to be a τ-prime set, as is illustrated in the following corollary.

Corollary 4.1. *Let D be an integral domain, τ a symmetric relation on $D^\sharp$ and M a proper ideal of D. Suppose that τ is a multiplicative relation, then $M - \{0\}$ is a τ-prime set if and only if $co_{D^\sharp}(M - \{0\})$ is a τ-multiplicative set.*

Proof. Since M is and ideal, if $\lambda a_1 \cdot a_2 \cdots a_n \in M - \{0\}$ is a τ-factorization, then $a_1 \cdot a_2 \cdots a_n \in M - \{0\}$. Consequently, by Theorem 4.1 the proof is concluded.

□

Of course, if $co_{D^\sharp}(M - \{0\})$ is a τ-ideal or a co-τ-saturated set, it was going to obtain also that $M - \{0\}$ is a τ-prime set. Then it is possible to think that since these two τ-sets are τ-multiplicative sets the reciprocal is also true for arbitrary relations. However, it is not true in general, but, there are relations where the reciprocal is also true.

Example 4.0.1. *Take $D = \mathbb{Z}$ and $I = (5)$. For any $x, y \in \mathbb{Z}$, define $x \tau y$ if and only if $2|x, y$. Observe that $4 \in co_{D^\sharp} I$ and $4\tau 20$, but $5|20$ and $5 \notin co_{D^\sharp}(I)$. Since I is a prime ideal, then I is a τ-prime set, but $co_{D^\sharp}(I)$ is not a τ-ideal. On the other* *hand, consider the relation τ_p define as follows: $x\tau_p y$ if and only if $x - y \in (p)$. Now consider $S = (p) - \{0\}$, and notice that S is a τ_p-ideal and also a τ-prime set.*

Theorem 4.2. *Let M be a proper ideal of D and $Z_\tau(t) \neq \varnothing$ for all $t \in M - \{0\}$, then $M = D^\sharp \cup \{0\}$ if and only if $M - \{0\}$ is a co-τ-saturated set.*

Proof. Suppose that $M = D^\sharp \cup \{0\}$, so that $M - \{0\} = D^\sharp$ and $M - \{0\}$ is obviously a co-τ-saturated set. Conversely, since M is a proper ideal, then $M \subseteq D^\sharp \cup \{0\}$. Now let $x \in D^\sharp$ and $y \in M - \{0\}$ be fixed but arbitrary, so $xy \in M - \{0\}$. By hypothesis $Z_\tau(xy) \neq \varnothing$, then there exists $t \in D^\sharp$ such that $xy\tau t$. Therefore $t \in M - \{0\}$, because $M - \{0\}$ is a co-τ-saturated set. But $t\tau xy$, and since $x \mid xy$, $x \in D^\sharp$ and $M - \{0\}$ is a co-τ-saturated set, $x \in M - \{0\}$. In conclusion, $D^\sharp \cup \{0\} \subseteq M$.

$\square$

In Theorem 4.2, $M = D^\sharp \cup \{0\}$ implies that D is a quasi-local ring, that is, a ring with a unique maximal ideal. Then 4.2 provide a sufficient and necessary condition about an integral domain being a quasi-local ring.

Example 4.0.2. *Let $p \in Z$ be a fixed positive prime number. Consider,*

$$\begin{aligned} Z_p &= \{\frac{a}{b} \in Q | b \in Z - (p)\} \\ &= \{\frac{a}{b} \in Q | b \notin (p)\} \\ &= \{\frac{a}{b} \in Q | p \nmid b\} \end{aligned} \tag{4.1}$$

Z_p is already an integral domain.

Let $M = (p)$ and define the relation: $x\tau y$ if and only if $x = y$. Clearly, τ is a reflexive relation, so that $Z_\tau(t) \neq \varnothing$ for all $t \in M - \{0\}$. Let $x \in M - \{0\}$ and $y \in Z_p^\sharp$ such that $x\tau y$, then $x = y \in M - \{0\}$. Hence $y \in (p)$ with $y \neq 0$. Then $M - \{0\}$

is a τ-ideal. Let $t \mid y$ be such that $t \in Z_p^\sharp$. If $t \notin (p)$, then $\frac{1}{t} \in Z_p$. So, $t \in U(Z_p)$ which is a contradiction. Finally $t \in M - \{0\}$, so $M - \{0\}$ is a co-τ-saturated set. By Theorem 4.2, $M - \{0\} = Z_p^\sharp = (p) - \{0\}$.

Theorem 4.3. *Let D be an integral domain, let τ be a symmetric relation on $D^\sharp$ and I a proper ideal of D. Suppose that $I - \{0\}$ is a τ-prime set, then $S = I - \{0\}$ is a τ-ideal if and only if $co_{D^\sharp}(S)$ is a co-τ-saturated set.*

Proof. ($\Longrightarrow$) Suppose that $S = I - \{0\}$ is a τ-ideal. By the fact that I is a τ-prime set and Corollary 4.1 $co_{D^\sharp}(S)$ is a τ-multiplicative set. Let $x \in co_{D^\sharp}(S)$ be fixed but arbitrary and $y \in D^\sharp$ such that $x\tau y$. Suppose $t \mid y$ for some $t \in D^\sharp$. If $t \notin co_{D^\sharp}(S)$, then $t \in S$, so that $y \in S$, because I is an ideal. By hypothesis $x \in S$, which is a contradiction. So, $co_{D^\sharp}(S)$ is a co-τ-saturated set.

($\Longleftarrow$) Suppose $co_{D^\sharp}(S)$ is a co-τ-saturated set. Since I is an ideal, then I is a τ-multiplicative set. Let $x \in S$ and $y \in D^\sharp$ such that $x\tau y$. If $y \notin S$, $y \in co_{D^\sharp}(S)$. Now $x \in co_{D^\sharp}(S)$, which is a contradiction. Therefore S is a τ-ideal.

□

Taking away zero from an arbitrary prime ideal a τ-prime ideal is obtained. So, Theorem 4.3 also holds if it is replaced prime ideal for a τ-prime ideal. It is not difficult to show that if assume that $co_{D^\sharp}(S)$ is a τ-ideal instead of a co-τ-saturated set the same result is obtained, even without considering τ to be a divisive relation. In fact, the complement in $D^\sharp$ of a proper ideal of an integral domain D is a τ-ideal if and only if it is a co-τ-saturated set.

Corollary 4.2. *Let I be an ideal on an integral domain D and τ a symmetric and multiplicative relation, then the following statements are equivalents.*

(1) *$co_{D^\sharp}(I-\{0\})$ is a τ-ideal.*

(2) *$I-\{0\}$ is a τ-prime set and $I-\{0\}$ is a τ-ideal.*

(3) *$co_{D^\sharp}(I-\{0\})$ is a co-τ-saturated set.*

Proof. Observe that if $co_{D^\sharp}(I-\{0\})$ is a τ-ideal and there is $y \in co_{D^\sharp}(I-\{0\})$ such that $Z_\tau(y) \neq \emptyset$, $t \in co_{D^\sharp}(I-\{0\})$ for any factor t of y (I is an ideal and $co_{D^\sharp}(I-\{0\})$ is a τ-ideal). So, (1) implies (2) and by Theorem (2.2), (2) implies (1). Finally, (2) and (3) are equivalents by Theorem 4.3.

□

One of the objectives is to find connections between known theories and the concepts defined, that gives other ways to understand the theory or τ-factorizations and concepts in commutative ring theory, see for example Theorem 4.2. Connections were also found using Definition 2.1(1), this time with the multiplicative sets.

Theorem 4.4. *Let D be an integral domain, M a subset of $D^\sharp$ and τ a symmetric relation on $D^\sharp$. Assume that $Z_\tau(x) \neq \emptyset$ for all $x \in co_{D^\sharp}(M)$. If $co_{D^\sharp}(M)$ is a co-τ-saturated set, then M is a multiplicative set.*

Proof. Let $x, y \in M$ be fixed but arbitrary. Suppose that $xy \in co_{D^\sharp}(M)$, then by hypothesis $Z_\tau(xy) \neq \emptyset$, i.e., there is $t \in D^\sharp$ such that $t\tau xy$. Since $co_{D^\sharp}(M)$ is a co-τ-saturated set $x, y \in co_{D^\sharp}(M)$, a contradiction. In conclusion, M is a multiplicative set.

□

Theorem 4.5. *Let D be an integral domain, τ a multiplicative symmetric relation on $D^\sharp$ and I a proper ideal of D.*

(1) *Suppose there is $t \in co_{D^\sharp}(I-\{0\})$ such that $Z_\tau(t) \supseteq co_{D^\sharp}(I-\{0\})$ and $co_{D^\sharp}(I-\{0\})$ is a τ-ideal, then $I-\{0\}$ is a τ-ideal and I is a prime ideal.*

(2) *If $co_{D^\sharp}(I) \times co_{D^\sharp}(I) \subseteq \tau$ and $co_{D^\sharp}(I)$ is a τ-multiplicative set, then I is a prime ideal.*

Proof. (1) Since τ is a multiplicative relation and $co_{D^\sharp}(I - \{0\})$ is a τ-ideal, by Theorem 4.2, $I - \{0\}$ is a τ-ideal. In fact, $I - \{0\}$ is a τ-prime set. Let $xy \in I - \{0\}$ be arbitrary. Suppose that $x, y \in co_{D^\sharp}(I - \{0\})$, then $x\tau t$ and $y\tau t$. Since τ is multiplicative, $t\tau xy$ and $xy \in co_{D^\sharp}(I - \{0\})$, which it is a contradiction.

(2) Let $xy \in I$ be arbitrary. Suppose neither x nor y are in I, then $x, y \in co_{D^\sharp}(I)$ and by hypothesis $x\tau y$. Finally, since $co_{D^\sharp}(I - \{0\})$ is a τ-multiplicative set, $xy \in co_{D^\sharp}(I - \{0\})$, which is a contradiction. Observe that since $x\tau y$, neither x nor b could be zero. Hence I a is prime ideal.

□

Theorem 4.6. *Let D be an integral domain, τ a symmetric relation on $D^\sharp$ and M a proper ideal of D. Assume that for any ideal I of D such that $M \subseteq I$, $co_{D^\sharp}(I)$ is a τ-ideal. If $Z_\tau(x) \cap Z_\tau(y) \neq \emptyset$ for all $x, y \in D^\sharp$, then M is a maximal ideal or D is local.*

Proof. Suppose that there is an ideal N such that $M \subseteq N \subseteq D$ and $M \neq N$. Let $x \in D^\sharp$ be fixed but arbitrary. Assume that $x \notin M$, then $x \in co_{D^\sharp}(M)$. Let $y \in N - M$ be fixed but arbitrary. So $x, y \in co_{D^\sharp}(M)$ and by hypothesis $Z_\tau(x) \cap Z_\tau(y) \neq \emptyset$. Then there is an element $t \in Z_\tau(x) \cap Z_\tau(y)$ and therefore $t\tau x$ and $t\tau y$. Since $co_{D^\sharp}(N)$ is a τ-ideal, $N - \{0\}$ is a τ-ideal, $t \in N$ and hence $x \in N$. Therefore $D^\sharp \subseteq N$. Then $D^\sharp = N - \{0\}$ or $D = N$. Then D is local or M is maximal.

□

In commutative ring theory, if S is a multiplicative closed set and I is an ideal maximal with respect to the exclusion of S, then I is a prime ideal [10]. An analogous of it was establish with τ-multiplicative sets and τ-prime ideals. The following theorem is the result of the previous idea.

Theorem 4.7. *Let D be an integral domain, τ a symmetry multiplicative relation on $D^\sharp$ and I a proper ideal of D. Let $M \subseteq D^\sharp$ be a τ-multiplicative set such that $Z_\tau(M) \cap M \neq \emptyset$ and suppose that I is maximal ideal with respect to the exclusion of $Z_\tau(M) \cap M$, then $I - \{0\}$ is a τ-prime set.*

Proof. Let $\lambda x_1 \cdot x_2 \cdots x_n \in I - \{0\}$ be a τ-factorization, then $x_1 \cdot x_2 \cdots x_n \in I - \{0\}$. Suppose $x_i \notin I - \{0\}$ for all $i = 1, 2, \ldots, n$, then $(I, x_i) \supsetneq M$ for all $i = 1, 2, \ldots, n$. By hypothesis $(I, x_i) \cap Z_\tau(M) \cap M \neq \emptyset$, therefore for all $i = 1, 2, \cdots, n$ there is $r_i \in (I, x_i) \cap M$ such that $M \subseteq Z_\tau(r_i)$. So, $r_i \tau r_j$ for all $i, j = 1, 2, \ldots, n$. Using induction, the fact that M is a τ-multiplicative set and τ is a multiplicative relation (see proof of Theorem 4.1), $\prod_1^n r_i \in M$, which is a contradiction.

□

Observations Let D be an integral domain, τ a symmetric relation on $D^\sharp$ and $M \subseteq D^\sharp$ a co-τ-saturated set.

(1) If $Z_\tau(x) \neq \emptyset$ for all $x \in M$, then for all $y \in D^\sharp$ such that $y \mid x$, $y \in M$.

(2) If $Z_\tau(x) \cap Z_\tau(y) \cap M \neq \emptyset$ for all $x, y \in M$ and τ is a multiplicative relation, then M is a saturated set in $D^\sharp$.

This chapter presented connections between commutative ring theory and our work. More specifically on how to use the τ-sets to obtain information about ideals, prime ideals, saturated sets and quasi-local domains.

Chapter 5
CONCLUSIONS AND FUTURE WORK

In this chapter, the reader will find a summary of the main results, their importance and suggested question for future research. In the future work section, the author wanted to include some quick results on suggested alternative definitions for the sets $D_\tau(S)$, $M_\tau(S)$ and $A_\tau(S)$, but leaves the rest to be studied later and opens an invitation to the reader to study such sets.

5.1 Conclusions

Let τ be a symmetric relation on $D^\sharp$. The τ-centralizer of an element $x \in D^\sharp$, turns to be an essential tool to further develop the theory of τ-factorizations. Because it gives a more natural way to visualize the concepts and it is a key to understand the most important types of relations so far. Moreover, together with the set $P_\tau(S)$ it suggested that any other type of relation depends of the τ-centralizer of x or $Z_\tau(S)$ where $x \in S$ for some $S \subseteq D^\sharp$. It was possible to give a necessary and sufficient condition for τ, so $Z_\tau(x) = Z_\tau(S) = S$ for some $x \in D^\sharp$. Theorem 2.5 gives a very good characterization of $Z_\tau(x)$ under several hypothesis.

During this investigation it was possible to characterize the main three types of relations, known as: divisive, multiplicative and associated-preserving. The idea emerged from the study of such properties locally at a singleton set, that is, looking for elements of $D^\sharp$ with such properties. The results of this idea give several equivalent statements to the definitions of a divisive, a multiplicative and an associated-preserving relation. Moreover, some of the statements are very similar and provide a generalization of this set for any type of relation. At the same time, this suggested that there are other type of relations to consider. This coincides with the fact that the author in [8] defined other types of relations as combinabled. Also, out of this work suggested to study properties similar as the definitions of type of elements. For example; a relation is called a prime relation if whenever $x\tau yz$, then $x\tau y$ or $x\tau z$. Turns out that a divisive relation is a prime relation, but the converse is false.

Last, a connection between ideals, multiplicative closed sets and the usual commutative ring theory with the theory of τ-factorizations and the τ-sets. The τ-sets defined in this research are the first attempted to define what could be the analogous of an ideal in terms of the τ-products. As a consequence several definitions were considered. This reports only contains those τ-sets than give better results and connections with the usual commutative ring theory.

5.2 Future work

It is suggested to keep finding not artificial (or synthetic) examples for multiplicative, divisive and associated-preserving relations, using the equivalences obtained and study the implications of the τ-sets in the comaximal factorization, bounded factorization domain, half-factorial domain, etc.

From the definition of an associated-preserving relation, for any element x of $D^\sharp$ with non-empty τ-centralizer is related to any element associated with some element in $Z_\tau(x)$. However, there are elements with such property but not for all the elements in its τ-centralizer. Let $x \in D^\sharp$ be arbitrary and consider the set

$$A'_\tau(x) = \{y \in D^\sharp \mid x\tau y' \ \ for \ \ all \ \ y' \sim y\}$$

. Let $y \in A'_\tau(x)$, by definition $y\tau x$ and $y \in Z_\tau(x)$. Then $A'_\tau(x) \subseteq Z_\tau(x)$ for all $x \in D^\sharp$. Analogously

$$D'_\tau(x) = \{y \in D^\sharp \mid x\tau t \ \ for \ \ all \ \ t \mid y \ \ with \ \ t \in D^\sharp\}$$

and

$$M'_\tau(x) = \{y \in Z_\tau(x) \mid x\tau yz \ \ for \ \ all \ \ z \in Z_\tau(x)\}$$

are defined. Also $D'_\tau(x) \subseteq Z_\tau(x)$ and $M'_\tau(x) \subseteq Z_\tau(x)$ for all $x \in D^\sharp$. The following theorem attempt to provide an idea of the definitions of $M'_\tau(x)$, $D'_\tau(x)$ and $A'_\tau(x)$.

Theorem 5.1. *Let τ be a symmetric relation on an integral domain D.*

(1) *The relation τ is associated-preserving if and only if $A'_\tau(x) = Z_\tau(x)$ for all $x \in D^\sharp$.*

(2) *The relation τ is divisive if and only if $D'_\tau(x) = Z_\tau(x)$ for all $x \in D^\sharp$.*

(3) *The relation τ is multiplicative if and only if $M'_\tau(x) = Z_\tau(x)$ for all $x \in D^\sharp$.*

Proof. (1) Suppose τ is an associated-preserving relation. Let $p \in Z_\tau(x)$ be fixed but arbitrary, then $p\tau x$. Since τ is an associated-preserving relation, $x\tau p'$ for all $p' \sim p$. Thus $p \in A'_\tau(x)$, so $Z_\tau(x) \subseteq A'_\tau(x)$. For the converse, let $x, y \in D^\sharp$ be such that $x\tau y$, then $y \in Z_\tau(x) = A'_\tau(x)$. By definition of $A'_\tau(x)$, $x\tau y$ for all $y \sim y'$. Therefore, τ is an associated-preserving relation.

(2) Suppose τ is a divisive relation. The containment $D'_\tau(x) \subseteq Z_\tau(x)$ follows from the definition. Let $y \in Z_\tau(x)$, then $y\tau x$. By definition of a divisive relation, $x\tau t$ for all $t \mid y$. Hence $y \in D'_\tau(x)$. Conversely, suppose $D'_\tau(x) = Z_\tau(x)$ for all $x \in D^\sharp$. Let $x, y \in D^\sharp$ be arbitrary such that $y \in Z_\tau(x)$. By hypothesis $y \in D'_\tau(x)$, then

$t \in Z_\tau(x)$ for all $t \mid y$ with $t \in D^\sharp$. Therefore, by Theorem 3.7 τ is a divisive relation.

(3) Suppose τ is a multiplicative relation. By definition, $M'_\tau(x) \subseteq Z_\tau(x)$. Let $y \in Z_\tau(x)$, then $y\tau x$. Since τ is a multiplicative relation $x\tau yz$ for all $x\tau z$, then $y \in M'_\tau(x)$. For the converse let $x, y, z \in D^\sharp$ such that $x\tau y$ and $x\tau z$. Then $y \in Z_\tau(x) = M'_\tau(x)$, $z \in Z_\tau(x)$ and $x\tau yz$. Therefore τ is a multiplicative relation.

□

Note that it would be nice to have a complete characterization of this new definition.

The connections between $A'_\tau(x)$ (respectively) $D'_\tau(x)$ and $M'_\tau(x)$) and the definitions of τ to be associated-preserving (respectively divisive and multiplicative) with respect to x is given in the following theorem.

Theorem 5.2. *Let τ be a symmetric relation on an integral domain $D^\sharp$ and $x \in D^\sharp$ fixed but arbitrary. Then*

(1) *τ is associated-preserving with respect to x if and only if $A'_\tau(x) = Z_\tau(x)$,*

(2) *τ is divisive with respect to x if and only if $D'_\tau(x) = Z_\tau(x)$,*

(3) *τ is multiplicative with respect to x if and only if $M'_\tau(x) = Z_\tau(x)$.*

Proof. (1) Suppose τ is associated-preserving with respect to x. Let $y \in Z_\tau(x)$, then $x\tau y$. By hypothesis $x\tau y'$ for all $y' \sim y$, hence $y \in A'_\tau(x)$ and $Z_\tau(x) \subseteq A'_\tau(x)$. Conversely, suppose $A'_\tau(x) = Z_\tau(x)$. Assume $y\tau x$, then $y \in A'_\tau(x)$. By definition $x\tau y'$ for all $y' \sim y$, therefore τ is associated-preserving with respect to x.

(2) Suppose τ is divisive with respect to x. Let $y \in Z_\tau(x)$, then $x\tau y$ and by hypothesis $x\tau t$ for all $t \mid y$ with $t \in D^\sharp$. By definition $y \in D'_\tau(x)$ and $Z_\tau(x) \subseteq D'_\tau(x)$. For the converse, assume $x\tau y$ and $t \mid y$ with $t \in D^\sharp$. By hypothesis $y \in Z_\tau(x) =$

$D'_\tau(x)$, then $x\tau t$. Hence τ is divisive with respect to x.

(3) Suppose τ is multiplicative with respect to x. Let $y \in Z_\tau(x)$, then $x\tau y$. Since τ is multiplicative with respect to x, $x\tau yz$ for all $z \in Z_\tau(x)$. Therefore $Z_\tau(x) \subseteq M'_\tau(x)$. For the other direction, let $y, z \in D^\sharp$ such that $x\tau y$ and $x\tau z$, then $x, y \in Z_\tau(x) = M'_\tau(x)$. Therefore $x\tau yz$ and τ is multiplicative with respect to x.

□

Using the definitions of $A'_\tau(x)$, $D'_\tau(x)$ and $M'_\tau(x)$, $A_\tau(S)$, $D_\tau(S)$ and $M_\tau(S)$ can be expressed, in the following way: $A_\tau(S) = S' \cup \{x \in S \mid A'_\tau(x) = Z_\tau(x)\}$, $D_\tau(S) = S' \cup \{x \in S \mid D'_\tau(x) = Z_\tau(x)\}$ and $M_\tau(S) = S' \cup \{x \in S \mid M'_\tau(x) = Z_\tau(x)\}$, where S' is the set of elements in S with empty τ-centralizer.

The author invites the reader to work on this sets and establish similar equivalent statements for divisive, multiplicative and associated-preserving relations as the ones given in chapter 3.

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τ-MULTIPLICATIVE SETS

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