COSEPARATION WITH RESPECT TO AN INTERIOR OPERATOR IN TOPOLOGY

By

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COSEPARACIÓN RESPECTO A UN OPERADOR DE INTERIOR EN TOPOLOGÍA

Por

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Motivados por los resultados obtenidos en el artículo [1], respecto a la noción de separación para un operador de interior en topología, se introduce la noción de I-coseparación para un operador de interior topológico I. Se presentan algunos ejemplos que ilustran el comportamiento de esta noción de coseparación para operadores de interior topológicos concretos. Posteriormente se determina bajo qué propiedades topológicas esta noción es cerrada, de donde se obtiene en particular que los espacios I-coseparados son cerrados bajo la imagen directa de funciones continuas y bajo espacios cocientes, pero no son cerrados bajo suma topológica y subspacios topológicos.

Se prueba que la noción de I-coseparación genera una conexión de Galois entre la clase de todos los operadores de interior topológicos y el conglomerado de todas las subclases de espacios topológicos y usando este resultado se presenta un diagrama conmutativo de conexiones de Galois que muestra la relación entre las nociones de I-separación e I-coseparación. Finalmente se prueba que una caracterización de los espacios I-coseparados, en términos de separadores, análoga a la presentada en [1] para la noción de I-separación, no es posible.

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Motivated by the results obtained in the paper [1], concerning the notion of separation for an interior operator in topology, the notion of I-coseparation for an interior operator I in topology is introduced. A few examples that illustrate the behavior of this notion are presented for concrete interior operators in topology. Subsequently, it is determined under which topological properties this notion is closed. Later, it is obtained that in particular the I-coseparated topological spaces are closed under direct images of continuous functions and under quotient spaces but they are not closed under topological sums and topological subspaces.

It is proved that the notion of I-coseparation generates a Galois connection between the class of all interior operators in topology and the conglomerate of all the subclasses of topological spaces. Using this result, a commutative diagram of Galois connections that shows the relationship between the notions of I-separation and I-coseparation is presented. Finally, it is proved that a characterization of the I-coseparated spaces in terms of separators, analogous to the one presented in [1] for the notion of I-separation, is not possible.

To my parents.

To mrs Maria, my second mother.

To you my beautiful negra.

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TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

In general in mathematics, there is a tendency to always try to generalize or extend basic concepts to more general ones. Due to the importance of Topology in mathematics, it is natural to think about generalizing some topological concepts, such as compactness, connectedness, separations axioms, among others. It is well known that a topology on a set can be defined using the notion of closure operator, or equivalently, the notion of interior operator.

In the books [8] and [12] some generalizations of topological concepts to an arbitrary category, by means of categorical closure operators were shown.

On the other hand, the concept of interior operator was introduced in an arbitrary category by Vorster [6] and subsequently some properties of general nature were studied by Castellini ([3], [4] and [5]). Interior operators defined on the concrete category **Top** of topological spaces and continuous functions were initially used in [2] to introduce and study notions of connectedness and disconnectedness with respect to a topological interior operator. Later, they have also been used to introduce a notion of separation with respect to a topological interior operator [1].

This thesis intends to continue the work done in [1] and [2], by introducing a new topological concept with respect to an interior operator in topology. Precisely the notion of coseparation with respect to an interior operator in topology is introduced, as the counterpart or dual of the notion of separation presented in [1]. More precisely, this work studies the notion of coseparation for a topological interior operator and its relation with the notion of separation given in [1], via Galois connections.

This thesis starts by introducing some definitions and results of General Topology, that are necessary for the development and understanding of this work. These results can be found in [10] and [11].

In Chapter 3, the concept of interior operator on the category **Top** of topological spaces is defined, a list of concrete examples of interior operators on the category **Top** are shown and it is established that the class of all topological interior operators ($IN(\mathbf{Top})$) forms a complete lattice (these results can be seen in [1] and [2]). This chapter is completed by introducing the notion of I-open and I-isolated sets, the latter being the concept by which the I-coseparated spaces are defined, with respect to a topological interior operator I.

Finally in Chapter 4 the notion of coseparation with respect to an interior operator on the category **Top** is defined. With the help of the list of examples of interior operators in topology, given in Chapter 3, several I-coseparated spaces are concretely characterized for each specific interior operator. Then, it is proved that the collection of all I-coseparated spaces is not closed under topological sums nor topological subspaces, but it is closed under direct images of continuous functions and under quotient spaces. Furthermore, it is proved that the notion of I-coseparation generates a Galois connection between the class of all interior operators in topology and the conglomerate of all the subclasses of topological spaces. This Galois connection, together with the one found in the paper [1] for the I-separated spaces, relates the notions of I-coseparation and I-separation through a commutative diagram of Galois connections.

Finally, it is shown that when trying to characterize the I-coseparated spaces in terms of separators, in an analogous way to the characterization presented for the

I-separated spaces in [1], a new notion of coseparation is introduced. We call this new notion I-$2nd$coseparation and its collection of objects is strictly contained in the collection of the original I-coseparated objects. Moreover, it is proved that this notion produces a Galois connection between the class of all interior operators in topology and the conglomerate of all the subclasses of topological spaces that differs from the Galois connection initially introduced for the notion of I-coseparation.

CHAPTER 2
PRELIMINARY CONCEPTS

This chapter introduces a few definitions and results that will be very useful for the development and understanding of this theory. Some subcategories of topological spaces are introduced together with a number of basic results of the concept of connectedness that are later used. These definitions and proofs of the results can be found in [10] and [11].

In order to clarify some terminology, a category is a mathematical entity that consists of objects, usually denoted by X, Y, Z etc. and morphisms, denoted by $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ etc., with a partial operation of composition defined. There are certain conditions to be satisfied but that is beyond the scope of this work and consequently we refer the reader interested in the theory of categories to [7], for instance.

Definition 2.1. *A topology on a set X is a collection $\Im$ of subsets of X having the following properties:*

1. *$\emptyset$ and X are in $\Im$.*
2. *The union of the elements of any subcollection of $\Im$ is in $\Im$.*
3. *The intersection of the elements of any finite subcollection of $\Im$ is in $\Im$.*

A set X for which a topology $\Im$ has been specified is called a topological space. One says that a subset U of X is an open set of X if U belongs to the collection $\Im$.

Definition 2.2. *Let X and Y be topological spaces. A function $X \xrightarrow{f} Y$ is said to be continuous is for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.*

The category of topological spaces will be denoted by **Top**, in this category, the objects are the topological spaces and the morphisms are the continuous functions.

Definition 2.3. *$X \in$ **Top** is said to be irreducible if for every pair of open disjoint sets U and V, $U = \emptyset$ or $V = \emptyset$. **Irred** denotes the category of all irreducible topological spaces.*

Definition 2.4. *$X \in$ **Top** is said to be closed irreducible if for every pair of closed disjoint sets C_1 and C_2, $C_1 = \emptyset$ or $C_2 = \emptyset$. **CIrred** denotes the category of all closed irreducible topological spaces .*

Definition 2.5. *A topological space X is said to be indiscrete if the only open sets in X are $\emptyset$ and X, **Ind** denotes the category of all indiscrete topological spaces.*

Definition 2.6. *Let $X, Y \in$ **Top** and let $X \xrightarrow{q} Y$ be a surjective map. The map q is said to be a quotient map if it satisfies that a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X.*

Definition 2.7. *Let $X \in$ **Top**, let M be a set and let $X \xrightarrow{q} M$ be a surjective map. Then, there exists exactly one topology $\Im$ on M such that q is a quotient map relative to $\Im$. $\Im$ is called the quotient topology induced by q and is defined by*

$$\Im = \left\{ U \subset M : q^{-1}(U) \text{ is open in } X \right\}.$$

That is $U \subseteq M$ is open with respect to $\Im$ if and only if $q^{-1}(U)$ is open in X.

Definition 2.8. *Let $X \in$ **Top**. A separation of X is a pair U, V of disjoint nonempty open subsets of X, such that $X = U \cup V$. The space X is said to be connected if a separation of X does not exist. The category of all connected topological spaces is denoted by* ***Conn****.*

Remark 2.9. *Note that if U, V is a separation of X, then they are disjoint nonempty open subsets of X, such that $X = U \cup V$. So $U = X - V$ and $V = X - U$. Hence U and V are open and closed sets in X, that is they are clopen sets in X.*

The following proposition shows another way of formulating the definition of connectedness.

Proposition 2.10. *A topological space X is connected if and only if the only subsets of X that are clopen in X are $\emptyset$ and X.*

Lemma 2.11. *Let $X \in$ **Top** and let Y be a connected subspace of X. If U and V form a separation of X, then Y lies entirely within either U or V.*

Theorem 2.12. *Let $X \in$ **Top** and let $\{A_i\}_{i \in I}$ be a family of connected subspaces of X, such that $\bigcap_{i\in I} A_i \neq \emptyset$. Then $\bigcup_{i \in I} A_i$ is connected.*

Proposition 2.13. *Let $X \in$ **Top**, $B \subseteq X$ and let A be a connected subspace of X. If $A \subseteq B \subseteq \bar{A}$ then B is also connected.*

Proposition 2.14. *The direct image of a connected space under a continuous function is a connected space.*

Theorem 2.15. *The product of connected spaces is connected.*

Definition 2.16. *Let $X \in$ **Top** and let $x, y \in X$. An equivalence relation on X is defined by setting $x \sim y$ if there is a connected subspace of X containing both of them. The equivalence classes are called the components or the connected components of X.*

Remark 2.17. *Note that from the definition, the components of a topological space X are disjoint subsets of X whose union is X, since they are the equivalence classes of an equivalence relation.*

Proposition 2.18. *Let $X \in$ **Top**, then the following statement are true:*

1. *If A is a nonempty connected subspace of X, then A intersects only one component of X.*
2. *The components are connected subspaces of X.*

Remark 2.19. *Note that the components are maximal connected subsets of X. Indeed, let C be a component of X and let A be a connected subset of X, such that $C \subseteq A$. Then by 2. of the previous proposition, C is a connected subspace of X, and as $A \cap C \neq \emptyset$, by 1. $A \subseteq C$. Hence $A = C$ and therefore C is not a subset of some other connected subspace of X.*

The following proposition is a result of the previous Remark.

Proposition 2.20. *A topological space X is connected if and only if X has only one connected component.*

Proposition 2.21. *Let $X \in$ **Top** and let $a, b \in X$. If $C_{(a,b)}$ is a connected component containing (a, b) on $X \times X$, then*

$$C_{(a,b)} = C_a \times C_b.$$

Where C_a and C_b are connected components containing a and b, respectively on X.

Definition 2.22. *Let $\{X_i\}_{i\in I}$ be a nonempty family of nonempty sets. The disjoint union of this family is defined by the set*

$$\coprod_{i\in I} X_i = \bigcup_{i\in I}(X_i \times \{i\})$$

where $X_i \times \{i\} = \{(x,i) : x \in X_i\}$, for all $i \in I$. If $X = X_i$ for all $i \in I$ then

$$\coprod_{i\in I} X_i = X \times \{i\}_{i\in I}.$$

Definition 2.23. *Let $\{X_i\}_{i\in I}$ be a nonempty family of topological spaces and let*

$$\coprod_{i\in I} X_i$$

be the disjoint union of the family $\{X_i\}_{i\in I}$. The canonical injection is defined by

$$\begin{cases} \varphi_j : X_j \longrightarrow \coprod_{i\in I} X_i \\ \qquad \varphi_j(x) = (x,j) \end{cases}$$

The topological sum (disjoint union topology) is defined by

$$\Im = \left\{ U \subseteq \coprod_{i\in I} X_i : \varphi_j^{-1}(U) \text{ is open in } X_j, \text{ for every } j \in I \right\}.$$

That is a subset U of $\coprod_{i\in I} X_i$ is open in $\coprod_{i\in I} X_i$ if and only if $\varphi_j^{-1}(U)$ is open in X_j for every $j \in I$.

The example below relates the concepts of connectedness and topological sum, which shows that connectedness is not closed under topological sums. This result will be useful later when we introduce the concept of coseparation with respect to an interior operator.

Example 2.24. *Let $\{X_i\}_{i\in I}$ be a family of connected topological spaces, it will be proved that the topological sum $\coprod_{i\in I} X_i$ is not connected. To do this it will be proved that there is a nonempty proper subset of $\coprod_{i\in I} X_i$ that is clopen and then Theorem 2.10 is applied. Indeed, consider the nonempty proper subset $X_k \times \{k\}$ of $\coprod_{i\in I} X_i$, for some $k \in I$. As*

$$\varphi_j^{-1}\left(X_k \times \{k\}\right) = \begin{cases} X_j & if \quad j = k \\ \emptyset & if \quad j \neq k \end{cases}$$

and $\emptyset$, X_j are open sets in X_j, for some arbitrary j in I, then $\varphi_j^{-1}\left(X_k \times \{k\}\right)$ is open set in X_j for every $j \in I$. So by definition of topological sum $X_k \times \{k\}$ is an open set in $\coprod_{i\in I} X_i$.

Now it will proved that the set

$$\coprod_{i\in I} X_i - (X_k \times \{k\})$$

is open and hence that $(X_k \times \{k\})$ is closed in $\coprod_{i\in I} X_i$. We have that

$$\varphi_j^{-1}\left(\coprod_{i\in I} X_i - (X_k \times \{k\})\right) = \begin{cases} X_j & if \quad j \neq k \\ \emptyset & if \quad j = k \end{cases}$$

Therefore

$$\varphi_j^{-1}\left(\coprod_{i\in I} X_i - (X_k \times \{k\})\right)$$

is a open set in X_j, for every $j \in I$. Hence the nonempty proper subset $X_k \times \{k\}$ is clopen in $\coprod_{i\in I} X_i$. In this manner by Theorem 2.10, $\coprod_{i\in I} X_i$ is not connected.

CHAPTER 3
INTERIOR OPERATORS

In this chapter the concept of interior operator in **Top** is defined, a few examples illustrating this concept are shown and the notion of isolated sets and open sets with respect to a topological interior operator are also introduced. Subsequently, the supremum and the infimum of a family of interior operators are defined and a result is shown that proves that they are indeed interior operators. Finally, the isolated sets with respect to the supremum of a family of interior operators in **Top** are characterized.

Definition 3.1. *Let $I = (i_X)_{X \in \mathbf{Top}}$ be an indexed family of functions on the subset lattices of* **Top**, *where every i_X is a function defined by*

$$\begin{cases} S(X) \xrightarrow{i_X} S(X) \\ M \longmapsto i_X(M), \end{cases}$$

and $S(X)$ is the collection of all subsets of X, ordered by inclusion.
The family I is called an interior operator on the category **Top**, *if for every $X \in$* **Top**, *i_X satisfies the following properties*

a. *Contractiveness: For every $M \in S(X)$, $i_X(M) \subseteq M$.*

b. *Monotonicity: For every $M_1, M_2 \in S(X)$ such that $M_1 \subseteq M_2$,*
   $i_X(M_1) \subseteq i_X(M_2)$.

c. *Continuity: For every* $X, Y \in \mathbf{Top}$ *and continuous function* $X \xrightarrow{f} Y$ *with* $N \in S(Y)$, $f^{-1}(i_Y(N)) \subseteq i_X(f^{-1}(N))$.

The class of all interior operators on $\mathbf{Top}$ *is denoted by* $IN(\mathbf{Top})$.

Definition 3.2. *Let* $I = (i_X)_{X \in \mathbf{Top}}$, $J = (j_X)_{X \in \mathbf{Top}}$ *be interior operators on* $\mathbf{Top}$. *A partial order* $\sqsubseteq$ *on* $IN(\mathbf{Top})$ *is defined by:*

$I \sqsubseteq J$ *if and only if for every* $X \in \mathbf{Top}$ *and every* $M \subseteq X$

$$i_X(M) \subseteq j_X(M).$$

Remark 3.3. *Note that the pair* $(IN(\mathbf{Top}), \sqsubseteq)$ *is a partially ordered class, since by definition, the relation* $\sqsubseteq$ *is reflexive, antisymmetric and transitive.*

Now a few examples of interior operators are presented, whose demonstrations can be found in [1]. For every one of them, let $X \in \mathbf{Top}$ and let $M \in S(X)$.

Examples 3.4.

1. $K = (k_X)_{X \in \mathbf{Top}}$, *where*

$$k_X(M) = \bigcup\{\mathcal{O} \subseteq M : \mathcal{O} \text{ is open in } X\}.$$

2. $L = (l_X)_{X \in \mathbf{Top}}$, *where*

$$l_X(M) = \{x \in X : C_x \subseteq M\},$$

and C_x *is the connected component of* x *in* X.

3. $Q = (q_X)_{X \in \mathbf{Top}}$, *where*

$$q_X(M) = \bigcup\{C \subseteq M : C \text{ is clopen in } X\}.$$

4. $B = (b_X)_{X \in \mathbf{Top}}$, *where*

$$b_X(M) = \left\{ x \in X : \exists\, U_x \text{ nbhd of } x \quad s.t.\, U_x \cap \overline{\{x\}} \subseteq M \right\}.$$

5. $\Theta = (\theta_X)_{X \in \mathbf{Top}}$, *where*

$$\theta_X(M) = \left\{ x \in M : \exists\, U_x \text{ nbhd of } x \quad s.t.\, \overline{U}_x \subseteq M \right\}.$$

6. $H = (h_X)_{X \in \mathbf{Top}}$, *where*

$$h_X(M) = \bigcup \{ C \subseteq M : C \text{ is closed in } X \}.$$

The following proposition shows the existence of infima and suprema in $IN(\mathbf{Top})$, which together with Remark 3.3 proves that $IN(\mathbf{Top})$ is a complete lattice and whose demonstration can be found in [2].

Proposition 3.5. *Let* $\{I_k\}_{k \in K}$ *be an indexed family of interior operators on* **Top**, *with* $I_k = ((i_k)_X)_{X \in \mathbf{Top}}$. *For* $X \in \mathbf{Top}$ *and every* $M \subseteq X$, *one defines:*

a. $\bigwedge_{k \in K} I_k = \left((i_{\wedge I_k})_X \right)_{X \in \mathbf{Top}}$, *by*

$$(i_{\wedge I_k})_X\,(M) = \bigcap_{k \in K} (i_k)_X\,(M).$$

b. $\bigvee_{k \in K} I_k = \left((i_{\vee I_k})_X \right)_{X \in \mathbf{Top}}$, *by*

$$(i_{\vee I_k})_X\,(M) = \bigcup_{k \in K} (i_k)_X\,(M).$$

Then $\bigwedge_{k \in K} I_k$ *and* $\bigvee_{k \in K} I_k$ *are interior operators on* **Top**, *and they are the infimum and supremum, respectively of the indexed family* $\{I_k\}_{k \in K}$.

Definition 3.6. *Let* $I = (i_X)_{X \in \mathbf{Top}}$ *be an interior operator on* **Top**. *Let* $X \in \mathbf{Top}$ *and let* $M \subseteq X$. M *is* I*-isolated if* $i_X(M) = \emptyset$, M *is called* I*-open if* $i_X(M) = M$.

The proof of the following result can be found in [2].

Proposition 3.7. *Let* $\{I_k\}_{k\in K} \subseteq IN(\mathbf{Top})$ *with* $I_k = ((i_k)_X)_{X\in\mathbf{Top}}$ *and* $X \in \mathbf{Top}$. *Then for every* $M \subseteq X$ *one has that* M *is* $\bigvee_{k\in K} I_k$*-isolated if and only if* M *is* I_k*-isolated for every* $k \in K$.

CHAPTER 4
I-COSEPARATION

The notion of I-coseparation for an interior operator on the category **Top** is introduced as the counterpart of the concept of I-separation developed in [1], which emerges as a natural extension of the notion of Hausdorff spaces for an interior operator on the category **Top**. For I-separation, the notion of the separator of two continuous functions being open was used. Then, in [1] it is also shown that a topological space X is I-separated if and only if the complement of the diagonal is I-open, that is $(i_X)\left(\complement\Delta_X\right) = \complement\Delta_X$.

Following this idea, the notion of I-coseparation with respect to an interior operator in **Top** is introduced as a topological space X such that the complement of the diagonal is I-isolated, that is $(i_X)\left(\complement\Delta_X\right) = \emptyset$. In this chapter some topological properties of this notion and its relation with I-separation will be studied.

Definition 4.1. *Let I be an interior operator on the category* **Top**. *$X \in$* **Top** *is I-coseparated if $\complement\Delta_X = X \times X - \Delta_X$ is I-isolated, that is if $i_{X^2}(\complement\Delta_X) = \emptyset$. $Cosep(I)$ will denote all coseparated objects with respect to I.*

In the following examples the interior operators defined in Examples 3.4 are considered. M will always denote a subset of the topological space X and $\overline{M}$ will denote the topological closure of M.

Example 4.2. *Let us consider the interior operator K, defined in Example 3.4(1), by*

$$k_X(M) = \bigcup\{\mathcal{O} \subseteq M : \mathcal{O} \textit{ is open in } X\}.$$

It is claimed that $Cosep(K) = \mathbf{Irred}$, that is, $X \in Cosep(K)$ if only if for every pair of disjoint open sets U, V, one has $U = \emptyset$ or $V = \emptyset$. Indeed, if $X \in Cosep(K)$ then X is K-coseparated or equivalently $k_{X^2}(\complement\Delta_X) = \emptyset$, where

$$k_{X^2}(\complement\Delta_X) = \bigcup\{\mathcal{O} \subseteq \complement\Delta_X : \mathcal{O} \textit{ is open in } \ X \times X\}.$$

Let U, V be disjoint open sets in X, then $U \times V$ is also open in $X \times X$ and $U \times V \subseteq \complement\Delta_X$. So, by definition of the interior operator K and the monotonicity, it follows that

$$U \times V = k_{X^2}(U \times V) \subseteq k_{X^2}(\complement\Delta_X) = \emptyset$$

therefore $U \times V = \emptyset$. Hence $U = \emptyset$ or $V = \emptyset$. Thus $Cosep(K) \subseteq$ **Irred**.

Suppose now that if U and V are any disjoint open sets in X, then $U = \emptyset$ or $V = \emptyset$. It will be proved that $k_{X^2}(\complement\Delta_X) = \emptyset$. Assume that $k_{X^2}(\complement\Delta_X) \neq \emptyset$. Then, there is a point $(x, y) \in X \times X$ such that $(x, y) \in k_{X^2}(\complement\Delta_X)$, hence there is a neighborhood $U_{(x,y)}$ of (x, y) such that $U_{(x,y)} \subseteq k_{X^2}(\complement\Delta_X)$. Since $(x, y) \in U_{(x,y)}$, there are open sets U, V in X, such that $x \in U$, $y \in V$ and

$$(x, y) \in U \times V \subseteq U_{(x,y)} \subseteq k_{X^2}(\complement\Delta_X) \subseteq \complement\Delta_X.$$

Thus U, V are disjoint open sets in X and it follows that $U = \emptyset$ or $V = \emptyset$. This implies that $U \times V = \emptyset$, that is a contradiction since $(x, y) \in U \times V$. Therefore $k_{X^2}(\complement\Delta_X) = \emptyset$ or equivalently $X \in Cosep(K)$. Thus **Irred** $\subseteq$ *$Cosep(K)$.*

Example 4.3. *Let us consider the interior operator L, defined in Example 3.4(2), by*

$$l_X(M) = \{x \in X : C_x \subseteq M\},$$

where C_x denotes the connected component of x in X. It is claimed that $Cosep(L) = \mathbf{Conn}$, that is, X is connected if and only if $l_{X^2}(\complement\Delta_X) = \emptyset$, where

$$l_{X^2}(\complement\Delta_X) = \{(x, y) \in X \times X : C_{(x,y)} \subseteq \complement\Delta_X\}$$

and $C_{(x,y)}$ is a connected component of (x, y) in $X \times X$.
Let us prove that $Cosep(L) \subseteq \mathbf{Conn}$. Note that by Proposition 2.20 a topological space is connected if and only if it has only one connected component. Let $X \notin \mathbf{Conn}$ and let $a, b \in X$. Then, X has at least two disjoint connected components C_a, C_b such that $a \in C_a$ and $b \in C_b$. Since $(a, b) \in X \times X$ then there is a connected component $C_{(a,b)}$ of (a, b) in $X \times X$. Thus by Proposition 2.21

$$C_{(a,b)} = C_a \times C_b,$$

where $C_a \cap C_b = \emptyset$ and therefore $C_{(a,b)} \subseteq \complement\Delta_X$. This implies that $(a, b) \in l_{X^2}\left(\complement\Delta_X\right)$ and so $l_{X^2}\left(\complement\Delta_X\right) \neq \emptyset$. That is $X \notin Cosep(L)$ and by contrapositive $Cosep(L) \subseteq$ **Conn**.

Now, in order to prove that $\mathbf{Conn} \subseteq Cosep(L)$. Suppose that $X \in \mathbf{Conn}$ but $X \notin Cosep(L)$ i.e. $l_{X^2}(\complement\Delta_X) \neq \emptyset$ then there is $(a, b) \in X \times X$ such that $C_{(a,b)} \subseteq \complement\Delta_X$, where $C_{(a,b)}$ is the connected component of (a, b) in $X \times X$. Since X is connected, by Theorem 2.15, $X \times X$ is also connected and thus $X \times X$ has only one connected component, namely $C_{(a,b)}$. In this manner $(x, x) \in X \times X$ implies that $(x, x) \in C_{(a,b)}$ and hence $C_{(a,b)} \cap \Delta_X \neq \emptyset$. That is a contradiction, because $C_{(a,b)} \subseteq \complement\Delta_X$. Therefore $X \in Cosep(L)$.

Example 4.4. *Let us consider the interior operator Q, defined in Example 3.4(3), by*

$$q_X(M) = \bigcup\{C \subseteq M : C \quad \text{is clopen in } X\}.$$

It is claimed that $Cosep(Q) = \mathbf{Conn}$, that is, X is connected if and only if $q_{X^2}(\complement\Delta_X) = \emptyset$, where

$$q_{X^2}(\complement\Delta_X) = \bigcup\{C \subseteq \complement\Delta_X : C \quad \text{is clopen in} \quad X \times X\}.$$

To see that $Cosep(Q) \subseteq \mathbf{Conn}$, we use the contrapositive approach. Assume that $X \notin \mathbf{Conn}$. Then, there is a separation U, V of X, that is by Remark 2.9 U, V are disjoint non empty clopen sets in X. This implies that $U \times V$ is clopen in $X \times X$ and $U \times V \subseteq \complement\Delta_X$, since U and V are disjoint. Hence $q_{X^2}(\complement\Delta_X) \neq \emptyset$ and therefore $X \notin Cosep(Q)$.

Now, in order to prove that $\mathbf{Conn} \subseteq Cosep(Q)$, assume that X is connected but $X \notin Cosep(Q)$ that is $q_{X^2}(\complement\Delta_X) \neq \emptyset$. Then, there is a subset $C \subseteq \complement\Delta_X$ that is not empty and such that C is clopen in $X \times X$. Since X is connected, $X \times X$ is also connected. By Proposition 2.10 this implies that $C = \emptyset$ or $C = X \times X$. However C is not empty and thus

$$X \times X = C \subseteq \complement\Delta_X.$$

This is a contradiction and therefore $X \in Cosep(Q)$.

Example 4.5. *Let us consider the interior operator B, defined in Example 3.4(4), by*

$$b_X(M) = \left\{x \in X : \exists\, U_x \text{ nbhd of } x \quad \text{s.t. } U_x \cap \overline{\{x\}} \subseteq M\right\}.$$

It is claimed that $Cosep(B) = \mathbf{Ind}$, that is, $X \in \mathbf{Ind}$ if and only if $b_{X^2}(\complement\Delta_X) = \emptyset$, where

$$b_{X^2}(\complement\Delta_X) = \left\{(x,y) \in X \times X : \exists\, U_{(x,y)} \text{ nbhd of } (x,y) \quad \text{s.t. } U_{(x,y)} \cap \overline{\{(x,y)\}} \subseteq \complement\Delta_X\right\}.$$

To see that $\mathbf{Ind} \subseteq Cosep(B)$, *suppose that* $X \in \mathbf{Ind}$ *but* $X \notin Cosep(B)$, *that is* $b_{X^2}(\complement\Delta_X) \neq \emptyset$. *Then, there is* $(x,y) \in X \times X$ *and a neighborhood* $U_{(x,y)}$ *of* (x,y) *such that*

$$U_{(x,y)} \cap \overline{\{(x,y)\}} \subseteq \complement\Delta_X.$$

Since X *is an indiscrete topological space, then* $X \times X$ *is also an indiscrete topological space and thus* $U_{(x,y)} = X \times X$ *is the only non-empty open set of* $X \times X$. *On the other hand, since* $X \times X \in \mathbf{Ind}$, *then*

$$\overline{\{(x,y)\}} = \overline{\{x\}} \times \overline{\{y\}} = X \times X.$$

and this implies that

$$X \times X = (X \times X) \cap (X \times X) = U_{(x,y)} \cap \overline{\{(x,y)\}} \subseteq \complement\Delta_X.$$

That is a contradiction and therefore $X \in Cosep(B)$.

To check that $Cosep(B) \subseteq \mathbf{Ind}$, *the following will be proved:*

a. *If* $X \in Cosep(B)$, *then for every* $(x,y) \in \complement\Delta_X$ *and every neighborhood* U_x *of* x *and* V_y *of* y, *one has that* $y \in U_x$ *and* $x \in V_y$.

b. *Let* $X \in \mathbf{Top}$ *satisfy that for any* $x, y \in X$ *with* $x \neq y$, *every neighborhood* U_x *of* x *contains* y *and every neighborhood* V_y *of* y *contains* x, *then* $X \in \mathbf{Ind}$.

Indeed,

a. *Let* $X \in Cosep(B)$, *that is* $b_{X^2}(\complement\Delta_X) = \emptyset$, *also for any* $(x,y) \in X \times X$ *and every neighborhood* $U_{(x,y)}$ *of* (x,y)

$$U_{(x,y)} \cap \overline{\{(x,y)\}} \nsubseteq \complement\Delta_X.$$

In particular for $(x,y) \in \complement\Delta_X$, *that is* $x \neq y$, *it follows that if* U_x *and* V_y *are neighborhoods of* x *and* y, *respectively such that*

$$U_x \times V_y \cap \overline{\{(x,y)\}} \cap \Delta_X \neq \emptyset,$$

then there is $(z,z) \in U_x \times V_y$, *that is* $z \in U_x \cap V_y$. *Moreover,* $(z,z) \in \overline{\{(x,y)\}}$, *and thus* $(x,y) \in U_{(z,z)}$ *for every neighborhood* $U_{(z,z)}$ *of* (z,z). *This implies that* $(x,y) \in U_z \times U_z$, *that is* $x, y \in U_z$ *for every neighborhood* U_z *of* z. *Since* $U_x \cap V_y$ *is a neighborhood of* z, *then* $x, y \in U_x \cap V_y$. *Hence* $y \in U_x$ *and* $x \in V_y$.

b. *Suppose that* X *satisfies the hypothesis of b. If* X *were not an indiscrete space, then there would be a non-trivial open set* U *in* X. *Now if* $x \in U$ *and* $y \in X - U$, *then* $(x,y) \in \complement\Delta_X$ *and* $x \in U$, $y \in X$ *but* $y \notin U$. *Therefore* X *must be an indiscrete topological space. Thus from a. and b.* $Cosep(B) \subseteq$ **Ind**.

Example 4.6. *Let us consider the interior operator* Θ, *defined in Example 3.4(5), by*

$$\theta_X(M) = \left\{ x \in M : \exists\, U_x,\ \text{nbhd of } x, \quad \text{s.t. } \overline{U}_x \subseteq M \right\}.$$

It will be proved that $X \in Cosep(\Theta)$ *if and only if for every pair of open sets* U *and* V *such that* $\overline{U} \cap \overline{V} = \emptyset$, *then* $\overline{U} = \emptyset$ *or* $\overline{V} = \emptyset$.
Let $X \in Cosep(\Theta)$, *then* $\theta_{x^2}\left(\complement\Delta_X\right) = \emptyset$, *where*

$$\theta_X\left(\complement\Delta_X\right) = \left\{ (x,y) \in \complement\Delta_X : \exists\, U_{(x,y)},\ \text{nbhd of } (x,y), \quad \text{s.t. } \overline{U}_{(x,y)} \subseteq \complement\Delta_X \right\}.$$

Assume that there are two non-empty open sets U *and* V, *such that* $\overline{U} \cap \overline{V} = \emptyset$ *but* $\overline{U} \neq \emptyset$ *and* $\overline{V} \neq \emptyset$. *Since* U *and* V *are not empty, there are elements* $x \in U$ *and* $y \in V$, *that is* $(x,y) \in U \times V$ *and consequently* $U \times V$ *is a neighborhood of* (x,y). *Since by hypothesis* $\overline{U}$ *and* $\overline{V}$ *are disjoint sets, then*

$$(x,y) \in U \times V \subseteq \overline{U \times V} = \overline{U} \times \overline{V} \subseteq \complement\Delta_X.$$

This implies that $(x,y) \in \theta_{x^2}(\complement\Delta_X)$ *and therefore* $\theta_{x^2}(\complement\Delta_X) \neq \emptyset$, *that is a contradiction. This concludes the first implication.*

To prove the converse, suppose that $X \notin Cosep(\Theta)$ *that is* $\theta_{x^2}(\complement\Delta_X) \neq \emptyset$*. Thus, there is* $(x,y) \in \complement\Delta_X$ *and a neighborhood* $U_{(x.y)}$ *of* (x,y) *such that* $\overline{U}_{(x.y)} \subseteq \complement\Delta_X$*. Since* $U_{(x,y)}$ *is a neighborhodd of* (x,y)*, there are open sets* U *and* V*, such that* $(x,y) \in U \times V \subseteq U_{(x,y)}$ *and consequently,*

$$(x,y) \in U \times V \subseteq \overline{U} \times \overline{V} = \overline{U \times V} \subseteq \overline{U}_{(x,y)} \subseteq \complement\Delta_X.$$

Therefore $\overline{U} \cap \overline{V} = \emptyset$ *and by hypothesis* $\overline{U} = \emptyset$ *or* $\overline{V} = \emptyset$*. This implies that* $\overline{U} \times \overline{V} = \emptyset$*, but*

$$(x,y) \in U \times V \subseteq \overline{U \times V} = \overline{U} \times \overline{V} = \emptyset,$$

that is a contradiction. Therefore $\theta_{x^2}(\complement\Delta_X) = \emptyset$ *or equivalently* $X \in Cosep(\Theta)$ *and consequently the desired result is proved.*

Example 4.7. *Let us consider the interior operator* H*, defined in Example 3.4(6), by*

$$h_X(M) = \bigcup\{C \subseteq M : C \text{ is closed in } X\}.$$

It is claimed that **Ind** $\subseteq Cosep(H) \subseteq$ **CIrred***.*
Assume that X *is an indiscrete topological space. Then,* $X \times X$ *is also indiscrete and thus the only closed subsets are* $\emptyset$ *and* $X \times X$*. It follows that there are no non-empty subsets of* $\complement\Delta_X$ *that are closed in* $X \times X$*. Therefore,*

$$h_{x^2}(\complement\Delta_X) = \bigcup\{C \subseteq \complement\Delta_X : C \text{ is closed in } X \times X\} = \emptyset.$$

Hence, $X \in Cosep(H)$*, that is* **Ind** $\subseteq Cosep(H)$*.*

To prove that $Cosep(H) \subseteq$ **CIrred***. We assume that* $X \in Cosep(H)$*, that is* X *is* H*-coseparated or equivalently* $h_{X^2}(\complement\Delta_X) = \emptyset$*. Let* C_1, C_2 *be closed disjoint sets in* X*, then* $C_1 \times C_2$ *is also closed in* $X \times X$ *and* $C_1 \times C_2 \subseteq \complement\Delta_X$*. Consequently, by* *definition of the interior operator H and monotonicity, it follows that*

$$C_1 \times C_2 = h_{X^2}(C_1 \times C_2) \subseteq h_{X^2}(\complement\Delta_X) = \emptyset$$

that is, $C_1 \times C_2 = \emptyset$. Hence $C_1 = \emptyset$ or $C_2 = \emptyset$. Thus $Cosep(H) \subseteq$ **CIrred**.

The following proposition shows that $Cosep(I)$ is closed under the direct image of a continuous function.

Proposition 4.8. *The direct image of an I-coseparated space under a continuous function is I-coseparated.*

Proof. Let $X, Y \in$ **Top**, with $X \in Cosep(I)$, i.e. X is an I-coseparated space and let $X \xrightarrow{f} Y$ be a continuous function. It will be proved that $Z = f(X)$ is I-coseparated. Since the surjective function $X \xrightarrow{g} Z$ obtained from f by restricting its range to the space Z is also continuous, then without loss of generality, one can assume that $X \xrightarrow{f} Y$ is a surjective continuous function, and prove that $Y \in Cosep(I)$. Suppose otherwise, that is, $i_{Y^2}(\complement\Delta_Y) \neq \emptyset$. Since $X \xrightarrow{f} Y$ is a continuous and surjective map, then the function h defined by

$$\begin{cases} h : X \times X \longrightarrow Y \times Y \\ h\left((x_1, x_2)\right) = (f(x_1), f(x_2)) \end{cases}$$

is also a continuous and surjective map. Let $\mathbf{y} \in i_{Y^2}(\complement\Delta_Y) \subseteq \complement\Delta_Y$, then there is $\mathbf{x} \in X \times X$ such that $h(\mathbf{x}) = \mathbf{y}$. Hence,

$$\mathbf{x} \in h^{-1}\left(i_{Y^2}(\complement\Delta_Y)\right) \subseteq i_{X^2}\left(h^{-1}(\complement\Delta_Y)\right) \subseteq i_{X^2}\left(\complement\Delta_X\right),$$

Where the first inclusion is obtained from the continuity property of interior operators. Consequently, $\mathbf{x} \in i_{X^2}\left(\complement\Delta_X\right)$ and thus $i_{X^2}\left(\complement\Delta_X\right) \neq \emptyset$. This contradicts the fact that $X \in Cosep(I)$. Hence $i_{Y^2}(\complement\Delta_Y) = \emptyset$ and therefore $Y \in Cosep(I)$.

Note that in this proof

$$i_{X^2}\left(h^{-1}(\complement\Delta_Y)\right) \subseteq i_{X^2}\left(\complement\Delta_X\right).$$

This is a consequence of $h^{-1}(\complement\Delta_Y) \subseteq \complement\Delta_X$ and this occurs because if $h^{-1}(\complement\Delta_Y)\cap \Delta_X \neq \emptyset$, there is $(x,x) \in h^{-1}(\complement\Delta_Y) = (f \times f)^{-1}\left(\complement\Delta_Y\right)$. Hence $(f(x), f(x)) \in \complement\Delta_Y$ and this is a contradiction. $\square$

An immediate consequence of the previous proposition, shown in the following corollary, is the fact that $Cosep(I)$ is closed under quotient spaces. This is a consequence of the fact that all quotient maps are continuous and surjective and therefore if $X \xrightarrow{q} Y$ is a quotient map and $X \in Cosep(I)$, then from Proposition 4.8, $Y \in Cosep(I)$.

Corollary 4.9. *$Cosep(I)$ is closed under quotient spaces.*

Remark 4.10. *Notice that from Examples 4.3 and 4.4, one has that $Cosep(L) =$* **Conn** *and $Cosep(Q) =$* **Conn**, *that is in both cases the coseparated spaces under their respective interior operators are connected topological spaces. Moreover, Example 2.24 shows that connected spaces are not closed under the formation of topological sums (disjoint union topology or coproduct topology). This yields the important result that $Cosep(I)$ is not closed under topological sums.*

From Examples 4.3 and 4.4 one can also conclude that $Cosep(I)$ is not closed under topological subspaces, since if one considers the topological space $\mathbb{R}$ with the usual topology, then $\mathbb{R}$ is connected but $\mathbb{Q} \subseteq \mathbb{R}$ is not a connected subspace.

From now on $S(\mathbf{Top})$ will denote the conglomerate of all subclasses of objects of $\mathbf{Top}$, ordered by inclusion and as already mentioned in Chapter 3, $IN(\mathbf{Top})$ denotes the class of all interior operators on $\mathbf{Top}$ ordered as in Definition 3.2. The following lemma and proposition show a relationship between $IN(\mathbf{Top})$ and $S(\mathbf{Top})^{op}$, where $S(\mathbf{Top})^{op}$ represents the same as $S(\mathbf{Top})$ but with inverted order. That is for $\mathcal{A}, \mathcal{B} \in S(\mathbf{Top})^{op}$ then

$$\mathcal{A} \leq \mathcal{B}, \quad \text{if and only if} \quad \mathcal{A} \supseteq \mathcal{B}.$$

Hence the union of elements of $S(\mathbf{Top})^{op}$ is seen as an intersection of elements in $S(\mathbf{Top})$ and the suprema of an indexed family in $S(\mathbf{Top})^{op}$ as the infima of the indexed family in $S(\mathbf{Top})$.

Lemma 4.11. *The function* $IN(\mathbf{Top}) \xrightarrow{C} S(\mathbf{Top})^{op}$ *defined by*

$$C(I) = Cosep(I) = \{X \in \mathbf{Top} : X \ is \ I\text{-}Coseparated\}$$

is order preserving

Proof. Let $I, J \in IN(\mathbf{Top})$ such that $I \sqsubseteq J$, it will be proved that $C(I) \leq C(J)$. Let $X \in C(J)$, that is $j_{X^2}(\complement\Delta_X) = \emptyset$ and since

$$i_{X^2}(\complement\Delta_X) \subseteq j_{X^2}(\complement\Delta_X) = \emptyset$$

then $i_{X^2}(\complement\Delta_X) = \emptyset$, that is $X \in C(I)$ and therefore $C(I) \leq C(J)$. ∎

Proposition 4.12. *The function* $IN(\mathbf{Top}) \xrightarrow{C} S(\mathbf{Top})^{op}$ *defined as in the previous lemma, preserves suprema.*

Proof. Let $\{I_k\}_{k\in K}$ be a family of interior operators in $\mathbf{Top}$. It will be proved that

$$C\left(\bigvee_{k\in K} I_k\right) = \bigvee_{k\in K} C(I_k).$$

By definition of supremum, $I_k \sqsubseteq \bigvee_{k\in K} I_k$, for every $k \in K$. From Lemma 4.11,

$$C(I_k) \leq C\left(\bigvee_{k\in K} I_k\right)$$

for every $k \in K$ and therefore

$$\bigvee_{k\in K} C(I_k) = \bigcap_{k\in K} C(I_k) \leq C\left(\bigvee_{k\in K} I_k\right)$$

that is

$$\bigvee_{k\in K} C(I_k) \leq C\left(\bigvee_{k\in K} I_k\right).$$

On the other hand if $X \in C\left(\bigvee_{k\in K} I_k\right)$ then by definition X is $\bigvee_{k\in K} I_k$-coseparated or equivalently $\complement\Delta_X$ is $\bigvee_{k\in K} I_k$-isolated. Thus from Proposition 3.7 this occurs if and only if $\complement\Delta_X$ is I_k-isolated for every $k \in K$, thus $(i_k)_{X^2}\left(\complement\Delta_X\right) = \emptyset$, for every $k \in K$. Hence $X \in C(I_k)$ for every $k \in K$ and therefore

$$X \in \bigcap_{k\in K} C(I_k) = \bigvee_{k\in K} C(I_k).$$

This implies that

$$C\left(\bigvee_{k\in K} I_k\right) \leq \bigvee_{k\in K} C(I_k).$$

Therefore

$$C\left(\bigvee_{k\in K} I_k\right) = \bigvee_{k\in K} C(I_k).$$

□

Next, the concept of Galois connection is introduced together with a result that will be very important for this theory, both recalled from [8].

Definition 4.13. *For pre-ordered classes $\mathcal{X} = (\mathbf{X}, \leq)$ and $\mathcal{Y} = (\mathbf{Y}, \leq)$ a Galois connection $\mathcal{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{Y}$ consists of order preserving functions f and g that satisfy $x \leq g(f(x))$ for every $x \in \mathbf{X}$ and $f(g(y)) \leq y$ for every $y \in \mathbf{Y}$.*

Proposition 4.14. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two pre-ordered classes and assume that suprema exist in $\mathcal{X}$. Let $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ be a function that preserves suprema. Define $\mathcal{Y} \xrightarrow{g} \mathcal{X}$ as follows: for every $y \in \mathcal{Y}$,*

$$g(y) = \bigvee \{x \in \mathcal{X} : f(x) \leq y\}.$$

Then, $\mathcal{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{Y}$ is a Galois connection.

Proposition 4.15. *Let $\mathcal{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{Y}$ be a Galois connection between partially ordered classes $\mathcal{X}$ and $\mathcal{Y}$. Then, the functions f and g uniquely determine each other.*

Remark 4.16. *We recall that:*

i. *From Remark 3.3, $IN(\mathbf{Top})$, the class of all interior operators in $\mathbf{Top}$, is a pre-ordered class in which by Proposition 3.5 suprema exist, that is, if $\{I_k\}$ is a family of interior operators in $\mathbf{Top}$, then the supremum of this family is $\bigvee_{k \in K} I_k$, the interior operator known from Proposition 3.5.*

ii. *$S(\mathbf{Top})^{op}$ denotes the conglomerate of all subclasses of objects of $\mathbf{Top}$, ordered by inverted inclusion.*

iii. *By Proposition 4.33 it is known that the function $IN(\mathbf{Top}) \xrightarrow{C} S(\mathbf{Top})^{op}$, defined as in Lemma 4.32, preserves suprema.*

Therefore from Proposition 4.14 one obtains that there exists a function $S(\mathbf{Top})^{op} \xrightarrow{D} IN(Top)$, defined by

$$D(\mathcal{B}) = \bigvee \{I \in IN(\mathbf{Top}) : \mathcal{B} \subseteq C(I)\}$$

such that

$$IN(\mathbf{Top}) \underset{D}{\overset{C}{\rightleftarrows}} S(\mathbf{Top})^{op}$$

is a Galois connection.

Remark 4.17. *We would like to observe that the fact that* $IN(\mathbf{Top}) \underset{D}{\overset{C}{\rightleftarrows}} S(\mathbf{Top})^{op}$ *is a Galois connection is an important result, not only because it establishes a relation between interior operators on* **Top** *and subclasses of topological spaces but also for the following interesting consequence.*

For any subclass of topological spaces $\mathcal{A}$*, we have that* $C(D(\mathcal{A})) \leq \mathcal{A}$*, in* $S(\mathbf{Top})^{op}$*, that is* $\mathcal{A} \subseteq C(D(\mathcal{A}))$*, in other words, all the topological spaces in* $\mathcal{A}$ *are coseparated with respect to the interior operator* $(D(\mathcal{A}))$*. This means that for any class of topological spaces* $\mathcal{A}$*, one can always find an appropriate interior operator* $(D(\mathcal{A}))$ *with respect to which all the spaces in* $\mathcal{A}$ *are coseparated.*

From now on, unless otherwise stated, the notation $X \xrightarrow{f} Y$ will always mean that $X, Y \in \mathbf{Top}$ and that f is a continuous function from X to Y.

Now, the next objective consists in finding a more practical characterization of the function D. To do this, the following definition is introduced.

Definition 4.18. *Let* $\mathcal{B} \in S(\mathbf{Top})^{op}$*,* $Y \in \mathbf{Top}$ *and* $M \subseteq Y$*. Define* $I_{\mathcal{B}} = ((i_{\mathcal{B}})_Y)_{Y \in \mathbf{Top}}$*, where*

$$(i_{\mathcal{B}})_Y(M) = \bigcup \left\{ N \subseteq M : \forall\, X \times X \xrightarrow{f} Y \text{ with } X \in \mathcal{B} \text{ and } f^{-1}(M) \subseteq \complement\Delta_X, f^{-1}(N) = \emptyset \right\}.$$

$$\begin{tikzcd}
X \times X \arrow[rr, "f"] & & Y & \\
\complement\Delta_X \arrow[u] & f^{-1}(M) \arrow[l] \arrow[r] & M \arrow[u] & \bigcup\{N \subseteq M\} \arrow[l] \\
& f^{-1}(N) \arrow[ul] \arrow[u] \arrow[r] & N \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Lemma 4.19. *For $\mathcal{B} \in S(\mathbf{Top})^{op}$, the assignment $I_{\mathcal{B}} = ((i_{\mathcal{B}})_Y)_{Y \in \mathbf{Top}}$, defined as in the previous definition, is an interior operator on* **Top**.

Proof. Clearly $(i_{\mathcal{B}})_Y$ satisfies the contractiveness property, that is $(i_{\mathcal{B}})_Y(M) \subseteq M$ for every $M \subseteq X$ with $X \in \mathbf{Top}$, since by definition it is a union of subsets of M.

Let now $M_1, M_2 \subseteq Y$, such that $M_1 \subseteq M_2$ and assume that $N \subseteq M_1$ occur in the construction of $(i_{\mathcal{B}})_Y(M_1)$, it must be shown that N also occurs in the construction of $(i_{\mathcal{B}})_Y(M_2)$. Since $N \subseteq M_1$ then $N \subseteq M_2$ and if $X \times X \xrightarrow{f} Y$ is a continuous function that satisfies $X \in \mathcal{B}$ and $f^{-1}(M_2) \subseteq \complement\Delta_X$, then from

$$f^{-1}(M_1) \subseteq f^{-1}(M_2) \subseteq \complement\Delta_X$$

it is concluded that $f^{-1}(N) = \emptyset$, because $X \in \mathcal{B}$, $f^{-1}(M_1) \subseteq \complement\Delta_X$ and N occur in the construction of $(i_{\mathcal{B}})_Y(M_1)$. Thus N also occurs in the construction of $(i_{\mathcal{B}})_Y(M_2)$ and therefore $(i_{\mathcal{B}})_Y(M_1) \subseteq (i_{\mathcal{B}})_Y(M_2)$.

To show the continuity property, let $Z \in \mathbf{Top}$, $M \subseteq Z$ and let $Y \xrightarrow{g} Z$ be a continuous function. It will be proved that

$$g^{-1}\left((i_{\mathcal{B}})_Z(M)\right) \subseteq (i_{\mathcal{B}})_Y\left(g^{-1}(M)\right).$$

Indeed

$$
\begin{aligned}
g^{-1}\left((i_{\mathcal{B}})_Z(M)\right) &= g^{-1}\Big(\bigcup\Big\{N \subseteq M : \forall\, X \times X \xrightarrow{f} Z \textit{ with } X \in \mathcal{B} \\
&\qquad\qquad \textit{and } f^{-1}(M) \subseteq \complement\Delta_X, f^{-1}(N) = \emptyset\Big\}\Big) \\
&= \bigcup\Big\{g^{-1}(N), N \subseteq M : \forall\, X \times X \xrightarrow{f} Z \textit{ with } X \in \mathcal{B} \\
&\qquad\qquad \textit{and } f^{-1}(M) \subseteq \complement\Delta_X, f^{-1}(N) = \emptyset\Big\} \\
&\subseteq \bigcup\Big\{g^{-1}(N), N \subseteq M : \forall\, X \times X \xrightarrow{h} Y \textit{ with } X \in \mathcal{B} \\
&\qquad\qquad \textit{and } h^{-1}\left(g^{-1}(M)\right) \subseteq \complement\Delta_X, h^{-1}\left(g^{-1}(N)\right) = \emptyset\Big\} \\
&\subseteq \bigcup\Big\{H \subseteq g^{-1}(M) : \forall\, X \times X \xrightarrow{h} Y \textit{ with } X \in \mathcal{B} \\
&\qquad\qquad \textit{and } h^{-1}\left(g^{-1}(M)\right) \subseteq \complement\Delta_X, h^{-1}(H) = \emptyset\Big\} \\
&= (i_{\mathcal{B}})_Y\left(g^{-1}(M)\right).
\end{aligned}
$$

Thus $g^{-1}\left((i_{\mathcal{B}})_Z(M)\right) \subseteq (i_{\mathcal{B}})_Y\left(g^{-1}(M)\right)$ and therefore $I_{\mathcal{B}} \in IN(\mathbf{Top})$.

$\square$

In the following proposition a more practical characterization of the function D of the Galois connection $IN(\mathbf{Top}) \underset{D}{\overset{C}{\rightleftarrows}} S(\mathbf{Top})^{op}$ is presented. Before, recall that for $\mathcal{B} \in S(\mathbf{Top})^{op}$, $S(\mathbf{Top})^{op} \xrightarrow{D} IN(\mathbf{Top})$ is defined by

$$D(\mathcal{B}) = \bigvee \{I \in IN(\mathbf{Top}) : \mathcal{B} \subseteq C(I)\}.$$

and $I_{\mathcal{B}} = ((i_{\mathcal{B}})_Y)_{Y \in \mathbf{Top}}$, is defined by

$$
\begin{aligned}
(i_{\mathcal{B}})_Y(M) = \bigcup\Big\{N \subseteq M : \forall\, X \times X \xrightarrow{f} Y \textit{ with } X \in \mathcal{B} \\
\textit{and } f^{-1}(M) \subseteq \complement\Delta_X, f^{-1}(N) = \emptyset\Big\}.
\end{aligned}
$$

Theorem 4.20. *Let $\mathcal{B} \in S(\mathbf{Top})^{op}$ and let $Y \in \mathbf{Top}$. Then for every $M \subseteq Y$ one has that*

$$D(\mathcal{B})(M) = (i_{\mathcal{B}})_Y\,(M).$$

Proof. First it will be proved that $I_{\mathcal{B}} \leq D(\mathcal{B})$. To see this note that

$$I_{\mathcal{B}} \in \{I \in IN(\mathbf{Top}) : \mathcal{B} \subseteq C(I)\}\,.$$

Indeed, from Lemma 4.19, $I_{\mathcal{B}} \in IN(\mathbf{Top})$ and thus it only remains to prove that $\mathcal{B} \subseteq C(I_{\mathcal{B}})$. That is, if $X \in \mathcal{B}$ then for $\complement\Delta_X \subseteq X \times X$, the existence of the function

$$\begin{cases} X \times X \xrightarrow{id_{X^2}} X \times X \\ id_{X^2}((x_1, x_2)) = (x_1, x_2) \end{cases}$$

implies that the only subset $N \subseteq \complement\Delta_X$ that satisfies $id_{X^2}^{-1}(N) = \emptyset$ is $N = \emptyset$. Equivalently, for every $N \subseteq \complement\Delta_X$, the function id_{X^2} is such that $id_{X^2}^{-1}\left(\complement\Delta_X\right) \subseteq \complement\Delta_X$, and $id_{X^2}^{-1}(N) \neq \emptyset$. Hence, by definition of I_B it is obtained that $(i_{\mathcal{B}})_{X^2}\left(\complement\Delta_X\right) = \emptyset$ and thus $X \in C(I_{\mathcal{B}})$. Therefore

$$I_{\mathcal{B}} \in \{I \in IN(\mathbf{Top}) : \mathcal{B} \subseteq C(I)\}$$

and this implies that

$$I_{\mathcal{B}} \leq \bigvee \{I \in IN(\mathbf{Top}) : \mathcal{B} \subseteq C(I)\} = D(\mathcal{B}).$$

On the other hand, for every $M \subseteq Y$ and every continuous function $X \times X \xrightarrow{f} Y$ with $X \in \mathcal{B}$ and $f^{-1}(M) \subseteq \complement\Delta_X$, from the continuity and monotonicity property of the interior operator D, it is obtained that

$$f^{-1}\left(D(\mathcal{B})(M)\right) \subseteq D(\mathcal{B})\left(f^{-1}(M)\right) \subseteq D(\mathcal{B})\left(\complement\Delta_X\right).$$

Now, since $X \in \mathcal{B}$, then $X \in C(I)$ that is $\complement\Delta_X$ is I-isolated. Thus by definition of D and Proposition 3.7, it is obtained that $\complement\Delta_X$ is $D(\mathcal{B})$-isolated or equivalently

$D(\mathcal{B})\left(\complement\Delta_X\right) = \emptyset$. Hence

$$f^{-1}\left(D(\mathcal{B})(M)\right) \subseteq D(\mathcal{B})\left(\complement\Delta_X\right) = \emptyset.$$

That is $f^{-1}\left(D(\mathcal{B})(M)\right) = \emptyset$. Notice that by contractiveness $D(\mathcal{B})(M) \subseteq M$ and furthermore, $X \in \mathcal{B}$ and $f^{-1}(M) \subseteq \complement\Delta_X$ imply $f^{-1}\left(\left(D(\mathcal{B})\left(M\right)\right) = \emptyset\right.$. Then, by definition of the interior operator $I_{\mathcal{B}}$, this implies that $D(\mathcal{B})(M)$ is one of the N's occurring in the construction of $(i_{\mathcal{B}})_Y(M)$ and so,

$$D(\mathcal{B})(M) \subseteq (i_{\mathcal{B}})_Y(M), \text{ for every } M \subseteq Y.$$

Therefore $D(\mathcal{B}) \leq I_{\mathcal{B}}$ and thus it is concluded that $D(\mathcal{B}) = I_{\mathcal{B}}$.

$\square$

Remark 4.21. *From now on, denote*

$$D(\mathcal{B}) = ((d_{\mathcal{B}})_Y)_{Y \in \mathbf{Top}}$$

where

$$(d_{\mathcal{B}})_Y(M) = \bigcup \Big\{ N \subseteq M : \forall\, X \times X \xrightarrow{f} Y \text{ with } X \in \mathcal{B} \\ \text{and } f^{-1}(M) \subseteq \complement \Delta_X, f^{-1}(N) = \emptyset \Big\}.$$

Next, a few results whose contribution is essential for the main objective of this theory are introduced. From [9] one has the following result

Proposition 4.22. *Let* $S(\mathbf{Top}) \xrightarrow{\Delta} S(\mathbf{Top})^{op}$ *and* $S(\mathbf{Top})^{op} \xrightarrow{\nabla} S(\mathbf{Top})$ *be functions defined as follows:*

- *For all* $\mathcal{A} \in S(\mathbf{Top})$

$$\Delta(\mathcal{A}) = \Big\{ Y \in \mathbf{Top} : \forall\, X \in \mathcal{A} \text{ and } X \xrightarrow{f} Y,\ f \text{ is constant} \Big\}$$

- *For all* $\mathcal{B} \in S(\mathbf{Top})^{op}$

$$\nabla(\mathcal{B}) = \{ X \in \mathbf{Top} : \forall\, Y \in \mathcal{B} \text{ and } X \xrightarrow{f} Y,\ f \text{ is constant} \}.$$

Then $S(\mathbf{Top}) \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} S(\mathbf{Top})^{op}$ *is a Galois connection.*

Now the concept of separation for an interior operator in topology is presented together with a result that will be very useful in the context of this work (cf. [1]).

Definition 4.23. *Let* $X, Y \in \mathbf{Top}$ *and let* $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ *be two functions. The separator of* f *and* g *is the set*

$$sep(f, g) = \{ x \in X : f(x) \neq g(x) \}.$$

Definition 4.24. *Let I be an interior operator on the category* **Top**. *$Y \in$* **Top** *is I-separated if and only if for every $X \in$* **Top** *and for every pair of continuous functions $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, $sep(f,g)$ is I-open.*

A characterization of this concept is showed in the following proposition.

Proposition 4.25. *Y is I-separated if and only if $\complement\Delta_Y$ is I-open.*

Proposition 4.26. *Let $S(\mathbf{Top}) \xrightarrow{T} IN(\mathbf{Top})$ and let $IN(\mathbf{Top}) \xrightarrow{S} S(\mathbf{Top})$ be functions defined as follows:*

- *For all $\mathcal{A} \in S(\mathbf{Top})$, $T_{\mathcal{A}} = ((t_{\mathcal{A}})_X)_{X \in \mathbf{Top}}$, where*

$$(t_{\mathcal{A}})_X (M) = \bigcup \left\{ sep(f,g) \subseteq M : X \underset{g}{\overset{f}{\rightrightarrows}} Y; Y \in \mathcal{A} \right\}.$$

- *For all $I \in IN(\mathbf{Top})$*

$$S(I) = \{X \in \mathbf{Top} : X \text{ is } I\text{-separated}\}$$

Then $S(\mathbf{Top}) \underset{S}{\overset{T}{\rightleftarrows}} IN(\mathbf{Top})$ is a Galois connection.

Furthermore, remember from Remark 4.16 and the characterization of Proposition 4.20 that:

The functions $IN(\mathbf{Top}) \xrightarrow{C} S(\mathbf{Top})^{op}$ and $S(\mathbf{Top})^{op} \xrightarrow{D} IN(\mathbf{Top})$ defined by

- For all $I \in IN(\mathbf{Top})$

$$C(I) = \{X \in \mathbf{Top} : X \ is \ I - Coseparated\}$$

- For all $\mathcal{B} \in S(\mathbf{Top})^{op}$ $D(\mathcal{B}) = ((d_{\mathcal{B}})_Y)_{Y \in \mathbf{Top}}$, where

$$(d_{\mathcal{B}})_Y(M) = \bigcup \Big\{ N \subseteq M : \forall\, X \times X \xrightarrow{f} Y \; with \; X \in \mathcal{B} \\ and \; f^{-1}(M) \subseteq \complement\Delta_X, f^{-1}(N) = \emptyset \Big\}$$

form the Galois connection $IN(\mathbf{Top}) \underset{D}{\overset{C}{\rightleftarrows}} S(\mathbf{Top})^{op}$.

For the purpose of the following theorem the order of $IN(\mathbf{Top})$ and $S(\mathbf{Top})^{op}$ in the last two Galois connections is inverted. That is

$$S(\mathbf{Top}) \underset{C}{\overset{D}{\rightleftarrows}} IN(\mathbf{Top})^{op} \quad \text{and} \quad IN(\mathbf{Top})^{op} \underset{T}{\overset{S}{\rightleftarrows}} S(\mathbf{Top})^{op}$$

are used instead.

Notice that in order to simplify the notation the symbols C, D, S and T are used instead of the more formally correct C^{op}, D^{op}, S^{op} and T^{op}.

The following theorem shows that the Galois connection

$$S(\mathbf{Top}) \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} S(\mathbf{Top})^{op}$$

factors through the Galois connections

$$S(\mathbf{Top}) \underset{C}{\overset{D}{\rightleftarrows}} IN(\mathbf{Top})^{op} \quad \text{and} \quad IN(\mathbf{Top})^{op} \underset{T}{\overset{S}{\rightleftarrows}} S(\mathbf{Top})^{op}$$

Theorem 4.27. *The following diagram of Galois connections is commutative.*

$$\begin{tikzcd}
S(\mathbf{Top}) \arrow[rr, shift left, "\Delta"] \arrow[dr, shift left, "D"] & & S(\mathbf{Top})^{op} \arrow[ll, shift left, "\nabla"] \arrow[dl, shift left, "T"] \\
& IN(\mathbf{Top})^{op} \arrow[ul, shift left, "C"] \arrow[ur, shift left, "S"] &
\end{tikzcd}$$

That is $\nabla = C \circ T$ and $\Delta = S \circ D$.

Proof. It will first be proved that $\nabla = C \circ T$. To see this it must be proved that for every $\mathcal{B} \in S(\mathbf{Top})^{op}$

$$C(T(\mathcal{B})) = \nabla(\mathcal{B}).$$

Indeed, if $X \in C(T(\mathcal{B}))$ then, X is $T(\mathcal{B})$-coseparated, where $T_{\mathcal{B}} = ((t_{\mathcal{B}})_X)_{X \in \mathbf{Top}}$, and

$$t(\mathcal{B})_X(M) = \bigcup \left\{ sep(f,g) \subseteq M : X \overset{f}{\underset{g}{\rightrightarrows}} Y; Y \in \mathcal{B} \right\}$$

Thus

$$t(\mathcal{B})_{X^2}\left(\complement \Delta_X\right) = \emptyset$$

Hence it must be proved that, if $X \in \mathbf{Top}$ satisfies for $\complement \Delta_X \subseteq X \times X$, $t(\mathcal{B})_{X^2}\left(\complement \Delta_X\right) = \emptyset$ then $X \in \nabla(\mathcal{B})$.

This is going to be done by contradiction. Suppose that $X \notin \nabla(\mathcal{B})$, then there is a $Y \in \mathcal{B}$ and a function $X \xrightarrow{h} Y$ that is continuous but not constant. Thus, there are at least two points $y_1, y_2 \in Y$ and $x_1, x_2 \in X$ with $y_1 \neq y_2$ and $x_1 \neq x_2$, such that $h(x_1) = y_1$ and $h(x_2) = y_2$. Thus, from

$$X \times X \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X \xrightarrow{h} Y,$$

where π_1 and π_2 are the projections on the first and second coordinates, the functions

$$\begin{cases} X \times X \xrightarrow{f} Y \\ f(\mathbf{x}) = h(\pi_1(\mathbf{x})) \end{cases} \quad \text{and} \quad \begin{cases} X \times X \xrightarrow{g} Y \\ g(\mathbf{x}) = h(\pi_2(\mathbf{x})) \end{cases}$$

are obtained. Notice that $X \times X \overset{f}{\underset{g}{\rightrightarrows}} Y$ are continuous functions with $Y \in \mathcal{B}$. On the other hand $sep(f,g) \neq \emptyset$, since $\mathbf{x}' = (x_1, x_2) \in X \times X$ is in $sep(f,g)$, because

$$f(\mathbf{x}') = h(\pi_1(\mathbf{x}')) = h(x_1) = y_1 \neq y_2 = h(x_2) = h(\pi_2(\mathbf{x}')) = g(\mathbf{x}'),$$

that is $f(\mathbf{x}') \neq g(\mathbf{x}')$. Furthermore $sep(f,g) \subseteq \complement\Delta_X$, because if $sep(f,g) \cap \Delta_X \neq \emptyset$ there is $\mathbf{x}'' = (x,x) \in sep(f,g)$ and by definition of separator $f(\mathbf{x}'') \neq g(\mathbf{x}'')$ but

$$f(\mathbf{x}'') = h(\pi_1(\mathbf{x}'')) = h(x) = h(\pi_2(\mathbf{x}'')) = g(\mathbf{x}'')$$

which is a contradiction. So, it is concluded that $X \times X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are a pair of continuous functions, where $Y \in \mathcal{B}$ and $\emptyset \neq sep(f,g) \subseteq \complement\Delta_X$. This implies that $(t\,(\mathcal{B}))_{X^2}\,(\complement\Delta_X) \neq \emptyset$ that is $X \notin C(T(\mathcal{B}))$. Therefore $C(T(\mathcal{B})) \subseteq \nabla(\mathcal{B})$.

Now in order to prove that $\nabla(\mathcal{B}) \subseteq C(T(\mathcal{B}))$, assume that $X \in \nabla(\mathcal{B})$. Then for every $Y \in \mathcal{B}$, if $X \xrightarrow{h} Y$ is a continuous function, then h is constant. Now suppose that $X \notin C(T(\mathcal{B}))$ that is $(t_{\mathcal{B}})_{X^2}(\complement\Delta_X) \neq \emptyset$. Then, there is a $sep(f,g) \subseteq \complement\Delta_X$, such that $sep(f,g) \neq \emptyset$ for $X \times X \underset{g}{\overset{f}{\rightrightarrows}} Y$ continuous functions with $Y \in \mathcal{B}$. Consequently there is at least $\mathbf{x}' = (x_1, x_2) \in \complement\Delta_X$ such that $f(\mathbf{x}') \neq g(\mathbf{x}')$. So define the continuous function

$$\begin{cases} X \xrightarrow{j} X \times X \\ j(x) = (x, x_2) \end{cases}$$

where x_2 is the second component of $\mathbf{x}'$. And from

$$X \xrightarrow{j} X \times X \underset{g}{\overset{f}{\rightrightarrows}} Y,$$

define

$$\begin{cases} X \xrightarrow{\bar{f}} Y \\ \bar{f}(x) = f(j(x)) \end{cases} \quad \text{and} \quad \begin{cases} X \xrightarrow{\bar{g}} Y \\ \bar{g}(x) = g(j(x)) \end{cases}.$$

Since j, f y g are continuous functions then $X \underset{\bar{g}}{\overset{\bar{f}}{\rightrightarrows}} Y$ with $Y \in \mathcal{B}$ are also continuous functions. Since $X \in \nabla(\mathcal{B})$, it follows by hypothesis that $\bar{f}$ and $\bar{g}$ are constant functions, that is there are $c, c' \in Y$, such that $\bar{f}(x) = c$ and $\bar{g}(x) = c'$ for all $x \in X$. Since $\bar{f} = f \circ j$ and $\bar{g} = g \circ j$ are constant functions, j is injective and the image of j is $X \times \{x_2\} \subseteq X \times X$, one obtains that f y g are constant on $X \times \{x_2\}$, that is

$$f\Big|_{X\times\{x_2\}}(\mathbf{x}) = c \qquad \text{and} \qquad g\Big|_{X\times\{x_2\}}(\mathbf{x}) = c'$$

for all $\mathbf{x} \in X \times \{x_2\}$. Now as $\mathbf{x}' = (x_1, x_2) \in X \times \{x_2\}$ and $f(\mathbf{x}') \neq g(\mathbf{x}')$, then

$$f\Big|_{X\times\{x_2\}}(\mathbf{x}) \neq g\Big|_{X\times\{x_2\}}(\mathbf{x})$$

for all $\mathbf{x} \in X \times \{x_2\}$. Hence

$$X \times \{x_2\} \subseteq sep(f, g) \subseteq \complement\Delta_X$$

That is a contradiction, since $(x_2, x_2) \in X \times \{x_2\} \subseteq \complement\Delta_X$. Hence, if $X \in \nabla(\mathcal{B})$ then $X \in C(T(\mathcal{B}))$, that is $\nabla(\mathcal{B}) \subseteq C(T(\mathcal{B}))$. So it is concluded that $C(T(\mathcal{B})) = \nabla(\mathcal{B})$.

Now, it will be proved that for every $\mathcal{A} \in S(\mathbf{Top})$.

$$S(D(\mathcal{A})) = \Delta(\mathcal{A})$$

First it will be proved that $S(D(\mathcal{A})) \geq \Delta(\mathcal{A})$ in $S(\mathbf{Top})^{op}$. By definition, $Y \in S(D(\mathcal{A}))$ if Y is $D(\mathcal{A})$-separated, i.e. if $D(\mathcal{A})\left(\complement\Delta_Y\right) = \complement\Delta_Y$, where $D(\mathcal{A}) = ((d_{\mathcal{A}})_Y)_{Y\in\mathbf{Top}}$, and

$$(d_{\mathcal{A}})_Y(M) = \bigcup\Big\{N \subseteq M : \forall\, X \times X \xrightarrow{f} Y \textit{ with } X \in \mathcal{A} \\ \textit{and } f^{-1}(M) \subseteq \complement\Delta_X, f^{-1}(N) = \emptyset\Big\}.$$

Thus, in particular

$$(d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right) = \bigcup\Big\{N \subseteq \complement\Delta_Y : \forall\, X \times X \xrightarrow{f} Y \times Y \textit{ with } X \in \mathcal{A} \\ \textit{and } f^{-1}\left(\complement\Delta_Y\right) \subseteq \complement\Delta_X, f^{-1}(N) = \emptyset\Big\}.$$

Suppose that $Y \in S(D(\mathcal{A}))$ but $Y \notin \Delta(\mathcal{A})$, then there is a topological space $X \in \mathcal{A}$ and a continuous function $X \xrightarrow{f} Y$ that is not constant, i.e. there are at least two points $y_1, y_2 \in Y$ and $x_1, x_2 \in X$ with $y_1 \neq y_2$ and $x_1 \neq x_2$, such that

$f(x_1) = y_1$ and $f(x_2) = y_2$. Hence one defines the function

$$\begin{cases} X \times X \xrightarrow{F} Y \times Y \\ F(\mathbf{x}) = (f(x), f(x')) \end{cases}$$

for every $\mathbf{x} = (x, x')$ in $X \times X$. Let $\mathbf{x}' = (x_1, x_2)$ and let $\mathbf{y}' = (y_1, y_2)$, so clearly

$$\emptyset \neq F^{-1}\left(\complement\Delta_Y\right) \subseteq \complement\Delta_X.$$

This is true because $\mathbf{x}' = (x_1, x_2) \in X \times X$ is in $F^{-1}\left(\complement\Delta_Y\right)$, that is

$$F(\mathbf{x}') = (f(x_1), f(x_2)) = (y_1, y_2) = \mathbf{y}' \in \complement\Delta_Y, \text{ since } y_1 \neq y_2.$$

Moreover, if $\mathbf{x}'' = (x, x) \in F^{-1}\left(\complement\Delta_Y\right)$, then $(f(x), f(x)) = F(\mathbf{x}'') \in \complement\Delta_Y$. This is a contradiction, hence $F^{-1}\left(\complement\Delta_Y\right) \subseteq \complement\Delta_X$.

Since $X \xrightarrow{f} Y$ is a continuous function then $X \times X \xrightarrow{F} Y \times Y$ with $X \in \mathcal{A}$ is also a continuous function, such that $\emptyset \neq F^{-1}\left(\complement\Delta_Y\right) \subseteq \complement\Delta_X$. It will be proved that

$$(d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right) \neq \complement\Delta_Y.$$

Since $D(\mathcal{A})$ is an interior operator, then $(d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right) \subseteq \complement\Delta_Y$, so it will be showed that there is at least $\mathbf{y} \in \complement\Delta_Y$, such that $\mathbf{y} \notin (d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right)$.

Let

$$\mathcal{N} = \Big\{N \subseteq \complement\Delta_Y : \forall\, X \times X \xrightarrow{f} Y \times Y \textit{ with } X \in \mathcal{A} \\ \textit{and } f^{-1}\left(\complement\Delta_Y\right) \subseteq \complement\Delta_X, f^{-1}(N) = \emptyset\Big\}$$

then $\mathbf{y} \notin (d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right)$ if and only if, for every $N \in \mathcal{N}$, $\mathbf{y} \notin N$.

Note that $\mathbf{y}' = (y_1, y_2) \in \complement\Delta_Y$ but $\mathbf{y}' \notin (d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right)$. Indeed, if there is $N \in \mathcal{N}$ such that $\mathbf{y}' \in N$, then the continuous function $X \times X \xrightarrow{F} Y \times Y$ defined before satisfies $X \in \mathcal{A}$ and $F^{-1}\left(\complement\Delta_Y\right) \subseteq \complement\Delta_X$. However, $F^{-1}(N) \neq \emptyset$, since $F(\mathbf{x}') = \mathbf{y}'$, that is $\mathbf{x}' \in F^{-1}(N)$. This contradicts that $N \in \mathcal{N}$. Thus $\mathbf{y}' \notin (d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right)$, hence $\complement\Delta_Y \nsubseteq (d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right)$ and therefore $(d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right) \neq \complement\Delta_Y$.

This contradicts the hypothesis that if $Y \in S(D(\mathcal{A}))$ then $(d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right) = \complement\Delta_Y$. Therefore if $Y \in S(D(\mathcal{A}))$, then $Y \in \Delta(\mathcal{A})$, that is $S(D(\mathcal{A})) \subseteq \Delta(\mathcal{A})$ and so $S(D(\mathcal{A})) \geq \Delta(\mathcal{A})$ in $S(\mathbf{Top})^{op}$.

Now to prove that $\Delta(\mathcal{A}) \geq S(D(\mathcal{A}))$ in $S(\mathbf{Top})^{op}$, let $Y \in \Delta(\mathcal{A})$ and assume that $Y \notin S(D(\mathcal{A}))$. Then, Y is not $D(\mathcal{A})$-separated, that is $(d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right) \neq \complement\Delta_Y$. This means that there is at least a point $\mathbf{y}' \in \complement\Delta_Y$, such that $\mathbf{y}' \notin (d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right)$, since $(d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right) \subseteq \complement\Delta_Y$ is true because $D(\mathcal{A})$ is an interior operator. Recall that $\mathbf{y}' \notin (d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right)$ if and only if, for every $N \in \mathcal{N}$, $\mathbf{y}' \notin N$, where $\mathcal{N}$ is defined as in the previous case. Thus define the set

$$M = \left\{\mathbf{y} \in \complement\Delta_Y : \mathbf{y} \notin (d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right)\right\}.$$

Clearly $M \neq \emptyset$, since $\mathbf{y}' \in M$, thus $M \subseteq \complement\Delta_Y$ but $M \notin \mathcal{N}$, since if $M \in \mathcal{N}$ this would contradict that $\mathbf{y}' \notin (d_{\mathcal{A}})_{Y^2}\left(\complement\Delta_Y\right)$. Since $M \notin \mathcal{N}$, then there is a continuous function $X \times X \xrightarrow{G} Y \times Y$, with $X \in \mathcal{A}$ and $G^{-1}(\complement\Delta_Y) \subseteq \complement\Delta_X$, such that $G^{-1}(M) \neq \emptyset$. Now $M \subseteq \complement\Delta_Y$ implies that

$$\emptyset \neq G^{-1}(M) \subseteq G^{-1}(\complement\Delta_Y)$$

that is $G^{-1}(\complement\Delta_Y) \neq \emptyset$.

Since $X \times X \xrightarrow{G} Y \times Y$ is a continuous function, then $G = (h, k)$, where $X \times X \overset{h}{\underset{k}{\rightrightarrows}} Y$ are continuous functions obtained by composing G with the projections $Y \times Y \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} Y$, that is $h = \pi_1 \circ G$ and $k = \pi_2 \circ G$. Consequently one has that $G^{-1}(\complement\Delta_X) = sep(h, k)$. Now, since $X \in \mathcal{A}$, $G^{-1}(\complement\Delta_Y) \neq \emptyset$ and $G^{-1}(\complement\Delta_Y) \subseteq \complement\Delta_X$, written in terms of the separator of h and k, one obtains that $\emptyset \neq sep(h, k) \subseteq \complement\Delta_Y$, where $X \times X \overset{h}{\underset{k}{\rightrightarrows}} Y$ are continuous function with $X \in \mathcal{A}$.

Let $\mathbf{x}' = (x_1, x_2) \in sep(h,k)$, then $h(\mathbf{x}') \neq k(\mathbf{x}')$. Reasoning as in the previous case, one defines the continuous functions

$$\begin{cases} X \xrightarrow{j} X \times X \\ j(x) = (x, x_2) \end{cases}$$

where x_2 is the second component of $\mathbf{x}' = (x_1, x_2) \in sep(h,k)$. And from

$$X \xrightarrow{j} X \times X \underset{k}{\overset{h}{\rightrightarrows}} Y,$$

define

$$\begin{cases} X \xrightarrow{\bar{h}} Y \\ \bar{h}(x) = h(j(x)) \end{cases} \quad \text{and} \quad \begin{cases} X \xrightarrow{\bar{k}} Y \\ \bar{k}(x) = k(j(x)) \end{cases}.$$

Since $\bar{h}$ and $\bar{k}$ are continuous functions, $X \in \mathcal{A}$ and $Y \in \Delta(\mathcal{A})$, then $\bar{h}$ and $\bar{k}$ are constant functions. One has that $\bar{h} = h \circ j$ and $\bar{k} = k \circ j$ are constant functions, j is injective and the image of j is $X \times \{x_2\} \subseteq X \times X$. This implies that h and k are constant on $X \times \{x_2\}$, that is

$$h\Big|_{X\times\{x_2\}}(\mathbf{x}) = c \qquad \text{and} \qquad k\Big|_{X\times\{x_2\}}(\mathbf{x}) = c'$$

for all $\mathbf{x} \in X \times \{x_2\}$. Now as $\mathbf{x}' = (x_1, x_2) \in X \times \{x_2\}$ and $h(\mathbf{x}') \neq k(\mathbf{x}')$, then

$$h\Big|_{X\times\{x_2\}}(\mathbf{x}) \neq k\Big|_{X\times\{x_2\}}(\mathbf{x})$$

for all $\mathbf{x} \in X \times \{x_2\}$. Hence

$$X \times \{x_2\} \subseteq sep(h,k) = G^{-1}(\complement\Delta_Y) \subseteq \complement\Delta_X.$$

That is a contradiction, since $(x_2, x_2) \in X \times \{x_2\} \subseteq \complement\Delta_X$. Hence if $Y \in \Delta(\mathcal{A})$ then $Y \in S(D(\mathcal{A}))$, that is $\Delta(\mathcal{A}) \geq S(D(\mathcal{A}))$ in $S(\mathbf{Top})^{op}$. So one concludes that $S(D(\mathcal{A})) = \Delta(\mathcal{A})$. $\square$

Remark 4.28. *Note that since Proposition 4.15 guarantees uniqueness of the two functions that occur in a Galois connection, in the previous theorem it would have been enough to prove either* $\nabla = C \circ T$ *or* $\Delta = S \circ D$ *and the other equality would automatically follow. However due to the importance of this theorem both results were shown.*

Remark 4.29. *Recall that in the paper [1], the concept of* I*-separated topological space was introduced. This has been formulated in Definition 4.24 where it is stated that a space* $Y \in \mathbf{Top}$ *is separated with respect to an interior operator* I *if for every* $X \in \mathbf{Top}$ *and for every pair of continuous functions* $X \underset{g}{\overset{f}{\rightrightarrows}} Y$*, sep*$(f,g)$ *is* I*-open, that is if* $(i_X)(sep(f,g)) = sep(f,g)$*. Later in Proposition 4.25, a characterization of this concept is shown by means of the complement of the diagonal, that is a space* $Y \in \mathbf{Top}$ *is* I*-separated if and only if* $\complement\Delta_Y$ *is* I*-open.*

It is natural to ask whether a similar characterization in terms of sep(f,g) *can be given, in an analogous way, for the coseparated spaces with respect to a topological interior operator. That is: is it true that a space* $Y \in \mathbf{Top}$ *is* I*-coseparated if and only if for every* $X \in \mathbf{Top}$ *and for every pair of continuous functions* $X \underset{g}{\overset{f}{\rightrightarrows}} Y$*, sep*$(f,g)$ *is* I*-isolated?.*

The answer to this question is no. To prove this fact one proceeds as follows.

Definition 4.30. *Let* I *be an interior operator on the category* $\mathbf{Top}$*.* $Y \in \mathbf{Top}$ *is* I*-2nd coseparated if and only if for every* $X \in \mathbf{Top}$ *and for every pair of continuous functions* $X \underset{g}{\overset{f}{\rightrightarrows}} Y$*, sep*$(f,g)$ *is* I*-isolated.* $2nd$*-Cosep*(I) *will denote all* $2nd$ *coseparated objects respect to* I*.*

The following proposition shows that $2nd$-$Cosep(I)$ and $Cosep(I)$ are different.

Proposition 4.31. *Let I be an interior operator on the category* **Top**. *Then $2nd\text{-}Cosep(I) \subseteq Cosep(I)$, but $2nd\text{-}Cosep(I) \neq Cosep(I)$.*

Proof. Let Y be an I-$2nd$ coseparated space. Then for every $X \in \mathbf{Top}$ and for every pair of continuous functions $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, $(i_X)(sep(f,g)) = \emptyset$. In particular, for $X = Y \times Y$, $f = \pi_1$ and $g = \pi_2$ it is follows that

$$(i_X)\left(\complement\Delta_Y\right) = (i_X)(sep(\pi_1, \pi_1)) = \emptyset.$$

That is $(i_X)\left(\complement\Delta_Y\right) = \emptyset$. This means that Y is an I-coseparated space. Hence $2nd\text{-}Cosep(I) \subseteq Cosep(I)$.

Now it will be proved that there are I-coseparated spaces that not are I-$2nd$ coseparated. Consider the Example 4.5, where $Cosep(B) = \mathbf{Ind}$.
Let $Y \in Cosep(B)$ with at least two points, then $Y \in \mathbf{Ind}$. Thus Y is an indiscrete topological space that contains at least the points $y_1, y_2 \in Y$. Then for every non-empty $X \in \mathbf{Top}$, two functions $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are defined by $f(x) = y_1$ and $g(x) = y_2$ for every $x \in X$. Clearly f and g are continuous functions, such that $sep(f,g) = X$, but

$$b_X(sep(f,g)) = b_X(X) = X.$$

This means that $b_X(sep(f,g)) \neq \emptyset$ and therefore $Y \notin 2nd\text{-}Cosep(B)$. $\square$

Next, a lemma and a proposition related to the concept of I-$2nd$ coseparated space that are analogous to Lemma 4.32 and Proposition 4.33, are introduced.

Lemma 4.32. *The function $IN(\mathbf{Top}) \xrightarrow{C_2} S(\mathbf{Top})^{op}$ defined by*

$$C_2(I) = \{X \in \mathbf{Top} : X \text{ is } I\text{-}2nd \text{ coseparated}\}$$

is order preserving.

Proof. Let $I, J \in IN(\mathbf{Top})$ be such that $I \sqsubseteq J$. It will be proved that $C_2(I) \leq C_2(J)$ Let $Y \in C_2(J)$, thus for every $X \in \mathbf{Top}$ and for every pair of continuous functions $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, $j_{X^2}(sep(f,g)) = \emptyset$. Since

$$i_{X^2}(sep(f,g)) \subseteq j_{X^2}(sep(f,g)) = \emptyset$$

then $i_{X^2}(sep(f,g)) = \emptyset$, that is $Y \in C_2(I)$ and therefore $C_2(I) \leq C_2(J)$. ☐

Proposition 4.33. *The function* $IN(\mathbf{Top}) \xrightarrow{C_2} S(\mathbf{Top})^{op}$ *defined as in the previous lemma, preserves suprema.*

Proof. Let $\{I_k\}_{k\in K}$ be a family of interior operators in **Top**. It will be proved that

$$C_2\left(\bigvee_{k\in K} I_k\right) = \bigvee_{k\in K} C_2(I_k).$$

By definition of supremum, $I_k \sqsubseteq \bigvee_{k\in K} I_k$, for every $k \in K$ and from Lemma 4.32,

$$C_2(I_k) \leq C_2\left(\bigvee_{k\in K} I_k\right)$$

for every $k \in K$. Therefore,

$$\bigvee_{k\in K} C_2(I_k) = \bigcap_{k\in K} C_2(I_k) \leq C_2\left(\bigvee_{k\in K} I_k\right).$$

On other hand if $Y \in C_2\left(\bigvee_{k\in K} I_k\right)$ then by definition Y is $\bigvee_{k\in K} I_k$-$2nd$ coseparated or equivalently for every $X \in \mathbf{Top}$ and for every pair of continuous functions $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, $sep(f,g)$ is $\bigvee_{k\in K} I_k$-isolated. Thus, from of Proposition, 3.7 this occurs if and only if $sep(f,g)$ is I_k-isolated for every $k \in K$ and so $(i_k)_{X^2}(sep(f,g)) = \emptyset$, for every $k \in K$. Hence $Y \in C_2(I_k)$ for every $k \in K$ and therefore

$$Y \in \bigcap_{k\in K} C_2(I_k) = \bigvee_{k\in K} C_2(I_k).$$

This implies that

$$C_2\left(\bigvee_{k\in K} I_k\right) \leq \bigvee_{k\in K} C_2(I_k).$$

Therefore

$$C_2\left(\bigvee_{k\in K} I_k\right) = \bigvee_{k\in K} C_2(I_k).$$

$\square$

Since $IN(\mathbf{Top}) \xrightarrow{C_2} S(\mathbf{Top})^{op}$ preserves suprema, from Proposition 4.14 one obtains that there exists a function $S(\mathbf{Top})^{op} \xrightarrow{D_2} IN(\mathbf{Top})$, defined by

$$D_2(\mathcal{B}) = \bigvee \left\{ I \in IN(\mathbf{Top}) : \mathcal{B} \subseteq C_2(I) \right\}.$$

Such that $IN(\mathbf{Top}) \underset{D_2}{\overset{C_2}{\rightleftarrows}} S(\mathbf{Top})^{op}$ is a Galois connection.

Remark 4.34. *Proposition 4.31 and the previous Galois connection provide an important result for the theory of I-coseparated spaces in contrast with the I-separated ones [1]. Precisely, the theory introduced in the latter is equivalent either presented by means of the concept of the separator of two continuous functions being I-open or by the complement of the diagonal being I-open. But on the contrary, the I-coseparated spaces defined by the complement of the diagonal being I-isolated contain the ones introduced by means of the I-isolated separators (I-2nd coseparated) but there are spaces whose complement of the diagonal is I-isolated (I-coseparated) that are not I-2nd coseparated.*

As a consequence, the map $IN(\mathbf{Top}) \xrightarrow{C_2} S(\mathbf{Top})^{op}$ which to each interior operator I assigns its I-2nd coseparated spaces, produces the Galois connection $IN(\mathbf{Top}) \underset{D_2}{\overset{C_2}{\rightleftarrows}} S(\mathbf{Top})^{op}$, that differs from the Galois connection

$IN(\mathbf{Top}) \underset{D}{\overset{C}{\rightleftarrows}} S(\mathbf{Top})^{op}$ *presented in Remark 4.16. However, in the theory developed in the paper [1] for the* I*-separated spaces, the corresponding Galois connections are the same.*

CHAPTER 5
CONCLUSIONS AND FUTURE WORK

CONCLUSIONS:

A notion of coseparation with respect to an interior operator on **Top** was introduced and it was proved that it is closed under direct images of continuous functions and quotient spaces, but it is not closed under topological sums and topological subspaces.

Examples of coseparated spaces for concrete interior operators in topology were presented.

A commutative diagram of Galois connection was shown and through which it was proved that the left-right constant Galois connection factorizes through the I-coseparated and I-separated Galois connections.

A new notion of coseparation with respect to an interior operator on **Top**, called I-$2nd$ coseparation, was introduced and it was proved that it is weaker than the notion of I-coseparation. Moreover, it was shown that it produces a Galois connection that differs from the Galois connection initially introduced by I-coseparation. However from the perspective of I-separation, the corresponding notions and Galois connections defined in term of separators or in term of the complement of the diagonal are exactly the same.

FUTURE WORK:

- Find a Galois connection between $IN(\mathbf{Top})$ and $S(\mathbf{Top})$ that together with $IN(\mathbf{Top}) \underset{D_2}{\overset{C_2}{\rightleftarrows}} S(\mathbf{Top})^{op}$ factorizes the left-right constant Galois connection, in an analogous way to what is presented in Theorem 4.27.

- Explore the possibility of introducing a notion of coseparation with respect to an interior operator in the category **Grp** of groups, with the purpose of trying to extend to that environment the results presented in this work.

  We would like to anticipate that this idea presents an initial challenge. Precisely, since an interior operator I on **Grp** acts on subgroups and for any non-trivial group X, $\complement\Delta_X$ is not a subgroup of $X \times X$, the notions of separation and coseparation with respect to an interior operator I on **Grp** cannot be defined directly as in the topological case. Consequently, in order to develop the ideas of this work in **Grp**, one must look for a different approach.

- Obtain an explicit characterization of $2nd\text{-}Cosep(I)$, for every interior operator I from Examples 3.4.

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