REISSNER'S PLATE THEORY IN THE FRAMEWORK OF ASYMMETRIC ELASTICITY

By

Pedro Joaquín Madrid

A thesis submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

APPLIED MATHEMATICS

UNIVERSITY OF PUERTO RICO
MAYAGÜEZ CAMPUS

November, 2007

Approved by:

Lev Steinberg, Ph.D President, Graduate Committee	Date
Paul E. Castillo, Ph.D Member, Graduate Committee	Date
Krzysztof Rózga, Ph.D Member, Graduate Committee	Date
Arsenio Cáceres, Ph.D Representative of Graduate Studies	Date
Julio Quintana, Ph.D Chairperson of the Department	Date

Abstract of Dissertation Presented to the Graduate School
of the University of Puerto Rico in Partial Fulfillment of the
Requirements for the Degree of Master of Science

REISSNER'S PLATE THEORY IN THE FRAMEWORK OF ASYMMETRIC ELASTICITY

By

Pedro Joaquín Madrid

November 2007

Chair: Dr. Lev Steinberg
Major Department: Department of Mathematical Sciences

ABSTRACT

The purpose of this thesis is to develop a new mathematical model for the bending of thin elastic plates with microstructure. Our approach is based on a generalization of the classical Reissner plate theory, which takes into account the transverse microrotation of the plates. Our model assumes polynomial approximations over the plate thickness of asymmetric stress, couple stress, displacement, and microrotation, which are consistent with the elastic equilibrium, boundary conditions and the constitutive relationships. We use a Cosserat free elastic energy function which includes the energy of the transverse shear couple stress. The application of the method of Lagrange multipliers to the free elastic energy function leads to a system of 9 equations describing the bending (6 equations) and the twisting (3 equations) of the plate. Analytical solutions for the deflection of the plate are calculated for a square plate made of syntatic foam. The Fourier series method is applied. The solutions are compared with a model developed by Eringen and also with solutions obtained from the classical theory. The results illustrate the influence of transverse microrotations on the bending of the rectangular plate.

Resumen de Disertación Presentado a Escuela Graduada
de la Universidad de Puerto Rico como requisito parcial de los
Requerimientos para el grado de Maestría en Ciencias

TEORÍA DE PLACAS DE REISSNER DESDE EL PUNTO DE VISTA DE LA TEORÍA DE ELASTICIDAD ASIMÉTRICA

Por

Pedro Joaquín Madrid

Noviembre 2007

Consejero: Dr. Lev Steinberg
Departamento: Departamento de Ciencias Matemáticas

RESUMEN

El propósito de esta tesis es desarrollar un nuevo modelo matemático que gobierne la deformacion de placas delgadas considerando los efectos de la microestructura. Nuestra metodología esta basada en la generalización de la teoría de placas de Reissner, tomando ahora en consideración el efecto de la microestructura. En nuestro modelo se asume que los esfuerzos y los momentos acoplados se pueden aproximar por medio de polinomios cuya variable dependiente se encuentra a lo largo del grosor de la placa. Los vectores de desplazamiento y microrotación también adquieren una representacion por medio de polinomios. Los grados del polinomio se eligen a modo las ecuaciones de equilibrio elástico, las condiciones de frontera y la ley de Hooke cumplan el principio de consistencia. El sistema de ecuaciones en derivadas parciales que gobierna la deformación de la placa se obtiene del funcional de energía elástica de Cosserat. El método de multiplicadores de Lagrange se aplica,

en total se obtienen 9 ecuaciones donde 6 de ellas describen la deflexión de la placa y las restantes 3 la torsión.

Las ecuaciones de deflexión las resolvemos por medio del método de series de Fourier. Como experimento consideramos una placa constituida de espuma sintética. Las soluciones analíticas son comparadas con un modelo desarrollado por Eringen y tambien con los resultados de la teoría clásica de elasticidad. Los resultados ilustran el efecto de la microestructura en la deflexión de la placa.

DEDICATORY

I dedicate this thesis to the following people:

my mother Martha Ramírez,

my late father José Arnulfo Madrid,

my sister Lizbeth Madrid,

my brothers Ramón Madrid and Luis Felipe Madrid.

ACKNOWLEDGMENTS

I want to express my gratitude to the following people, without their help this work would not be possible:

Professor Lev Steinberg for posing me the problem of elasticity theory, for his patience, for guiding me in the elaboration of the mathematical model and for giving me advices about my career and personal life.

Professor Paul Castillo for giving me the suggestion to study at the University of Puerto Rico and for his permanent care of my academic and personal life.

Professor Krzysztof Rózga and Professor Arsenio Cáceres for their valuable corrections to my thesis.

Professor Adalid Gutierrez for his inspiring teaching of undergraduate math courses and for motivating me to continue graduate studies in mathematics.

Professor Salvador Llopis for teaching me the basic techniques of mathematical proofs.

Professor Concepción Ferrufino for presenting me the beauty of mathematics.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF ABBREVIATIONS

PDE'S	Partial Differential Equations.
IHL	Isotropic Homogeneous Linear.

LIST OF SYMBOLS

E	Young's Modulus.
ν	Poisson's ratio.
D	Flexural rigidity.
G	Shear Modulus.
σ_{ij}	Component ij of stress tensor.
γ_{ij}	Component ij of the strain tensor.
μ_{ij}	Component ij of couple stress tensor.
φ_i	The i^{th} component of microrotation vector.
u_i	The i^{th} component of displacement vector.
F	Free elastic energy.
C	Bulk energy.
$\lambda,\ \mu$	The Lamé constants.
$\alpha,\ \beta,\ \gamma,\ \epsilon$	Complementary elasticity constants.
ϵ_{ijk}	Levi Civita tensor.
δ_{ij}	Delta Dirac tensor.
Γ	Boundary of the middle plane of a plate.
N	Coupling number.
l_t	Characteristic torsion.
l_b	Characteristic bending.
Ψ	Polar ratio.
∂R	Boundary of region R.
χ_{ij}	Component ij of the gradient of microrotation vector.
T	Top face of a plate.
B	Bottom face of a plate.
$\Gamma_1 \setminus \Gamma_2$	Differernce between set Γ_1 and Γ_2.
$\Gamma_1 \cup \Gamma_2$	Union of the sets Γ_1 and Γ_2.
$\delta\ I$	First variation applied to functional I.
p	External force per unit area.
t	External momentum per unit area.
Ω_i^0	Microrotation in the middle plane.
W	vertical deflection of the middle plane.
Ψ_i	Macrorotation of the middle plane.

CHAPTER 1
HISTORY OF ELASTICITY THEORY

In general terms, the elasticity theory studies the resistance to deformation of solid bodies subjected to a given set of forces. The first person who studied the nature of the resistance of solids was Galileo Galilei (1564 - 1642). In all of his work he treated solids as inelastic, that is, solids undergoing deformations never recover their original shape. This consideration made impossible the hypothesis of connecting applied forces to a body with their relative displacements. The most important contribution Galileo made in this field was posing a problem consisting of the determination of the axis on which a beam built into a wall would tend to turn. His investigations motivated many people to continue research in this direction. The next big advances where made by Robert Hooke (1635 - 1703) and Henri Navier (1785 - 1836). In 1660 Hooke discovered an experimental relation between the forces and the strain (relative displacements) of a body known today as Hooke's Law. His law states that the strain of a body is directly proportional to the set of forces applied to it, mathematically this means that applied force and strain follow a linear relation.

The term stress is understood as force per unit area and strain as a measure of deformation. The most important fact of Hooke's Law is that it gives the foundations of the linear theory of Elasticity. Between Hooke's and Navier's period mathematicians like Leonhard Euler (1707 - 1783), Daniel Bernoulli (1700 - 1782), James Bernoulli (1759 - 1789) and others were trying to develop theory for beams, plates, shells and vibrations.

The first mathematician to investigate the general equations of equilibrium and vibrations of elastic solids was Navier. He formulated equations of motion of a displaced molecule by developing expressions for the component in any direction of all the forces acting upon the molecule. He also obtained expressions for the work done by all forces to the molecule and with the use of calculus of variations he obtained a system of pde's together with its boundary conditions. The type of materials Navier studied were assumed to be isotropic and the equations for equilibrium contained only one constant dependent of the elastic properties of the material. Today this constant is known as Young's modulus.

Augustin Jean Fresnel (1788 - 1827) related the study of interference of polarised light with the theoretical results of vibrations in elasticity. This attracted the attention of the mathematicians Augustin Louis Cauchy (1789 - 1857) and Denis Poisson (1781 - 1840). By the year 1822 Cauchy discovered most of the elements of pure theory of elasticity. He introduced the notion of stress at a point which depends of the cross sectional area that contains the point, the principal axes of strain and the principal planes of stress. Also some of his important contributions were the formulation of equations of motion in terms of the stress-components and the acting body forces (Force per unit volume), the description of stress and strain in terms of six components.

In all of Cauchy's work the following assumptions were made: Relations between stress and strain are linear and the principal planes of stress are normal to the principal axes of strain. Both assumptions are supported by Hooke's law. It's interesting to know that Cauchy never made reference to Hooke's law. There is one central difference from Navier's results and Cauchy's : Navier's equations contained a single constant (Young's modulus) to express the elastic behaviour of a body, while Cauchy's contain only two. Poisson's results were equivalent to the ones obtained by Cauchy's, the only difference is that he required different hypothesis.

The next great advances were made by George Green (1793 - 1841), George Stokes (1819 - 1903), Robert Kirchhoff (1824 - 1887) , Gabriel Lamé (1795 - 1870), Lord Kelvin (1824 - 1907) and others. Their studies were concentrated more in the *Principle of Conservation of Energy*. New arguments were employed in their analysis, for example the use of the First and Second Laws of Thermodynamics. The potential energy of the strained elastic body per unit volume was expressed in terms of the components of strain. The resistance to deformation of an elastic body was also well classified into two types, resistance to shearing and to compression. Many of the terminology these last researchers made is still used today.

CHAPTER 2
ELASTICITY THEORY

2.1 Classical Elasticity Theory

The mathematical foundation of elasticity theory deals with the calculation of the relative displacements (deformation) of a solid body which is subject to the action of a system of forces. The deformations in a solid body depend of the type of material the body is made and the nature of the forces applied to it. Elasticity theory concern the situations where after the removal of forces producing the deformation of the body implies a complete recovery of the undeformed state. These type of deformations are known as *elastic* and materials satisfying this property are also known as elastic. Many materials can undergo elastic deformations for instance concrete, steel, aluminum, rubber, etc. From now on when the term deformation is employed it will be understood that it is elastic.

Deformations can be classified as of linear and nonlinear type. We are interested only in the linear case. Typically linear deformations are *very small* and the mathematical theory for its study requires the use of linear partial differential equations (pde's). The area of elasticity that deals with this type of deformations is known as *Linear Elasticity Theory.* Applications of this theory are very important for engineering, architecture and all other areas which deal with solids as material.

In the classical theory of elasticity only macroscopic effects are taken into consideration, that is, all solid bodies are assumed to be made of a continuous medium. It happens that the elastic properties of a body are described by some constants dependent on the structure of the material and known as *elastic coefficients*. The measurement of the elastic coefficients of a material at a given point is done by calculating some ratios between stress (force per unit area in a given direction) and the strain (deformation in a specific direction). In Cosserat's theory the measurement of these coefficients is not an easy task, this is why we don't get deep into this situation.

Materials can be classified according to the properties of their elastic coefficients. A material is said to be homogeneus if its elastic coefficients are independent of the spatial coordinates. If the calculation of the material's elastic coefficients are the same in every set of reference axes at any point then we say its isotropic. In the classical theory of elasticity it's known that all isotropic materials are described with exactly two elastic constants. The type of constants is not neccesary unique but it has been shown that different choices are equivalent. Some examples of isotropic materials are concrete and steel. In the rest of this work we deal with materials that are homogeneous and isotropic.

2.2 Asymmetric Elasticity Theory

As we saw in the previous section, the classical theory of elasticity is based in the model of an elastic continuum in which the transfer of forces through an interior element of area of the body occurs only by means of the stress vector. This type of assumption leads to a mathematical description of stress and strain by means of asymmetric tensors. Now stress and strain at a point require the description of nine components.

Classical elasticity theory showed satisfactory results with experimentation in many structural materials such as aluminum, steel and iron. There were other cases of elastic materials in which theory had discrepances with experimentation. Some of these are polymers, biological materials, cellular materials and nano materials. These differences seemed to become significative for problems where large stress gradients occur (near holes or cracks), for vibrational problems where waves have a very high frequency or small wavelength and for materials that possess granular structure. These type of observations suggested that the influence of microstructure should be taken into account.

Voigt was the first to consider in his work the effect of couple stress (local momentum) in granular bodies. His results lead to a description of stress and strain as nonsymmetric tensors. In 1909 the brothers E. and F. Cosserat published a work where the effects of couple stress and rotation of particles was taken into account. Now deformation of a granular body was not only described by means of a displacement vector, also by a rotation vector. Thus, the Cosserat brothers defined an elastic medium (known today as *Cosserat's Continuum*) where each material point has six degrees of freedon, 3 for displacement and the other 3 for rotations. In Cosserat's Continuum, displacements and rotations are considered to be independent vectors. Today Cosserat's theory is also known with the name of micropolar elasticity.

During the lifetime of the Cosserat's brothers, no special attention was given to their theory. The main reasons could be that the theory treated problems far away from elasticity theory and also the notation employed was very difficult to understand. In the last fourty years Cosserat's theory has attracted the attention of many researchers such as C. Truesdell, C.A. Eringen, W. Nowacki, E.S. Suhubi, R.A. Toupin and others. The first studies were made for materials where only the effect of couple stress was taken into account (no rotations). This type of medium is known as *Pseudo-Cosserat Continuum.* In most recent works made by C.A. Eringen, W. Nowacki and R.A. Toupin, a complete treatment of Cosserat's Continuum has been made[1].

2.3 Basic Equations in Asymmetric Elasticity theory

We explain some mathematical notation before showing some important general results of Asymmetric Elasticity theory. In all expressions the subindexes k, l or m are understood to take values 1, 2 or 3. When i or j are employed as subindexes, we assume they take the values 1 or 2. When repeated subindexes are employed it will be assumed that addition is performed all over their range. For example, if we have γ_{kk} we mean $\gamma_{11}+\gamma_{22}+\gamma_{33}$ and $\sigma_{ij}\mu_i$ means $\sigma_{1j}\mu_1+\sigma_{2j}\mu_2$. When a comma is placed as subindex it will be understood that a derivative has been taken, for example $\sigma_{1,2}$ means $\frac{\partial\sigma_1}{\partial x_2}$ and $\mu_{,1}$ means $\frac{\partial\mu}{\partial x_1}$. The tensor δ_{ij} is defined as 1 if $i=j$ and 0 otherwise. The symbol ϵ_{klm} is defined as

$$\epsilon_{klm} = \begin{cases} 1 & \text{when } (k,l,m) \text{ is an even permutation,} \\ -1 & \text{when } (k,l,m) \text{ is an odd permutation,} \\ 0 & \text{otherwise.} \end{cases}$$

2.3.1 Equations in Asymmetric Elasticity Theory

In this section we describe the equations of equilibrium, constitutive relations and the free volume energy function. Next we give some boundary conditions related to the previous equations.

The Cosserat elasticity equilibrium equations without the presence of body forces have the following form [2]:

$$\begin{aligned} \sigma_{lk,l}(x_1,x_2,x_3) &= 0, \\ \epsilon_{mlk}\sigma_{lk}(x_1,x_2,x_3)+\mu_{lm,l}(x_1,x_2,x_3) &= 0, \end{aligned} \tag{2.1}$$

where σ_{lk} and μ_{lk} are known as the stress and couple stress tensors respectively (see appendix A). All 18 functions σ_{lk} and μ_{lk} may depend also of time t but our problem doesn't require this dependence. The stress σ_{lk} is understood to be contained in the plane whose normal is x_l and in the direction of axis x_k.The same applies to the couple stress μ_{lk}.

The following linear equations for isotropic materials, known as Hooke's Law (also as constitutive relations), relate the deformations of displacements and rotations with the stress and couple stress:

$$\begin{aligned}\sigma_{lk} &= (\mu+\alpha)\gamma_{lk} + (\mu-\alpha)\gamma_{kl} + \lambda\delta_{kl}\gamma_{mm},\\ \mu_{lk} &= (\gamma+\epsilon)\chi_{lk} + (\gamma-\epsilon)\chi_{kl} + \beta\delta_{kl}\chi_{mm},\end{aligned} \tag{2.2}$$

here $\gamma_{kl} = u_{l,k} - \epsilon_{mkl}\varphi_m$ and $\chi_{kl} = \varphi_{l,k}$. The tensor γ_{kl} is known as the micropolar strain tensor, u_k and φ_k are the displacement and microrotation vectors respectively. The coefficients μ, λ, α, γ, ϵ and β are known as the elastic constants of the material. Microstructure is strictly related with α, γ, ϵ and β, while λ and μ are the Lamé constants known from classical elasticity theory.

Equations (2.2) can also be expressed in terms of the following technical constants [3] :

$$\begin{aligned}\text{Young's modulus } E &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu},\\ \text{Poisson's ratio } \nu &= \frac{\lambda}{2(\lambda+\mu)},\\ \text{Characteristic length for torsion } l_t &= \sqrt{\frac{\gamma}{\mu}},\\ \text{Characteristic length for bending } l_b &= \frac{1}{2}\sqrt{\frac{\gamma+\epsilon}{\mu}},\\ \text{Coupling number } N &= \sqrt{\frac{\alpha}{\mu+\alpha}},\\ \text{Polar ratio } \Psi &= \frac{2\gamma}{\beta+2\gamma},\end{aligned} \tag{2.3}$$

some numerical values of (2.3) for different kind of materials appear in [4].

The displacement vector u_k is a measure of the relative positions of material points in the deformed state respect to the undeformed state. Figure 2–1 illustrates this situation. In this figure P represents a specific material point of a body in the deformed state and P^* the new position of P in the deformed state. The micro-rotation vector φ_k is assumed to be positive according to the convention shown in figure 2–2. Formally φ_k is not a vector by the fact that rotations don't satisfy the commutative property of addition of vectors, but for linear deformations (φ_k will assume very small values) rotations behave almost like vectors.

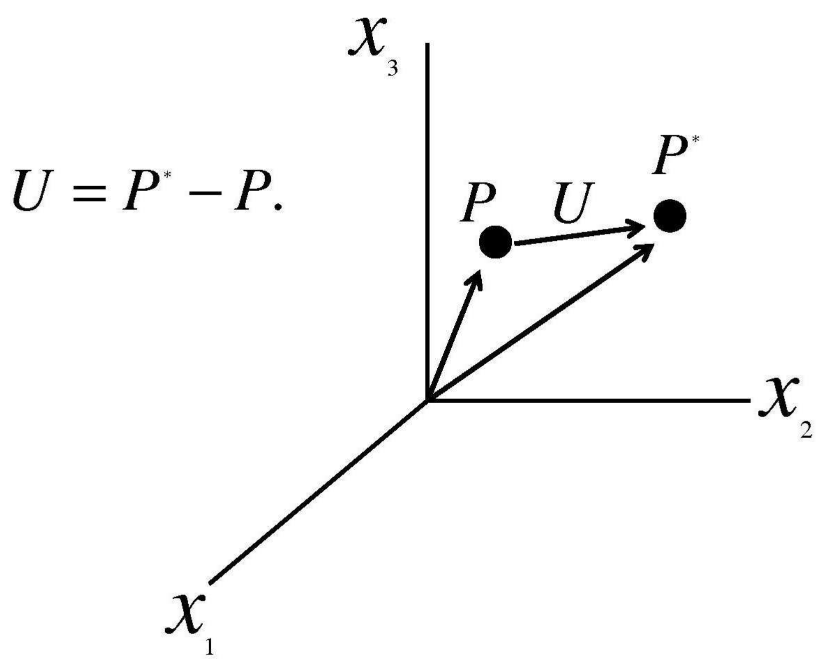

Figure 2–1: Displacement vector.

The strain energy density F in terms of the strain components has the following form [1]:

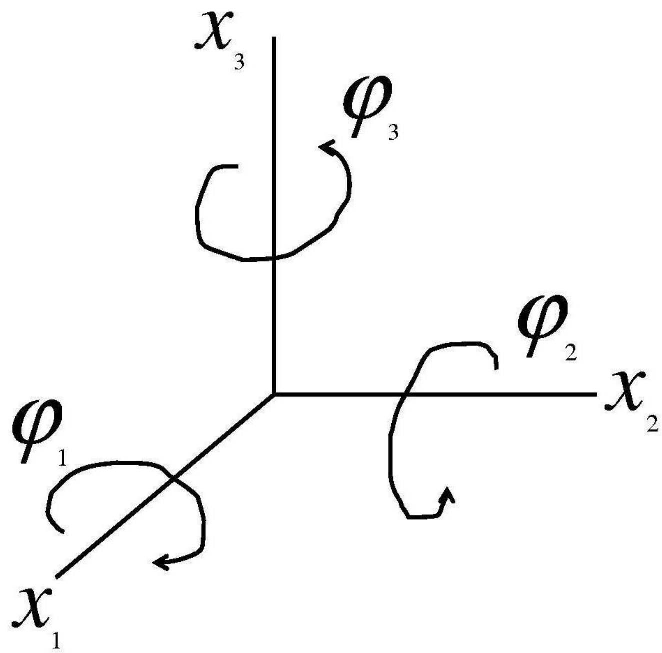

Figure 2–2: Positive orientation for rotation vector.

$$F = \frac{\mu+\alpha}{2}\gamma_{ij}\gamma_{ij} + \frac{\mu-\alpha}{2}\gamma_{ij}\gamma_{ji} + \frac{\lambda}{2}\gamma_{kk}\gamma_{nn} + \frac{\gamma+\epsilon}{2}\chi_{ij}\chi_{ij} + \frac{\gamma-\epsilon}{2}\chi_{ij}\chi_{ji} + \frac{\beta}{2}\chi_{kk}\chi_{nn}, \tag{2.4}$$

From the non-negativity of (2.4) the elastic constants in (2.2) satisfy the inequalities

$$\begin{aligned}
\mu &> 0, \quad 3\lambda + 2\mu > 0, \\
\gamma &> 0, \quad 3\beta + 2\gamma > 0, \\
\alpha &> 0, \quad \mu + \alpha > 0, \\
\epsilon &> 0, \quad \gamma + \epsilon > 0.
\end{aligned}$$

For future convenience we express the strain energy density function (2.4) in terms of σ_{lk} and μ_{lk}. This can be done after solving for γ_{lk} and χ_{lk} in (2.2) and substituting the results in (2.4)

$$
\begin{aligned}
F &= \frac{\mu' + \alpha'}{2}\sigma_{ij}\sigma_{ij} + \frac{\mu' - \alpha'}{2}\sigma_{ij}\sigma_{ji} + \frac{\lambda'}{2}\sigma_{kk}\sigma_{nn} + \\
&\quad \frac{\gamma' + \epsilon'}{2}\mu_{ij}\mu_{ij} + \frac{\gamma' - \epsilon'}{2}\mu_{ij}\mu_{ji} + \frac{\beta'}{2}\mu_{kk}\mu_{nn},
\end{aligned}
\tag{2.5}
$$

where $\mu' = \frac{1}{4\mu}$, $\alpha' = \frac{1}{4\alpha}$, $\gamma' = \frac{1}{4\gamma}$, $\epsilon' = \frac{1}{4\epsilon}$, $\lambda' = \frac{-\lambda}{6\mu(\lambda + \frac{2\mu}{3})}$ and $\beta' = \frac{-\beta}{6\mu(\beta + \frac{2\gamma}{3})}$.

Suppose we have a Cosserat elastic body R with boundary $\partial R = \partial R_d \cup \partial R_\sigma$, where ∂R_d and ∂R_σ are disjoint. In most problems of elasticity, the equilibrium equations (2.1) together with Hooke's law (2.2) are combined with the following boundary conditions:

$$
\begin{aligned}
\sigma_{lk} n_l &= \sigma_{ok},\ \mu_{lk} n_l = \mu_{ok} \text{ on } \partial R_\sigma \\
u_l &= u_{ol},\ \varphi_\alpha = \varphi_{o\alpha} \text{ on } \partial R_d,
\end{aligned}
\tag{2.6}
$$

where σ_{ok} and μ_{ok} are prescribed on ∂R_σ, and u_{ol}, $\varphi_{o\alpha}$ are prescribed on ∂R_s. The coefficients n_l appearing in (2.6) denote the components of the exterior unit normal vector to ∂R.

CHAPTER 3
MATHEMATICAL MODEL

In this chapter we develop a mathematical model for calculation of bending and twisting of a thin plate subject to some perpendicular distributed forces and momentums. Before showing the details of the development of the mathematical model we first explain briefly the meaning of some technical expressions like middle plane, rigidity constant and shear modulus. Next we explain the type of problem we solve.

3.1 Introduction

The well known classic bending theory of elastic plates [5], [6] [7], was first presented by Kirchhoff in his thesis (1850) and is described by a bi-harmonic differential equation [8]. The usual assumption of this theory is that the normal to the middle plane remains normal during deformation. Thus the theory neglects transverse shear strain effects. A system of equations, which takes into account the transverse shear deformation, has been developed by E. Reissner (1945) [9].

One of the advantages of Reissner's model is that it is able to determine the reactions along the edges of a simply supported rectangular plate, where classical theory leads to a concentrated reaction at the corners of the plate. The Reissner theory has been applied to thin walled structures with moderate thickness. The study of the relationships between these two models has proved [10] that the solution of the clamped Reissner plate approaches the solution of the Kirchhoff plate when the thickness approaches to zero. The numerical calculations of bending behavior of the plate of moderate thickness [11] show high level agreement between 3D and

Reissner models. More remarks on the history of the modeling of classic linear elastic plates can be found in [5], [11], [12].

In order to describe deformation of elastic plates that possess grains, particles, fibers, and cellular structures A. C. Eringen (1967) proposed a theory of plates in the framework of Cosserat (micropolar) Linear Elasticity [2]. His theory is based on the integration of the linearized three-dimensional Cosserat Elasticity and assumes variation of micro-rotations along the middle plane. The use of the averages, the first moments of stress, couple stress combined with constitutive relationships provides the model system of equations of Eringen's theory. This technique is similar to the technique used for Kirchhoff plate. In fact, the Eringen plate equations asymptotically produce the Kirchhoff plate bi-harmonic equation for zero microrotations, i.e. it reduces to the classic bending problem. In this chapter we propose to use the Reissner plate theory as a foundation for the modeling of elastic plates with microstructure. Our approach, in addition to the transverse shear deformation, also takes into account the second order approximation of couple stresses and the variation of micropolar rotations in the thickness direction. A governing system of equations is obtained for the bending and twisting of the Cosserat plate, a proof for the existence of the governing system is also developed.

3.2 The Cosserat Plate Assumptions

In this section we formulate the stress, couple stress and kinematic assumptions of the Cosserat plate. We consider the thin plate P that appears in Figure 3–1, here h is the thickness of the plate and $x_3 = 0$ contains its middle plane. The sets T and B are the top and bottom surfaces contained in the planes $x_3 = h/2$, $x_3 = -h/2$ respectively and Γ is the boundary of the middle plane of the plate. The set of points $\partial P = \{\Gamma \times [-h/2, h/2]\} \cup T \cup B$ forms the entire surface of the plate.

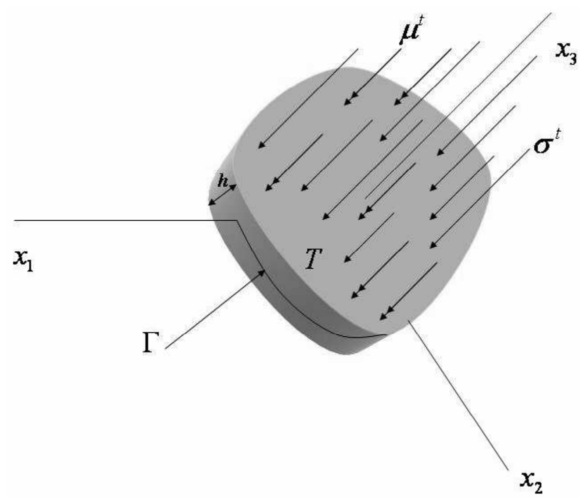

Figure 3–1: A plate element.

We assume that plate P is subjected to some perpendicular distributed load of stress and couple stress along faces T and B. These conditions are described in the following form:

$$\begin{aligned} \sigma_{33}(x_1, x_2, h/2) &= \sigma^t(x_1, x_2), \quad \sigma_{33}(x_1, x_2, -h/2) = \sigma^b(x_1, x_2), \\ \sigma_{3j}(x_1, x_2, \pm h/2) &= 0, \qquad\qquad\quad \mu_{3j}(x_1, x_2, \pm h/2) = 0, \\ \mu_{33}(x_1, x_2, h/2) &= \mu^t(x_1, x_2), \quad \mu_{33}(x_1, x_2, -h/2) = \mu^b(x_1, x_2). \end{aligned} \tag{3.1}$$

here σ^t and μ^t are the normal loads of stress and couple stress acting at the top of the plate. The functions σ^b and μ^b, describe the normal load of stress and couple stress acting along B.

Assumptions for Stress

Our approach, which is in the spirit of the Reissner's theory of plates [9], assumes that the variation of stress σ_{kl} and coupled stress μ_{kl} components across the thickness can be represented by means of polynomials of x_3 in such a way that it will be consistent with the equilibrium equations (2.1). Like in standard theory of plates, first we assume the following form for some stress components [9]:

$$\sigma_{ij} = n_{ij}(x_1, x_2) + x_3 m_{ij}(x_1, x_2). \tag{3.2}$$

where n_{ij} and m_{ij} are functions to be determined. The difference between our and Reissner's assumptions is that the functions n_{ij}, m_{ij} are not necessarily symmetric. From (3.2) by means of equilibrium equations we obtain the following form for the shear stress components:

$$\sigma_{3j} = q_j(x_1, x_2)\left(1 - \left(\frac{x_3}{h/2}\right)^2\right), \tag{3.3}$$

where q_j are functions to be determined. For the remaining shear stress components we assume they have following form:

$$\sigma_{j3} = q_j^*(x_1, x_2)\left(1 - \left(\frac{x_3}{h/2}\right)^2\right). \tag{3.4}$$

In order to preserve consistency, assumption (3.4) seems the most natural. The function q_j^* is also unknown and in the classical case it should be the same as q_j. After substituting expressions (3.4) in the remaining equilibrium equations of stress , we obtain the following form for the transverse normal stress:

$$\sigma_{33} = \frac{x_3}{h/2}\left(\frac{1}{3}\left(\frac{x_3}{h/2}\right)^2 - 1\right) k^*(x_1, x_2) + m^*(x_1, x_2), \tag{3.5}$$

(3.6)

where k^* and m^* are functions to be determined. The functions k^* and m^* in (3.5) can be determined with the boundary conditions (3.1). It's easy to check that $k^* = -\frac{3}{4}(\sigma^t - \sigma_b)$ and $m^* = \frac{\sigma^t + \sigma^b}{2}$, therefore equation (3.5) takes the following form:

$$\sigma_{33} = -\frac{3}{4}\left(\frac{1}{3}\left(\frac{x_3}{h/2}\right)^3 - \left(\frac{x_3}{h/2}\right)\right)p + \sigma_0,$$

where $p = \sigma^t - \sigma^b$ and $\sigma_0 = \frac{1}{2}(\sigma^t + \sigma^b)$.

The assumptions for μ_{kl} follow from the stress assumptions made above and the equilibrium equations for couple stress:

$$\mu_{ij} = \left(1 - \left(\frac{x_3}{h/2}\right)^2\right) r_{ij}(x_1, x_2), \tag{3.7}$$

$$\mu_{j3} = \left(\frac{x_3}{h/2}\right) s_j^*(x_1, x_2) + m_j^*(x_1, x_2), \tag{3.8}$$

$$\mu_{3j} = \left(\frac{1}{3}\left(\frac{x_3}{h/2}\right)^3 - \left(\frac{x_3}{h/2}\right)\right) s_j(x_1, x_2) + m_j(x_1, x_2), \tag{3.9}$$

here the functions s_j^*, m_j^*, r_{ij}, s_j and m_j are also to be determined. Like the case for σ_{3j}, the boundary conditions (3.1)) are enough to identify s_j and m_j, hence it's not difficult to show that $s_j = 0$, $m_j = 0$, therefore

$$\mu_{3j} = 0.$$

Now substituting (3.8) and (3.2) on the third equilibrium equation of (2.1), we conclude that μ_{33} should be of the following form

$$\mu_{33} = \frac{1}{2}\left(\frac{x_3}{h/2}\right)^2 a^*(x_1, x_2) + \left(\frac{x_3}{h/2}\right) b^*(x_1, x_2) + c^*(x_1, x_2),$$

where the functions a^*, b^* and c^* should satisfy the boundary conditions (3.1). It happens that conditions (3.1) are not enough to determine all coefficients of μ_{33}, hence a^* and c^* can be any arbitrary function. For simplicity on the approximation of μ_{33} we make $a^* = 0$ and therefore assume that is a first order polynomial in the variable x_3,

$$\mu_{33} \quad = \quad \left(\frac{x_3}{h/2}\right) b^*(x_1, x_2) + c^*(x_1, x_2).$$

Under this new assumption of μ_{33} we find that $b^* = \frac{\mu^t - \mu^b}{2}$ and $c^* = \frac{\mu^t + \mu^b}{2}$. Finally the couple stress μ_{33} takes the following form:

$$\mu_{33} = \frac{x_3}{h/2} v + t, \tag{3.10}$$

where $t(x_1, x_2) = \frac{1}{2}\left(\mu^t(x_1, x_2) + \mu^b(x_1, x_2)\right)$ and $v(x_1, x_2) = \frac{1}{2}\left(\mu^t(x_1, x_2) - \mu^b(x_1, x_2)\right)$.

Up to this point all components of stress and couple stress are represented in terms of the 20 unknown functions n_{ij}, m_{ij}, q_j, q_j^*, r_{ij}, s_j^* and m_j^*. The next step is to substitute the assumptions of stress and couple stress in (2.1) and obtain a new system of equilibrium equations. For simplicity we make an average of (2.1) in the following form:

$$\begin{aligned} \int_{-h/2}^{h/2} \sigma_{lk,l}(x_1, x_2, x_3) dx_3 \quad &= \quad 0, \\ \int_{-h/2}^{h/2} \left(\epsilon_{mlk}\sigma_{lk}(x_1, x_2, x_3) + \mu_{lm,l}(x_1, x_2, x_3)\right) dx_3 \quad &= \quad 0, \end{aligned} \tag{3.11}$$

this simplification is good enough to describe deformation along the middle plane since we assume h to be very small compared to the plate dimensions [2]. After substituting the assumptions for stress and couple stress in (3.11) we obtain a new system of nine equilibrium equations. The resulting system is classified into two parts: The bending system which is composed of 6 equations and the twisting system which has 3 equations.

The bending system of equations has the following form:

$$
\begin{aligned}
M_{11,1} + M_{21,2} - Q_1 &= 0, \\
M_{12,1} + M_{22,2} - Q_2 &= 0, \\
Q^*_{1,1} + Q^*_{2,2} + p &= 0, \\
R_{11,1} + R_{21,2} + Q^*_2 - Q_2 &= 0, \\
R_{12,1} + R_{22,2} + Q_1 - Q^*_1 &= 0, \\
S^*_{1,1} + S^*_{2,2} + M_{12} - M_{21} &= 0, \qquad (3.12)
\end{aligned}
$$

with traction boundary conditions at Γ_σ:

$$
\begin{aligned}
M_{ij} n_j &= \Pi_{0j}, \; R_{ij} n_j = M_{0j}, \\
Q^*_i n_i &= \Pi_{03}, \; S^*_i n_i = M_{03},
\end{aligned}
$$

where

$$
\begin{aligned}
M_{ij} &= \frac{h^3}{12} m_{ij}, & R_{ij} &= \frac{2h}{3} r_{ij}, \\
\Pi_{0j} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 \sigma_{0j} dx_3, & M_{0j} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \mu_{0j} dx_3, \\
Q_j &= \frac{2h}{3} q_j, & Q^*_j &= \frac{2h}{3} q^*_j, \\
\Pi_{03} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\sigma_{03} - \sigma_0\right) dx_3, & M_{03} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 \left(\mu_{03} - \frac{x_3}{h/2} v\right) dx_3, \\
S^*_j &= \frac{h^2}{6} s^*_j
\end{aligned}
$$

The twisting equilibrium equations have the following form:

$$
\begin{aligned}
N_{ij,i} &= 0, \\
M^*_{1,1} + M^*_{2,2} + N_{12} - N_{21} + v &= 0
\end{aligned}
\tag{3.13}
$$

with traction boundary conditions at Γ_σ:

$$
\begin{aligned}
N_{ij}n_1 + N_{2j}n_2 &= \Sigma_{0,j}, \\
M^*_1 n_1 + M^*_2 n_2 &= M_{03},
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma_{0,j} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{0j} dx_3, \;\; M_{03} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\mu_{03} - tn_3) dx_3, \\
N_{ij} &= hn_{ij}, \qquad\qquad M^*_j = hm^*_j.
\end{aligned}
$$

The boundary conditions for the bending and twisting systems are obtained after the substitution of the stress and couple stress assumptions in (2.6). The boundary conditions at $\Gamma_d = \Gamma \setminus \Gamma_\sigma$ will be given after the kinematic assumptions are stated.

3.3 Kinematic Assumptions

Similarly to the case of stress and couple stress assumptions, the choice of kinematic assumptions (assumptions made for displacement and microrotation vectors) is based on their compatibility with the constitutive relationships (2.2). Like in Eringen's work [2] and in [13], we make a linear approximation for the displacement vector in the following form:

$$
\begin{aligned}
u_i(x_1, x_2, x_3) &= U_i(x_1, x_2) - x_3 V_i(x_1, x_2), \\
u_3(x_1, x_2, x_3) &= w(x_1, x_2),
\end{aligned}
\tag{3.14}
$$

where U_i, V_i and w are unknown functions. The microrotations are approximated in the following form:

$$\begin{aligned}
\varphi_i(x_1,x_2,x_3) &= \Theta_i^0(x_1,x_2)\left(1-\left(\frac{x_3}{h/2}\right)^2\right), \\
\varphi_3(x_1,x_2,x_3) &= \Theta_3^0(x_1,x_2)+\frac{x_3}{h/2}\left(1-\frac{1}{3}\left(\frac{x_3}{h/2}\right)^2\right)\Theta_3(x_1,x_2), \qquad (3.15)
\end{aligned}$$

where the functions Θ_k^0 are also unknown. Up to this point we can appreciate that the deformation of the middle plane is completely described in terms of the 9 functions U_i, V_i, w, Θ_i^0, Θ_3^0 and Θ_3.

In [2] microrotations are assumed to be constant along the thickness of the plate while (3.15) indicate that microrotations are dependent of plate thickness. We believe this difference makes our approach more convenient for relatively thick plates [14].

3.4 Free energy expression

The total energy of the plate is calculated in the following form [1]:

$$I_F = \int\int_R \int_{-h/2}^{h/2} F dx_3 dA - \int_{\Gamma_d}\int_{-h/2}^{h/2} F_s dx_3 ds, \qquad (3.16)$$

where $\Gamma_d = \Gamma \backslash \Gamma_\sigma$ is the portion of Γ on which edge displacements and microrotations are prescribed. In order to develop a system of governing equations of our plate, the method we follow is based on the Lagrange's method applied to (3.16). A total amount of 9 Lagrange's multipliers will appear, each of them has a physical meaning that in further sections will be explained.

The first thing we do is evaluate the free bulk energy

$$C = \int_{-\frac{h}{2}}^{\frac{h}{2}} F \; dx_3 \qquad (3.17)$$

of the plate. After integrating expression (3.17) and substituting the stress and couple stress assumptions (3.2) - (3.9) in (2.5), expression (3.17) takes the following form:

$$
\begin{aligned}
C \;=\;& \frac{\lambda+\mu}{2h\mu(3\lambda+2\mu)}\left[\left(N_{11}^2+N_{22}^2\right)+\frac{12}{h^2}\left(M_{11}^2+M_{22}^2\right)\right]-\frac{\lambda}{2h\mu(3\lambda+2\mu)}N_{11}N_{22}+\\
& \frac{6\lambda}{h^3\mu^2(3\lambda+2\mu)}M_{11}M_{22}+\frac{\alpha+\mu}{8h\alpha\mu}\left[\left(N_{12}^2+N_{21}^2\right)+\frac{3}{4h^2}\left(M_{12}^2+M_{21}^2\right)\right]+\\
& \frac{3(\alpha+\mu)}{20h\alpha\mu}\left((Q_1^*)^2+Q_2^2+(Q_2^*)^2\right)+\frac{3(\alpha-\mu)}{10h\alpha\mu}\left(Q_1Q_1^*+Q_2Q_2^*\right)+\\
& \frac{3(\alpha-\mu)}{h^3\alpha\mu}M_{12}M_{21}+\frac{3\lambda}{5h\mu(3\lambda+2\mu)}\left[M_{11}Q_{1,1}^*+M_{11}Q_{2,2}^*+M_{22}Q_{1,1}^*\right]+\\
& \frac{3\lambda}{5h\mu(3\lambda+2\mu)}M_{22}Q_{2,2}^*+\frac{3(\epsilon-\gamma)}{10h\gamma\epsilon}R_{12}R_{21}+\frac{3(\beta+\gamma)}{5h\gamma(3\beta+2\gamma)}\left(R_{11}^2+R_{22}^2\right)-\\
& \frac{3\beta}{5h\gamma(3\beta+2\gamma)}R_{11}R_{22}+\frac{17h(\lambda+\mu)}{280\mu(3\lambda+2\mu)}\left[\left(Q_{1,1}^*\right)^2+\left(Q_{2,2}^*\right)^2+2Q_{1,1}^*Q_{2,2}^*\right]+\\
& \frac{\alpha-\mu}{4h\alpha\mu}N_{12}N_{21}-\frac{\gamma+\epsilon}{h\gamma\epsilon}\left[\frac{3}{2h^2}\left((S_1^*)^2+(S_2^*)^2\right)+\frac{3}{20}\left(R_{12}^2+R_{21}^2\right)\right]+\\
& \frac{h(\lambda+\mu)}{2\mu(3\lambda+2\mu)}\sigma_0^2-\frac{\beta}{2\gamma(3\beta+2\gamma)}\left(R_{11}t+R_{22}t\right)+\frac{h(\beta+\gamma)}{2\gamma(3\beta+2\gamma)}t^2-\\
& \frac{\gamma+\epsilon}{8h\gamma\epsilon}(M_1^*)^2+\frac{3(\alpha+\mu)}{20h\alpha\mu}Q_1^2+\frac{\lambda}{2\mu(3\lambda+2\mu)}\left(N_{11}\sigma_0+N_{22}\sigma_0\right)-\frac{\gamma+\epsilon}{8h\gamma\epsilon}(M_2^*)^2+\\
& \frac{h(\beta+\gamma)}{6\gamma(3\beta+2\gamma)}v^2. \qquad (3.18)
\end{aligned}
$$

Now we evaluate the surface integral of (3.16) on the boundary $\Gamma_d\times[-h/2,h/2]$ by means of the following formula

$$\int_{\Gamma_d}\int_{-h/2}^{h/2}F_s dx_3 ds=\int_{\Gamma_d}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_\nu\cdot\mathbf{u}+\mu_\nu\cdot\varphi_\nu\right)dx_3 ds. \qquad (3.19)$$

The vectors σ_ν and μ_ν in (3.19) are the components of the stress and couple stress acting along $\Gamma_d\times[-\frac{h}{2},\frac{h}{2}]$ and coplanar to the middle plane of the plate. Representing σ_ν and μ_ν in the form $\sigma_\nu=\sigma_{\nu_1}\boldsymbol{\nu_1}+\sigma_{\nu_2}\boldsymbol{\nu_2}$ and $\mu_\nu=\mu_{\nu_1}\boldsymbol{\nu_1}+\mu_{\nu_2}\boldsymbol{\nu_2}$, where $\boldsymbol{\nu_1}$ and $\boldsymbol{\nu_2}$ are unit vectors normal and tangential to $\Gamma_d\times[-\frac{h}{2},\frac{h}{2}]$ respectively, equation (3.19) takes the following form:

$$\int_{\Gamma_d}\int_{-\frac{h}{2}}^{\frac{h}{2}}(n_{\nu_i}u_i + q^*_{\nu_3}\left(1-\left(\frac{x_3}{h/2}\right)^2\right)u_3 + m_{\nu_i}u_i x_3 +$$
$$r_{\nu_i}\left(1-\left(\frac{x_3}{h/2}\right)^2\right)\varphi_i + \left(\left(\frac{x_3}{h/2}\right)s^*_{\nu_3} + m^*_{\nu_3}\right)\varphi_3)dx_3 ds,$$

or equivalently:

$$\int_{\Gamma_d}(N_{\nu_i}U_i + Q^*_{\nu_3}W + M_{\nu_i}\Psi_i + R_{\nu_i}\Omega^0_i + S^*_{\nu_3}\Omega_3 + M^*_{\nu_3}\Omega^0_3)ds,$$

where

$$\begin{aligned}
W &= \frac{3}{2h}\int_{-h/2}^{h/2} u_3(x_1,x_2,x_3)\left(1-\left(\frac{x_3}{h/2}\right)^2\right)dx_3,\\
\Psi_i &= \frac{12}{h^3}\int_{-h/2}^{h/2} x_3 u_i(x_1,x_2,x_3)dx_3,\\
\Omega^0_i &= \frac{3}{2h}\int_{-h/2}^{h/2} \varphi_i(x_1,x_2,x_3)\left(1-\left(\frac{x_3}{h/2}\right)^2\right)dx_3,\\
\Omega_3 &= \frac{12}{h^3}\int_{-h/2}^{h/2} x_3\varphi_3(x_1,x_2,x_3)dx_3,\\
\Omega^0_3 &= \frac{1}{h}\int_{-h/2}^{h/2} \varphi_3(x_1,x_2,x_3)dx_3,\\
U_i &= \frac{1}{h}\int_{-h/2}^{h/2} u_i(x_1,x_2,x_3)dx_3.
\end{aligned} \tag{3.20}$$

The functions appearing in (3.20) will be the Lagrange multipliers that make free elastic energy a minimum. The calculation of W, U_i and Ψ in (3.20) is based on the same methodology of Reissner's work [12] while the expressions for Ω^0_i, Ω_3, and Ω^0_3 are assumptions we make that later we verify they are correct. In next section we give more details about (3.20). After applying (3.14) and (3.15) in (3.20) we obtain the following expressions:

$$\begin{aligned} W &= w(x_1, x_2), \qquad \Psi_i = V_i(x_1, x_2), \\ \Omega_i^0 &= k_1 \Theta_i^0(x_1, x_2), \quad \Omega_3 = \frac{k_2}{h}\Theta_3(x_1, x_2), \\ \Omega_3^0 &= \Theta_3^0(x_1, x_2), \qquad U_i = U_i(x_1, x_2). \end{aligned} \tag{3.21}$$

here coefficients k_1 and k_2 depend on the variation of microrotations. Under the conditions (3.15) we have that $k_1 = \frac{4}{5}$ and $k_2 = \frac{24}{15}$.

The physical interpretation of the functions appearing in (3.20) can be verified with equations (3.21), (3.14) and (3.15). We summarize this in the following way:

- W : Vertical deflection of the middle plate.

- Ψ_i : Angle of deflection of the middle plane respect to the horizontal.

- Ω_k^0 : Microrotation around axis x_k of the material points of the middle plate.

- U_i : Displacement of the middle plane along axis x_i.

- Ω_3 : Instant rate of change of φ_3 along x_3,

figure (3–2) illustrates the physical interpretation of some of the above parameters.

3.5 Lagrange Equations and Constitutive Relations

Following Reissner methodology [9] we consider the zero variation of the total strain energy (3.16) due to the volume and surface area of the Cosserat plate under the constraint of equilibrium equations (3.12) and (3.13). According to the rules of the calculus of variations, this is accomplished by combining (3.16) with (3.12) and (3.13) in the following manner:

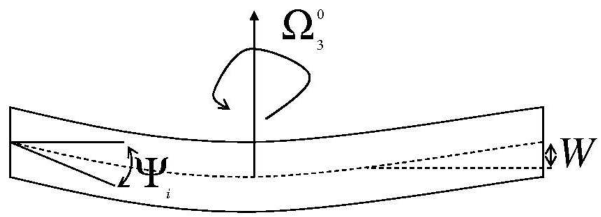

Figure 3–2: Illustration of some parameters appearing in equations (3.20).

$$\delta[I_F] \quad + \quad \int\int_R (U_j \delta[N_{ij,i}] \quad + \quad \Psi_j \left(\delta[M_{ij,i}] - \delta[Q_j]\right) \; + \; W\delta[Q^*_{j,j}] \quad +$$
$$\Omega^0_i \delta[(-1)^{(i+1)}(Q^*_{i+1} - Q_{i+1}) + R_{ji,j}] + \Omega^0_3 \delta[N_{12} - N_{21} + M^*_{1,1} + M^*_{2,2}] +$$
$$\Omega_3 \delta[M_{12} - M_{21} + S^*_{1,1} + S^*_{2,2}]) = 0. \tag{3.22}$$

After applying integration by parts and Stoke's theorem to (3.22) we obtain

$$
\begin{aligned}
0 \;=\; & \int\int_R ((\frac{\partial C}{\partial N_{11}} - U_{1,1})\delta N_{11} + (\frac{\partial C}{\partial N_{22}} - U_{2,2})\delta N_{22} + (\frac{\partial C}{\partial N_{12}} - U_{2,1})\delta N_{12} + \\
& (\frac{\partial C}{\partial N_{21}} - U_{1,2} - \Omega_3^0)\delta N_{21} + (\frac{\partial C}{\partial M_{11}} - \Psi_{1,1})\delta M_{11} + (\frac{\partial C}{\partial M_{22}} - \Psi_{2,2})\delta M_{22} + \\
& (\frac{\partial C}{\partial M_{12}} - \Psi_{2,1} + \Omega_3)\delta M_{12} + (\frac{\partial C}{\partial M_{21}} - \Psi_{1,2} - \Omega_3)\delta M_{21} + (\frac{\partial C}{\partial Q_1^*} - W_{,1})\delta Q_1^* + \\
& (\frac{\partial C}{\partial Q_2^*} - W_{,2} + \Omega_1^0)\delta Q_2^* + (\frac{\partial C}{\partial Q_1} - \Psi_1 + \Omega_2^0)\delta Q_1 + (\frac{\partial C}{\partial Q_2} - \Psi_2)\delta Q_2 - \\
& \Omega_1^0 \delta Q_2 + (\frac{\partial C}{\partial R_{11}} - \Omega_{1,1}^0)\delta R_{11} + (\frac{\partial C}{\partial R_{22}} - \Omega_{2,2}^0)\delta R_{22} + (\frac{\partial C}{\partial R_{21}} - \Omega_{1,2}^0)\delta R_{21} + \\
& (\frac{\partial C}{\partial R_{12}} - \Omega_{2,1}^0)\delta R_{12} + (\frac{\partial C}{\partial M_1^*} - \Omega_{3,1}^0)\delta M_1^* + (\frac{\partial C}{\partial M_2^*} - \Omega_{3,2}^0)\delta M_2^* + \frac{\partial C}{\partial S_1^*}\delta S_1^* - \\
& \Omega_{3,1}\delta S_1^* + (\frac{\partial C}{\partial S_2^*} - \Omega_{3,2})\delta S_2^* + \Omega_3^0 \delta N_{12} - \Omega_2^0 \delta Q_1^*) + \oint_\Gamma (\delta N_{\nu_1} U_1 + \delta N_{\nu_2} U_2 + \\
& \delta M_{\nu_1}\Psi_1 + \delta M_{\nu_2}\Psi_2 + \delta R_{\nu_1}\Omega_1^0 + \delta R_{\nu_2}\Omega_2^0 + \delta Q_{\nu_1}^* W + \delta Q_{\nu_2}^* W + \delta M_{\nu_1}^*\Omega_3^0 + \\
& \delta M_{\nu_2}^*\Omega_3^0 + \delta S_{\nu_1}^*\Omega_3 + \delta S_{\nu_2}^*\Omega_3) - \int_{\Gamma_p} (\delta N_{\nu_1} U_1 + \delta N_{\nu_2} U_2 + \delta Q_{\nu_3}^* W + \delta M_{\nu_1}\Psi_1 + \\
& \delta M_{\nu_2}\Psi_2 + \delta R_{\nu_1}\Omega_1^0 + \delta R_{\nu_2}\Omega_2^0 + \delta S_{\nu_3}^*\Omega_3 + \delta M_{\nu_3}^*\Omega_3^0). \qquad (3.23)
\end{aligned}
$$

With the use of the boundary conditions associated to (3.12) and (3.13) and with the application of expression (3.18), it can be shown that the first variation (3.23) is zero if and only if

$$\begin{aligned}
\frac{\partial C}{\partial N_{11}} &= U_{1,1} = \frac{\lambda+\mu}{h\mu(3\lambda+2\mu)}N_{11} - \frac{\lambda}{2h\mu(3\lambda+2\mu)}N_{22} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_0, \\
\frac{\partial C}{\partial N_{22}} &= U_{2,2} = \frac{\lambda+\mu}{h\mu(3\lambda+2\mu)}N_{22} - \frac{\lambda}{2h\mu(3\lambda+2\mu)}N_{11} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_0, \\
\frac{\partial C}{\partial N_{12}} &= U_{2,1} - \Omega_3^0 = \frac{\alpha+\mu}{4h\alpha\mu}N_{12} + \frac{\alpha-\mu}{4h\alpha\mu}N_{21}, \\
\frac{\partial C}{\partial N_{21}} &= U_{1,2} + \Omega_3^0 = \frac{\alpha+\mu}{4h\alpha\mu}N_{21} + \frac{\alpha-\mu}{4h\alpha\mu}N_{12}, \\
\frac{\partial C}{\partial M_{12}} &= \Psi_{2,1} - \Omega_3 = \frac{3(\alpha+\mu)}{h^3\alpha\mu}M_{12} + \frac{3(\alpha-\mu)}{h^3\alpha\mu}M_{21}, \\
\frac{\partial C}{\partial M_{21}} &= \Psi_{1,2} + \Omega_3 = \frac{3(\alpha+\mu)}{h^3\alpha\mu}M_{21} + \frac{3(\alpha-\mu)}{h^3\alpha\mu}M_{12}, \\
\frac{\partial C}{\partial Q_1^*} &= W_{,1} + \Omega_2^0 = \frac{3(\alpha-\mu)}{10h\alpha\mu}Q_1 + \frac{3(\alpha+\mu)}{10h\alpha\mu}Q_1^*, \\
\frac{\partial C}{\partial Q_2^*} &= W_{,2} - \Omega_1^0 = \frac{3(\alpha-\mu)}{10h\alpha\mu}Q_2 + \frac{3(\alpha+\mu)}{10h\alpha\mu}Q_2^*, \\
\frac{\partial C}{\partial Q_1} &= \Psi_1 - \Omega_2^0 = \frac{3(\alpha-\mu)}{10h\alpha\mu}Q_1^* + \frac{3(\alpha+\mu)}{10h\alpha\mu}Q_1, \\
\frac{\partial C}{\partial Q_2} &= \Psi_2 + \Omega_1^0 = \frac{3(\alpha-\mu)}{10h\alpha\mu}Q_2^* + \frac{3(\alpha+\mu)}{10h\alpha\mu}Q_2,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial C}{\partial M_{11}} &= \Psi_{1,1} = \frac{12(\lambda+\mu)}{h^3\mu(3\lambda+2\mu)}M_{11} - \frac{6\lambda}{h^3\mu(3\lambda+2\mu)}M_{22} + \\
&\quad \frac{3\lambda}{5h\mu(3\lambda+2\mu)}\left(\frac{\partial Q_1^*}{\partial x_1} + \frac{\partial Q_2^*}{\partial x_2}\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial C}{\partial M_{22}} &= \Psi_{2,2} = \frac{12(\lambda+\mu)}{h^3\mu(3\lambda+2\mu)}M_{22} - \frac{6\lambda}{h^3\mu(3\lambda+2\mu)}M_{11} + \\
&\quad \frac{3\lambda}{5h\mu(3\lambda+2\mu)}\left(\frac{\partial Q_1^*}{\partial x_1} + \frac{\partial Q_2^*}{\partial x_2}\right),
\end{aligned}$$

$$\frac{\partial C}{\partial R_{11}} = \Omega^0_{1,1} = \frac{6(\beta+\gamma)}{5h\gamma(3\beta+2\gamma)}R_{11} - \frac{3\beta}{5h\gamma(3\beta+2\gamma)}R_{22} - \frac{\beta}{2\gamma(3\beta+2\gamma)}t,$$

$$\frac{\partial C}{\partial R_{22}} = \Omega^0_{2,2} = \frac{6(\beta+\gamma)}{5h\gamma(3\beta+2\gamma)}R_{22} - \frac{3\beta}{5h\gamma(3\beta+2\gamma)}R_{11} - \frac{\beta}{2\gamma(3\beta+2\gamma)}t,$$

$$\frac{\partial C}{\partial R_{21}} = \Omega^0_{1,2} = \frac{3(\epsilon-\gamma)}{10h\gamma\epsilon}R_{12} + \frac{3(\gamma+\epsilon)}{10h\gamma\epsilon}R_{21},$$

$$\frac{\partial C}{\partial R_{12}} = \Omega^0_{2,1} = \frac{3(\epsilon-\gamma)}{10h\gamma\epsilon}R_{21} + \frac{3(\gamma+\epsilon)}{10h\gamma\epsilon}R_{12},$$

$$\frac{\partial C}{\partial M_1^*} = \Omega^0_{3,1} = \frac{\gamma+\epsilon}{4h\gamma\epsilon}M_1^*,$$

$$\frac{\partial C}{\partial M_2^*} = \Omega^0_{3,2} = \frac{\gamma+\epsilon}{4h\gamma\epsilon}M_2^*,$$

$$\frac{\partial C}{\partial S_1^*} = \Omega_{3,1} = \frac{3(\gamma+\epsilon)}{h^3\gamma\epsilon}S_1^*,$$

$$\frac{\partial C}{\partial S_2^*} = \Omega_{3,2} = \frac{3(\gamma+\epsilon)}{h^3\gamma\epsilon}S_2^*.$$

After solving this system for M_{ij}, Q_j, Q_j^*, R_{ij}, S_j^* and M_j^* in terms of the Lagrange Multipliers (3.20) we obtain :

$$M_{11} = D\left(\Psi_{1,1} + \nu\Psi_{2,2}\right) + \frac{\nu h^2}{10(1-\nu)}p, \tag{3.24}$$

$$M_{22} = D\left(\Psi_{2,2} + \nu\Psi_{1,1}\right) + \frac{\nu h^2}{10(1-\nu)}p, \tag{3.25}$$

$$M_{12} = \frac{D}{2}\frac{(1+v)}{(1-N^2)}\left(\Psi_{1,2} + \Psi_{2,1} - 2N^2\left(\Omega_3 + \Psi_{1,2}\right)\right), \tag{3.26}$$

$$M_{21} = \frac{D}{2}\frac{(1+v)}{(1-N^2)}\left(\Psi_{2,1} + \Psi_{1,2} + 2N^2\left(\Omega_3 - \Psi_{2,1}\right)\right), \tag{3.27}$$

$$R_{12} = \frac{5Gh(l_t^2 - 2l_b^2)}{3}\Omega^0_{1,2} + \frac{10Ghl_b^2}{3}\Omega^0_{2,1},$$

$$R_{21} = \frac{5Gh(l_t^2 - 2l_b^2)}{3}\Omega^0_{2,1} + \frac{10Ghl_b^2}{3}\Omega^0_{1,2}, \tag{3.28}$$

$$R_{11} = \frac{5Ghl_t^2}{3}\left(\Omega^0_{1,1} + (1-\Psi)\left(\Omega^0_{1,1} + \Omega^0_{2,2}\right)\right) + \frac{2Gl_t^2(1-\Psi)}{\Psi}t, \tag{3.29}$$

$$R_{22} = \frac{5Ghl_t^2}{3}\left(\Omega^0_{2,2} + (1-\Psi)\left(\Omega^0_{2,2} + \Omega^0_{1,1}\right)\right) + \frac{2Gl_t^2(1-\Psi)}{\Psi}t, \tag{3.30}$$

$$Q_1 = \frac{5Gh}{6(1-N^2)}\left(W_{,1} + \Psi_1 - 2N^2\left(W_{,1} + \Omega^0_2\right)\right), \tag{3.31}$$

$$Q_2 = \frac{5Gh}{6(1-N^2)}\left(W_{,2} + \Psi_2 - 2N^2\left(W_{,2} - \Omega^0_1\right)\right), \tag{3.32}$$

(3.33)

$$Q_1^* = \frac{5Gh}{6(1-N^2)}\left(W_{,1} + \Psi_1 - 2N^2\left(\Psi_1 - \Omega_2^0\right)\right), \tag{3.34}$$

$$Q_2^* = \frac{5Gh}{6(1-N^2)}\left(W_{,2} + \Psi_2 - 2N^2\left(\Psi_2 + \Omega_1^0\right)\right), \tag{3.35}$$

$$S_1^* = \frac{Gl_t^2(4l_b^2 - l_t^2)h^3}{12l_b^2}\Omega_{3,1}, \tag{3.36}$$

$$S_2^* = \frac{Gl_t^2(4l_b^2 - l_t^2)h^3}{12l_b^2}\Omega_{3,2}, \tag{3.37}$$

$$N_{11} = \frac{Eh}{(1-\nu^2)}\left(U_{1,1} + \nu U_{2,2}\right) + \frac{h\nu}{1-\nu}\sigma_0, \tag{3.38}$$

$$N_{22} = \frac{Eh}{(1-\nu^2)}\left(U_{2,2} + \nu U_{1,1}\right) + \frac{h\nu}{1-\nu}\sigma_0, \tag{3.39}$$

$$N_{12} = \frac{Gh}{(1-N^2)}\left(U_{2,1} + U_{1,2} - 2N^2\left(U_{1,2} + \Omega_3^0\right)\right), \tag{3.40}$$

$$N_{21} = \frac{Gh}{(1-N^2)}\left(U_{1,2} + U_{2,1} - 2N^2\left(U_{2,1} - \Omega_3^0\right)\right), \tag{3.41}$$

$$M_1^* = \frac{Gl_t^2(4l_b^2 - l_t^2)h}{l_b^2}\Omega_{3,1}^0, \tag{3.42}$$

$$M_2^* = \frac{Gl_t^2(4l_b^2 - l_t^2)h}{l_b^2}\Omega_{3,2}^0. \tag{3.43}$$

Substituting (3.24) - (3.43) in (3.12) and (3.13) we obtain the following governing system:

Twisting System:

$$h(\mu+\alpha)\frac{\partial^2 U_1}{\partial x_2^2} + \frac{4h\mu(\lambda+\mu)}{\lambda+2\mu}\frac{\partial^2 U_1}{\partial x_1^2} + \left(h(\mu-\alpha) + \frac{2h\lambda\mu}{\lambda+2\mu}\right)\frac{\partial^2 U_2}{\partial x_1 \partial x_2} + 2h\alpha\frac{\partial \Omega_3^0}{\partial x_2} = -\frac{h\lambda}{\lambda+2\mu}\frac{\partial \sigma_0}{\partial x_1}$$

$$h(\mu+\alpha)\frac{\partial^2 U_2}{\partial x_1^2} + \frac{4h\mu(\lambda+\mu)}{\lambda+2\mu}\frac{\partial^2 U_2}{\partial x_2^2} + \left(h(\mu-\alpha) + \frac{2h\lambda\mu}{\lambda+2\mu}\right)\frac{\partial^2 U_1}{\partial x_1 \partial x_2} - 2h\alpha\frac{\partial \Omega_3^0}{\partial x_1} = -\frac{h\lambda}{\lambda+2\mu}\frac{\partial \sigma_0}{\partial x_2}$$

$$\frac{4h\gamma\epsilon}{\gamma+\epsilon}\Delta\Omega_3^0 + 2h\alpha\left(\frac{\partial U_2}{\partial x_1} - \frac{\partial U_1}{\partial x_2}\right) - 4h\alpha\Omega_3^0 = -2v$$

Bending System:

$$\frac{h^3\mu(\lambda+\mu)}{3(\lambda+2\mu)}\frac{\partial^2 \Psi_1}{\partial x_1^2} + \frac{h^3(\mu+\alpha)}{12}\frac{\partial^2 \Psi_1}{\partial x_2^2} + \frac{h^3}{12}\left(-\alpha + \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}\right)\frac{\partial^2 \Psi_2}{\partial x_1 \partial x_2} + \frac{5h(\alpha-\mu)}{6}\frac{\partial W}{\partial x_1} + \frac{h^3\alpha}{6}\frac{\partial \Omega_3}{\partial x_2} + \frac{5h\alpha}{3}\Omega_2^0 - \frac{5h(\mu+\alpha)}{6}\Psi_1 = -\frac{h^2\lambda}{10(\lambda+2\mu)}\frac{\partial p}{\partial x_1}$$

$$\frac{h^3\mu(\lambda+\mu)}{3(\lambda+2\mu)}\frac{\partial^2 \Psi_2}{\partial x_2^2} + \frac{h^3(\mu+\alpha)}{12}\frac{\partial^2 \Psi_2}{\partial x_1^2} + \frac{h^3}{12}\left(-\alpha + \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}\right)\frac{\partial^2 \Psi_1}{\partial x_1 \partial x_2} + \frac{5h(\alpha-\mu)}{6}\frac{\partial W}{\partial x_2} - \frac{h^3\alpha}{6}\frac{\partial \Omega_3}{\partial x_1} - \frac{5h\alpha}{3}\Omega_1^0 - \frac{5h(\mu+\alpha)}{6}\Psi_2 = -\frac{h^2\lambda}{10(\lambda+2\mu)}\frac{\partial p}{\partial x_2}$$

$$\frac{5h(\alpha+\mu)}{6}\Delta W + \frac{5h\alpha}{3}\left(\frac{\partial \Omega_2^0}{\partial x_1} - \frac{\partial \Omega_1^0}{\partial x_2}\right) + \frac{5h(\mu-\alpha)}{6}\left(\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_2}\right) = -p$$

$$\frac{10h\gamma(\beta+\gamma)}{3(\beta+2\gamma)}\frac{\partial^2 \Omega_1^0}{\partial x_1^2} + \frac{5h(\gamma+\epsilon)}{6}\frac{\partial^2 \Omega_1^0}{\partial x_2^2} - \frac{10h\alpha}{3}\Omega_1^0 - \frac{5h\alpha}{3}\Psi_2 + \frac{5h\alpha}{3}\frac{\partial W}{\partial x_2} + \frac{5h}{6}\left(\gamma - \epsilon + \frac{2\beta\gamma}{\beta+2\gamma}\right)\frac{\partial^2 \Omega_2^0}{\partial x_1 \partial x_2} = -\frac{5h\beta}{6(\beta+2\gamma)}\frac{\partial t}{\partial x_1} \tag{3.44}$$

$$-\frac{10h\gamma(\beta+\gamma)}{3(\beta+2\gamma)}\frac{\partial^2\Omega_2^0}{\partial x_2^2}-\frac{5h(\gamma+\epsilon)}{6}\frac{\partial^2\Omega_2^0}{\partial x_1^2}+\frac{10h\alpha}{3}\Omega_2^0-\frac{5h\alpha}{3}\Psi_1+$$
$$\frac{5h\alpha}{3}\frac{\partial W}{\partial x_1}-\frac{5h}{6}\left(\gamma-\epsilon+\frac{2\beta\gamma}{\beta+2\gamma}\right)\frac{\partial^2\Omega_1^0}{\partial x_1\partial x_2}=\frac{5h\beta}{6(\beta+2\gamma)}\frac{\partial t}{\partial x_2}$$

$$\frac{h^3\gamma\epsilon}{3(\gamma+\epsilon)}\Delta\Omega_3-\frac{h^3\alpha}{3}\Omega_3+\frac{h^3\alpha}{6}\left(\frac{\partial\Psi_2}{\partial x_1}-\frac{\partial\Psi_1}{\partial x_2}\right)=0 \tag{3.45}$$

The previous governing system in terms of the technical constants (2.3), takes the following form:

Twisting System:

$$\frac{E}{2(1+\nu)}\frac{\partial^2U_1}{\partial x_2^2}+\frac{E(1-N^2)}{1-\nu^2}\frac{\partial^2U_1}{\partial x_1^2}+\frac{E(1+\nu-2N^2)}{2(1-\nu^2)}\frac{\partial^2U_2}{\partial x_1\partial x_2}+$$
$$\frac{N^2E}{(1+\nu)}\frac{\partial\Omega_3^0}{\partial x_2}=-\frac{\nu(1-N^2)}{1-\nu}\frac{\partial\sigma_0}{\partial x_1}$$

$$\frac{E}{2(1+\nu)}\frac{\partial^2U_2}{\partial x_1^2}+\frac{E(1-N^2)}{1-\nu^2}\frac{\partial^2U_2}{\partial x_2^2}+\frac{E(1+\nu-2N^2)}{2(1-\nu^2)}\frac{\partial^2U_1}{\partial x_1\partial x_2}-$$
$$\frac{N^2E}{(1+\nu)}\frac{\partial\Omega_3^0}{\partial x_1}=-\frac{\nu(1-N^2)}{1-\nu}\frac{\partial\sigma_0}{\partial x_2}$$

$$\frac{Ehl_t^2(4l_b^2-l_t^2)(1-N^2)}{4l_b^2(1+\nu)}\Delta\Omega_3^0+\frac{N^2Eh}{2(1+\nu)}\left(\frac{\partial U_2}{\partial x_1}-\frac{\partial U_1}{\partial x_2}-2\Omega_3^0\right)=(N^2-1)v$$

<u>Bending System</u>:

$$D(1-N^2)\frac{\partial^2\Psi_1}{\partial x_1^2}+\frac{D(1-\nu)}{2}\frac{\partial^2\Psi_1}{\partial x_2^2}+\frac{D(1+\nu-2N^2)}{2}\frac{\partial^2\Psi_2}{\partial x_1\partial x_2}+$$
$$\frac{5h(2N^2-1)E}{12(1+\nu)}\frac{\partial W}{\partial x_1}+DN^2(1-\nu)\frac{\partial\Omega_3}{\partial x_2}+\frac{5EhN^2}{6(1+\nu)}\Omega_2^0-$$
$$\frac{5Eh}{12(1+\nu)}\Psi_1=-\frac{h^2\nu(1-N^2)}{10(1-\nu)}\frac{\partial p}{\partial x_1}$$

$$D(1-N^2)\frac{\partial^2\Psi_2}{\partial x_2^2}+\frac{D(1-\nu)}{2}\frac{\partial^2\Psi_2}{\partial x_1^2}+\frac{D(1+\nu-2N^2)}{2}\frac{\partial^2\Psi_1}{\partial x_1\partial x_2}+$$
$$\frac{5h(2N^2-1)E}{12(1+\nu)}\frac{\partial W}{\partial x_2}-DN^2(1-\nu)\frac{\partial\Omega_3}{\partial x_1}-\frac{5EhN^2}{6(1+\nu)}\Omega_1^0-$$
$$\frac{5Eh}{12(1+\nu)}\Psi_2=-\frac{h^2\nu(1-N^2)}{10(1-\nu)}\frac{\partial p}{\partial x_2}$$

$$\frac{5N^2Eh}{6(1+\nu)}\left(\frac{\partial\Omega_2^0}{\partial x_1}-\frac{\partial\Omega_1^0}{\partial x_2}\right)+\frac{5(1-2N^2)Eh}{12(1+\nu)}\left(\frac{\partial\Psi_1}{\partial x_1}+\frac{\partial\Psi_2}{\partial x_2}\right)+$$
$$\frac{5Eh}{12(1+\nu)}\Delta W=(N^2-1)p$$

$$\frac{El_t^2(2-\Psi)}{(1+\nu)}\frac{\partial^2\Omega_1^0}{\partial x_1^2}+\frac{2El_b^2}{(1+\nu)}\frac{\partial^2\Omega_1^0}{\partial x_2^2}-\frac{2EN^2}{(1+\nu)(1-N^2)}\Omega_1^0+$$
$$\frac{E(l_t^2(2-\Psi)-2l_b^2)}{(1+\nu)}\frac{\partial^2\Omega_2^0}{\partial x_1\partial x_2}-\frac{EN^2}{(1+\nu)(1-N^2)}\Psi_2+$$
$$\frac{EN^2}{(1+\nu)(1-N^2)}\frac{\partial W}{\partial x_2}=(\Psi-1)\frac{\partial t}{\partial x_1}$$

$$-\frac{El_t^2(2-\Psi)}{(1+\nu)}\frac{\partial^2\Omega_2^0}{\partial x_2^2}-\frac{2El_b^2}{(1+\nu)}\frac{\partial^2\Omega_2^0}{\partial x_1^2}+\frac{2EN^2}{(1+\nu)(1-N^2)}\Omega_2^0-$$
$$\frac{E(l_t^2(2-\Psi)-2l_b^2)}{(1+\nu)}\frac{\partial^2\Omega_1^0}{\partial x_1\partial x_2}-\frac{EN^2}{(1+\nu)(1-N^2)}\Psi_1+$$
$$\frac{EN^2}{(1+\nu)(1-N^2)}\frac{\partial W}{\partial x_1}=(1-\Psi)\frac{\partial t}{\partial x_2}$$

$$\frac{l_t^2(4l_b^2-l_t^2)(1-N^2)}{4l_b^2(1+\nu)}\Delta\Omega_3+\frac{N^2}{2(1+\nu)}\left(\frac{\partial\Psi_2}{\partial x_1}-\frac{\partial\Psi_1}{\partial x_2}-2\Omega_3\right)=0 \tag{3.46}$$

In matrix form we also write the system for the bending in the form:

$$\mathbf{L}\left(\partial_{\mathbf{x}}\right)\mathbf{H}-\mathbf{F}=\mathbf{0},\ \mathbf{x}\in R, \tag{3.47}$$

where $\mathbf{L}\left(\partial_{\mathbf{x}}\right)=\mathbf{L}\left(\frac{\partial}{\partial x_a}\right)$,

$$\mathbf{L}\left(\xi\right)=\mathbf{L}\left(\xi_\alpha\right)=\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & 0 & L_{16} \\ L_{12} & L_{22} & L_{23} & L_{24} & -L_{16} & 0 \\ -L_{13} & -L_{23} & L_{33} & 0 & L_{35} & L_{36} \\ -L_{14} & L_{24} & 0 & L_{44} & 0 & 0 \\ 0 & L_{16} & -L_{35} & 0 & L_{55} & L_{56} \\ L_{16} & 0 & L_{36} & 0 & -L_{56} & L_{66} \end{bmatrix},$$

$$\mathbf{H}^T=\begin{bmatrix} \Psi_1 & \Psi_2 & W & \Omega_3 & \Omega_1^0 & \Omega_2^0 \end{bmatrix},$$

and

$$\mathbf{F}^T=\begin{bmatrix} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 \end{bmatrix}.$$

In the above $L_{11}=L_{11}(\xi_1,\xi_2)=k_1\xi_1^2+k_2\xi_2^2-k_3$, $L_{22}=L_{11}(\xi_2,\xi_1)$, $L_{33}=k_4\Delta$, $L_{44}=k_5\Delta-k_6$, $L_{55}=L_{55}(\xi_1,\xi_2)=k_7\xi_1^2+k_8\xi_2^2-k_9$, $L_{66}=-L_{55}(\xi_2,\xi_1)$, $L_{12}=k_{10}\xi_1\xi_2$, $L_{13}=k_{11}\xi_1$, $L_{14}=k_{12}\xi_2$, $L_{16}=k_{13}$, $L_{23}=k_{11}\xi_2$, $L_{24}=k_{12}\xi_1$, $L_{35}=-k_{13}\xi_2$, $L_{36}=k_{13}\xi_1$, $L_{56}=k_{14}\xi_1\xi_2$, $\Delta=\xi_1^2+\xi_2^2$ and $F_1=-\frac{h^2\nu(1-N^2)}{10(1-\nu)}\frac{\partial p}{\partial x_1}$, $F_2=-\frac{h^2\nu(1-N^2)}{10(1-\nu)}\frac{\partial p}{\partial x_2}$, $F_3=-(1-N^2)p$, $F_4=0$, $F_5=-\frac{5h(1-N^2)}{6}(1-\Psi)\frac{\partial t}{\partial x_1}$, $F_6=\frac{5h(1-N^2)}{6}(1-\Psi)\frac{\partial t}{\partial x_2}$. Here $k_1=D(1-N^2)$, $k_2=\frac{D(1-\nu)}{2}$, $k_3=-\frac{5Gh}{6}$, $k_4=\frac{5Gh}{6}$, $k_5=\frac{D(1-\nu)l_t^2(4l_b^2-l_t^2)(1-N^2)}{2l_b^2}$, $k_6=$

$2N^2D(1-\nu)$, $k_7 = \frac{5h(1-N^2)Gl_t^2(2-\Psi)}{3}$, $k_8 = \frac{10h(1-N^2)Gl_b^2}{3}$, $k_9 = \frac{10hGN^2}{3}$, $k_{10} = \frac{D(1+\nu-2N^2)}{2}$, $k_{11} = \frac{5Gh(2N^2-1)}{6}$, $k_{12} = DN^2(1-\nu)$, $k_{13} = \frac{5GhN^2}{3}$, $k_{14}\frac{5h(1-N^2)G\left(l_t^2(2-\Psi)-2l_b^2\right)}{3}$;

The correspondent boundary traction conditions are

$$\mathbf{T}(\partial_x)\mathbf{H} - \mathbf{F}^* = \mathbf{0},, \tag{3.48}$$

where differential operator $\mathbf{T}\left(\partial_{\mathbf{x}}\right) = \mathbf{T}\left(\frac{\partial}{\partial x_a}\right)$,

$$\mathbf{T}\left(\xi\right) = \mathbf{T}\left(\xi_\alpha\right) = \begin{bmatrix} T_{11} & T_{12} & 0 & T_{14} & 0 & 0 \\ T_{21} & T_{22} & 0 & T_{24} & 0 & 0 \\ T_{31} & T_{32} & T_{33} & 0 & 0 & T_{36} \\ 0 & 0 & 0 & T_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & T_{55} & T_{56} \\ 0 & 0 & 0 & 0 & T_{65} & T_{66} \end{bmatrix},$$

and

$$\left(F^*\right)^T = \begin{bmatrix} F_1^* & F_2^* & F_3^* & F_4^* & F_5^* & F_6^* \end{bmatrix}.$$

In the above $T_{11} = T_1(\xi_1, \xi_2), T_{22} = T_1(\xi_2, \xi_1)$, $T_1(\xi_1, \xi_2) = Dn_1\xi_1 + \frac{D(1+\nu)}{2(1-N^2)}n_2\xi_2$, $T_{33} = \frac{5Gh}{6(1-N^2)}\left(n_1\xi_1 + n_2\xi_2\right)$, $T_{44} = \frac{Gl_t^2(4l_b^2-l_t^2)h^3}{12l_b^2}(n_1\xi_1 + n_2\xi_2)$, $T_{55} = \frac{5Gh}{3}(l_t^2n_1(2 - \Psi)\xi_1 + 2l_b^2n_2\xi_2)$, $T_{66} = \frac{5Gh}{3}(2l_b^2n_1\xi_1 + l_t^2(2-\Psi)n_2\xi_2)$, $T_{12} = D\nu n_1\xi_2 + \frac{D(1+\nu)(1-2N^2)}{2(1-N^2)}n_2\xi_1$, $T_{14} = \frac{D(1+\nu)N^2}{1-N^2}n_2$, $T_{21} = D\nu n_2\xi_1 + \frac{D(1+\nu)(1-2N^2)}{2(1-N^2)}n_1\xi_2$, $T_{24} = -\frac{D(1+\nu)N^2}{1-N^2}n_1$, $T_{31} = \frac{5Gh(1-2N^2)}{6(1-N^2)}n_1$, $T_{32} = \frac{5Gh(1-2N^2)}{6(1-N^2)}n_2$, $T_{36} = \frac{5GhN^2}{3(1-N^2)}(n_1 - n_2)$, $T_{56} = T_2(\xi_1, \xi_2)$, $T_{65} = T_2(\xi_2, \xi_1)$, $T_2(\xi_1, \xi_2) = \frac{5Gh}{3}(l_t^2n_1(1-\Psi)\xi_2 + (l_t^2 - 2l_b^2)n_2\xi_1)$, $F_1^* = -\frac{\nu h^2}{10(1-\nu)}n_1p$, $F_2^* = -\frac{\nu h^2}{10(1-\nu)}n_2p$, $F_3^* = 0$, $F_4^* = 0$, $F_5^* = -\frac{2Gl_t^2(1-\Psi)}{\Psi}n_1t$, $F_6^* = -\frac{2Gl_t^2(1-\Psi)}{\Psi}n_2t$.

The governing system for the twisting case is

$$\tilde{\mathbf{L}}\left(\partial_{\mathbf{x}}\right)\tilde{\mathbf{H}} - \tilde{\mathbf{F}} = \mathbf{0},\ \mathbf{x} \in R, \tag{3.49}$$

where

$$\tilde{\mathbf{L}}(\xi) = \tilde{\mathbf{L}}(\xi_\alpha) = \begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} & \tilde{L}_{13} \\ \tilde{L}_{21} & \tilde{L}_{22} & \tilde{L}_{23} \\ \tilde{L}_{31} & \tilde{L}_{32} & \tilde{L}_{33} \end{bmatrix}$$

and

$$(\tilde{F})^T = \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 & \tilde{F}_3 \end{bmatrix}.$$

Here $\tilde{T}_{11} = \kappa_1\xi_1^2 + \kappa_2\xi_2^2$, $\tilde{T}_{12} = \kappa_3\xi_1\xi_2$, $\tilde{T}_{13} = 2\kappa_4\xi_2$, $\tilde{T}_{21} = \tilde{T}_{12}$, $\tilde{T}_{22} = \tilde{T}_{11}$, $\tilde{T}_{23} = 2\kappa_4\xi_1$, $\tilde{T}_{31} = -\kappa_4\xi_2$, $\tilde{T}_{32} = \kappa_4\xi_1$, $\tilde{T}_{33} = \kappa_5(\xi_1^2 + \xi_2^2) - \kappa_2$, $\tilde{F}_1^* = -\frac{\nu\kappa_1}{2G}\frac{\partial\sigma_0}{\partial x_1}$, $\tilde{F}_2^* = -\frac{\nu\kappa_1}{2G}\frac{\partial\sigma_0}{\partial x_2}$, $\tilde{F}_3^* = -\frac{(1-N^2)}{Gh}\upsilon$, $\kappa_1 = \frac{2(1-N^2)}{1-\nu}$, $\kappa_2 = 2N^2$, $\kappa_3 = 1 - \kappa_1 = \frac{(1+\nu-2N^2)}{(1-\nu)}$, $\kappa_4 = N^2$, $\kappa_5 = \frac{l_t^2(4l_b^2-l_t^2)(1-N^2)}{2l_b^2}$.

The boundary conditions for the twisting system has the following form:

$$\tilde{\mathbf{T}}(\partial_x)\tilde{\mathbf{H}} - \tilde{\mathbf{F}}^* = \mathbf{0},, \tag{3.50}$$

where differential operator $\tilde{\mathbf{T}}(\partial_{\mathbf{x}}) = \tilde{\mathbf{T}}\left(\frac{\partial}{\partial x_a}\right)$,

$$\tilde{\mathbf{T}}(\xi) = \tilde{\mathbf{T}}(\xi_\alpha) = \begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} & \tilde{T}_{13} \\ \tilde{T}_{21} & \tilde{T}_{22} & \tilde{T}_{23} \\ 0 & 0 & \tilde{T}_{33} \end{bmatrix},$$

$$\tilde{\mathbf{H}}^T = \begin{bmatrix} U_1 & U_2 & \Omega_3^0 \end{bmatrix},$$

and

$$\left(\tilde{\mathbf{F}}^*\right)^T = \begin{bmatrix} \tilde{F}_1{}^* & \tilde{F}_2{}^* & \tilde{F}_3{}^* \end{bmatrix}.$$

In the above $\tilde{T}_{11} = \frac{Ehn_1}{1-\nu^2}\xi_1 + \frac{Ghn_2}{1-N^2}\xi_2$, $\tilde{T}_{12} = \frac{Eh\nu n_1}{1-\nu^2}\xi_2 + \frac{Ghn_2(1-2N^2)}{1-N^2}\xi_1$, $\tilde{T}_{13} = \frac{2N^2Ghn_2}{1-N^2}$, $\tilde{T}_{21} = \frac{Eh\nu n_2}{1-\nu^2}\xi_1 + \frac{Ghn_1(1-2N^2)}{1-N^2}\xi_2$, $\tilde{T}_{22} = \frac{Ehn_2}{1-\nu^2}\xi_1 + \frac{Ghn_1}{1-N^2}\xi_1$, $\tilde{T}_{23} = -\frac{2N^2Ghn_1}{1-N^2}$, $\tilde{T}_{33} = \frac{Gl_t^2(4l_b^2-l_t^2)h}{l_b^2}(\xi_1 n_1 + \xi_2 n_2)$, $\tilde{F}_1^* = \Sigma_{0,1} - \frac{h\nu n_1}{1-\nu}\sigma_0$, $\tilde{F}_2^* = \Sigma_{0,2} - \frac{h\nu n_2}{1-\nu}\sigma_0$, $\tilde{F}_3{}^* = M_{03}$.

3.6 Reduction to Classical Case

In the classical case, that is, when the effect of microrotation is neglected, the proposed model given in the previous section reduces to the first three bending equations of (3.46). The equations take the following form:

$$D\frac{\partial^2\Psi_1}{\partial x_1^2}+\frac{D(1-\nu)}{2}\frac{\partial^2\Psi_1}{\partial x_2^2}+\frac{D(1+\nu)}{2}\frac{\partial^2\Psi_2}{\partial x_1\partial x_2}-\frac{5hE}{12(1+\nu)}\frac{\partial W}{\partial x_1}-\frac{5Eh}{12(1+\nu)}\Psi_1=-\frac{h^2\nu}{10(1-\nu)}\frac{\partial p}{\partial x_1}, \tag{3.51}$$

$$D\frac{\partial^2\Psi_2}{\partial x_2^2}+\frac{D(1-\nu)}{2}\frac{\partial^2\Psi_2}{\partial x_1^2}+\frac{D(1+\nu)}{2}\frac{\partial^2\Psi_1}{\partial x_1\partial x_2}-\frac{5hE}{12(1+\nu)}\frac{\partial W}{\partial x_2}-\frac{5Eh}{12(1+\nu)}\Psi_2=-\frac{h^2\nu}{10(1-\nu)}\frac{\partial p}{\partial x_2}, \tag{3.52}$$

$$\frac{5Eh}{12(1+\nu)}\Delta W+\frac{5Eh}{12(1+\nu)}\left(\frac{\partial\Psi_1}{\partial x_1}+\frac{\partial\Psi_2}{\partial x_2}\right)=-p, \tag{3.53}$$

after some manipulation of equations (3.51) - (3.53) it can be shown that the governing system for the vertical deflection can take the following form:

$$\Delta^2 W=\frac{p}{D}-\frac{h^2(\nu+2)}{10D(1-\nu)}\Delta p. \tag{3.54}$$

which is exactly Reissner's model [12].

3.7 Uniqueness of solutions

In this section we prove that if we have a solution of (3.47) and (3.49) that satisfies the boundary conditions at $\Gamma=\Gamma_\sigma\cup\Gamma_d$ and that satisfies the equilibrium equations (3.12), (3.13), together with all kinematic assumptions then the solution must be unique. During the proof we assume that all functions satisfy the Green - Gauss theorem requirements.

For the proof of the uniqueness we assume that the solution of the Cosserat plate is not unique. We suppose that there are two different solutions that satisfy the previous requirements, if this is the case then the difference of the solutions must satisfy the systems (3.47) and (3.49) with zero loads, then boundary conditions take the following form:

$$
\begin{aligned}
M_{ij}n_j &= 0,\ R_{ij}n_j = 0, \\
Q^*_j n_j &= 0,\ \ S^*_j n_j = 0, \\
N_{ij}n_i &= 0,\ M^*_j n_j = 0, \quad on \ \ \Gamma_\sigma
\end{aligned} \tag{3.55}
$$

and

$$
\begin{aligned}
W &= 0, \quad V_i = 0, \\
\Omega^0_k &= 0, \ \ \Omega_3 = 0, \\
U_i &= 0, \ \ on \ \ \Gamma_d.
\end{aligned} \tag{3.56}
$$

We shall show that under these conditions the strain in the plate should vanish and therefore the solution represents the plate deformation as a rigid body. It can be shown that for the homogeneous system associated to (3.47) and (3.49), that is, for a plate with zero loads, the free energy expression (3.16) can be represented in the following form:

$$
\begin{aligned}
I_F = & \int\int_R (N_{11}U_{1,1} + N_{12}(U_{2,1} - \Omega^0_3) + N_{21}(U_{1,2} + \Omega^0_3) + M_{11}\Psi_{1,1} + Q_2\Omega^0_1 + \\
& M_{21}(\Psi_{1,2} + \Omega_3) + Q^*_1(W_{,1} + \Omega^0_2) + Q^*_2(W_{,2} - \Omega^0_1) + Q_1(\Psi_1 - \Omega^0_2) + Q_2\Psi_2 + \\
& R_{11}\Omega^0_{1,1} + R_{22}\Omega^0_{2,2} + R_{21}\Omega^0_{1,2} + R_{12}\Omega^0_{2,1} + M^*_1\Omega^0_{3,1} + M^*_2\Omega^0_{3,2} + S^*_1\Omega_{3,1} + \\
& S^*_2\Omega_{3,2} + M_{12}(\Psi_{2,1} - \Omega_3) + N_{22}U_{2,2} + M_{22}\Psi_{2,2})dA,
\end{aligned} \tag{3.57}
$$

after some manipulations of (3.57) we obtain the following expression for I_F:

$$\begin{aligned} I_F &= \int\int_R \{[R_{ij}\Omega_i^0 + (M_{ij}+N_{ij})U_i + S_j^*\Omega_3 + Q_j^*W + M_i^*\Omega_3^0]_{,j} - \\ &(M_{11,1}+M_{21,2}-Q_1)\Psi_1 - (M_{12,1}+M_{22,2}-Q_2)\Psi_2 - \\ &(Q_{1,1}^*+Q_{2,2}^*)W - (R_{11,1}+R_{21,2}+Q_2^*-Q_2)\Omega_1^0 - \\ &(R_{12,1}+R_{22,2}+Q_1-Q_1^*)\Omega_2^0 - (S_{1,1}^*+S_{2,2}^*+M_{12}-M_{21})\Omega_3 - \\ &(N_{11,1}+N_{21,2})U_1 - (N_{12,1}+N_{22,2})U_2 - (M_{1,1}^*+M_{2,2}^*+N_{12}-N_{21})\Omega_3^0\}dA, \end{aligned} \tag{3.58}$$

now if we consider the equilibrium equations (3.12) and (3.13) with zero load in expression (3.58) and Green's theorem we obtain:

$$I_F = \oint_\Gamma (R_{ij}\Omega_i^0 + (M_{ij}+N_{ij})U_i + S_j^*\Omega_3 + Q_j^*W + M_i^*\Omega_3^0)n_j ds = 0. \tag{3.59}$$

The integral expression (3.59) vanishes because (3.55) implies $\int_{\Gamma_\sigma} \equiv 0$ and (3.56) implies $\int_{\Gamma_d} \equiv 0$.

With lots of calculation it can be shown that expressions (3.24) - (3.43) substituted in (3.18) represents a positive definite quadratic form in terms of the kinematic variables, therefore expression (3.59) and (3.23) imply:

$$\begin{aligned} U_{1,1} &= 0,\ U_{2,2}=0,\ U_{2,1}-\Omega_3^0=0,\ U_{1,2}+\Omega_3^0=0, \\ \Psi_{1,1} &= 0,\ \Psi_{2,2}=0,\ \Psi_{2,1}-\Omega_3=0,\ \Psi_{1,2}+\Omega_3=0, \\ W_{,1}+\Omega_2^0 &= 0,\ W_{,2}-\Omega_1^0=0,\ \Psi_1-\Omega_2^0=0,\ \Psi_2+\Omega_1^0=0, \\ \Omega_{1,1}^0 &= 0,\ \Omega_{2,2}^0=0,\ \Omega_{1,2}^0=0,\ \Omega_{2,1}^0=0,\ \Omega_{3,1}^0=0, \\ \Omega_{3,2}^0 &= 0,\ \Omega_{3,1}=0,\ \Omega_{3,2}=0, \end{aligned} \tag{3.60}$$

after integration of (3.60) we notice that the difference of any two distinct solutions of the deformation of the Cosserat plate is represented in the following form:

$$U_1(x_1, x_2) = -x_2\Omega_3^0 + U_1^0,\ U_2 = x_1\Omega_3^0 + U_2^0, \tag{3.61}$$

$$\Psi_1(x_1, x_2) = -x_2\Omega_3 + \Psi_1^0,\ \Psi_2 = x_1\Omega_3 + \Psi_2^0, \tag{3.62}$$

$$W(x_1, x_2) = \Omega_1^0 x_2 - \Omega_2^0 x_1 + W^0, \tag{3.63}$$

where U_i^0, Ω_i^0, W^0, Ω_3, Ψ_i^0 are constants. The solutions (3.61) - (3.63) describe pure translation and rotation of the plate, therefore there is no deformation. Since we know that in general we have deformations, the solution must be unique.

CHAPTER 4
ANALITICAL SOLUTIONS

4.1 Description of Experiments

In this section we are interested to solve analytically the bending system of equations (3.47) for a thin square plate of height h and length a. The plate is described by the set of points $[0,a]\times[0,a]\times[-\frac{h}{2},\frac{h}{2}]$. We assume the plate is subjected to a load p described by $p(x_1,x_2)=\sin(\frac{\pi}{a}x_1)\sin(\frac{\pi}{a}x_2)\ \frac{N}{m^2}$. We consider the plate to be made of syntatic foam. Numerical values of the elastic constants for this material can be found in [4] and are given as follows:

$$E = 2758\ MPa,\ G=1033\ MPa, \quad \nu=0.34, \tag{4.1}$$

$$l_t = 65\ \mu m, \qquad l_b = 33\times 10^{-3}, \quad \Psi = 1.5\ rad, \tag{4.2}$$

$$N^2 = 0.1\ . \tag{4.3}$$

The methodology to follow consists in assuming that each unknown function can be represented in terms of Fourier series. Considering that the load is represented by $p(x_1,x_2)=\sin(\frac{\pi}{a}x_1)\sin(\frac{\pi}{a}x_2)$, then the structure of the system (3.47) requires unknown functions to have the following representations:

$$W=\sum_{m,n} A_{mn}\sin(\frac{\pi m}{a}x_1)\sin(\frac{\pi n}{a}x_2), \quad \Psi_1=\sum_{m,n} B_{mn}\cos(\frac{\pi m}{a}x_1)\sin(\frac{\pi n}{a}x_2), \tag{4.4}$$

$$\Psi_2 = \sum_{m,n} C_{mn} \sin(\frac{\pi m}{a}x_1)\cos(\frac{\pi n}{a}x_2), \quad \Omega_1^0 = \sum_{m,n} D_{mn} \sin(\frac{\pi m}{a}x_1)\cos(\frac{\pi n}{a}x_2), \tag{4.5}$$

$$\Omega_2^0 = \sum_{m,n} E_{mn} \cos(\frac{\pi m}{a}x_1)\sin(\frac{\pi n}{a}x_2), \quad \Omega_3 = \sum_{m,n} F_{mn} \cos(\frac{\pi m}{a}x_1)\cos(\frac{\pi n}{a}x_2), \tag{4.6}$$

where all coefficients appearing in (4.4) - (4.6) are to be determined. It can be shown that after substitution of (4.4) - (4.6) in (3.47), a linear system of 6×6 is obtained with the property that all Fourier coefficients are zero for $m \neq 1$ or $n \neq 1$. The only case where they can't be zero is when $m = n = 1$. Solving the 6×6 system provides an analytical solution of (3.47).

It's important to notice that the nature of force p and the homogenity of the Cosserat plate imply that the maximum deflection of the plate occurs at its center, hence p is maximum at $(\frac{a}{2}, \frac{a}{2})$.

Experiment 1

In this experiment we compare W_P/W_E versus a/h, where W_E is the maximum deflection of the plate calculated with Eringen's model (appendix A), W_P is the maximum deflection calculated with the proposed model (3.47) and a/h is the number of times the plate dimensions are more bigger than its thickness. The main purpose of this comparison is to study the effect of microstructure and the effect of the plate thickness on the calculation of the maximum deflection of the plate.

In this experiment the effect of microstructure is appreciated after calculating W_P and W_E for different values of the constants appearing in (4.2) - (3.53), which correspond to the microstructure of the plate. The values of these constants are reduced by a factor of $\frac{1}{10}$ and $\frac{1}{100}$. The effect of the plate thickness is appreciated after calculating W_P and W_E for $h = 0.1m$ and by increasing a/h from 5 up to 30. Figure 4–1 shows the results of the experiment.

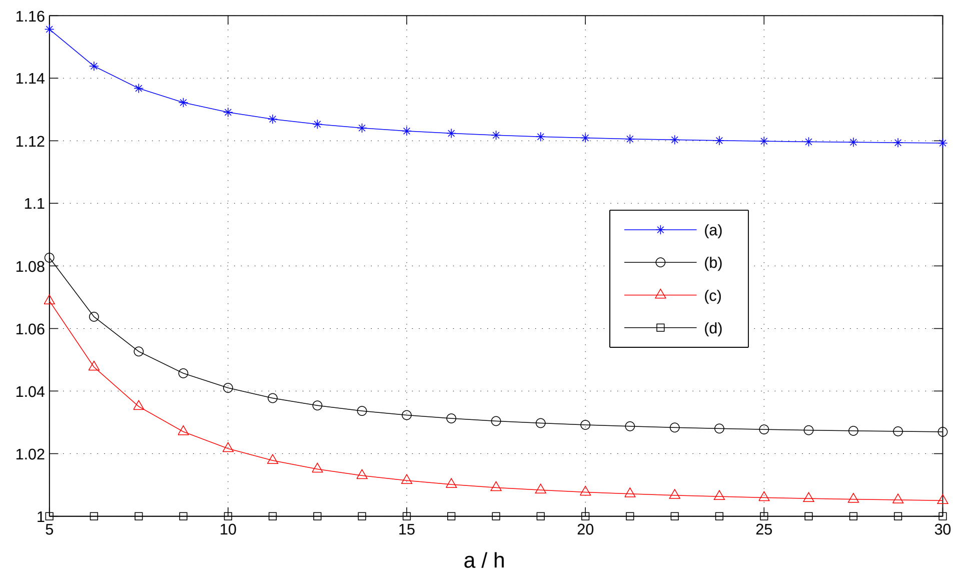

Figure 4–1: Asymmetric Effect

Comments of Experiment 1

As we can appreciate, figure 4–1 shows 3 plots which are described as follows:

- Plot (a) compares W_P and W_E for different values of a/h. The effect of microstructure is considered at 100%, that is the material characteristics are described by (4.1) - (4.3). As we can see, the values of W_P and W_E tend to be closer as h becomes alot smaller compared to size a. For all plate sizes, $W_P > W_E$ by a net difference of at least 12% and at most of 16% seems to appear. This result is expected since the model proposed in this thesis is based on Reissner's approach and Eringen's model in Kirchhoff's assumptions (see appendix A).

- Plot (b) compares W_P and W_E for different values of a/h. The effect of microstructure is considered at 10%, that is the material constants (4.2) - (4.3) are all reduced to a factor of 1/10. As we can see, the values of W_P and W_E tend to be closer than in (a), but the behaviour $W_P > W_E$ still remains. The net difference now is approximately at least 3% and at most of 8%.

- Plot (c) compares W_P and W_E for different values of a/h. The effect of microstructure is considered at 1%, that is the material constants (4.2) - (4.3) are all reduced to a factor of 1/100. As we can see, the values of W_P and W_E tend to be closer than in (b), and the behaviour $W_P > W_E$ still remains. The net difference now is less than 1% and at most of 7%. Notice also that when the plate thickness is 10 times smaller than its size, $W_P > W_E$ by approximately 2%.

Experiment 2

In this experiment we compare W_P/W_C versus a/h, where W_C is the maximum deflection of the plate calculated with Reissner's model(3.54) and W_P is the maximum deflection calculated with the proposed model (3.47). The main purpose of this comparison is to show that microstructure has a significant effect in the plate deformation. This is appreciated in the same way we did in experiment 1.

Before showing the results of this experiment, it's important to realize that now $W_P < W_C$. The explanation of this phenomena is done with energy principles. It happens that in the classical case the total free energy of the plate considers only the effect of W_C. From the other side, in the case where microstructure plays an important role, the total free energy of the plate is additionally considers microstructural deformations, therefore W_P should be smaller than W_C. Another important thing to realize is that microstructure should be taken into account when the plate thickness is alot smaller than its dimensions. Figure (4–2) shows and validates previous observations:

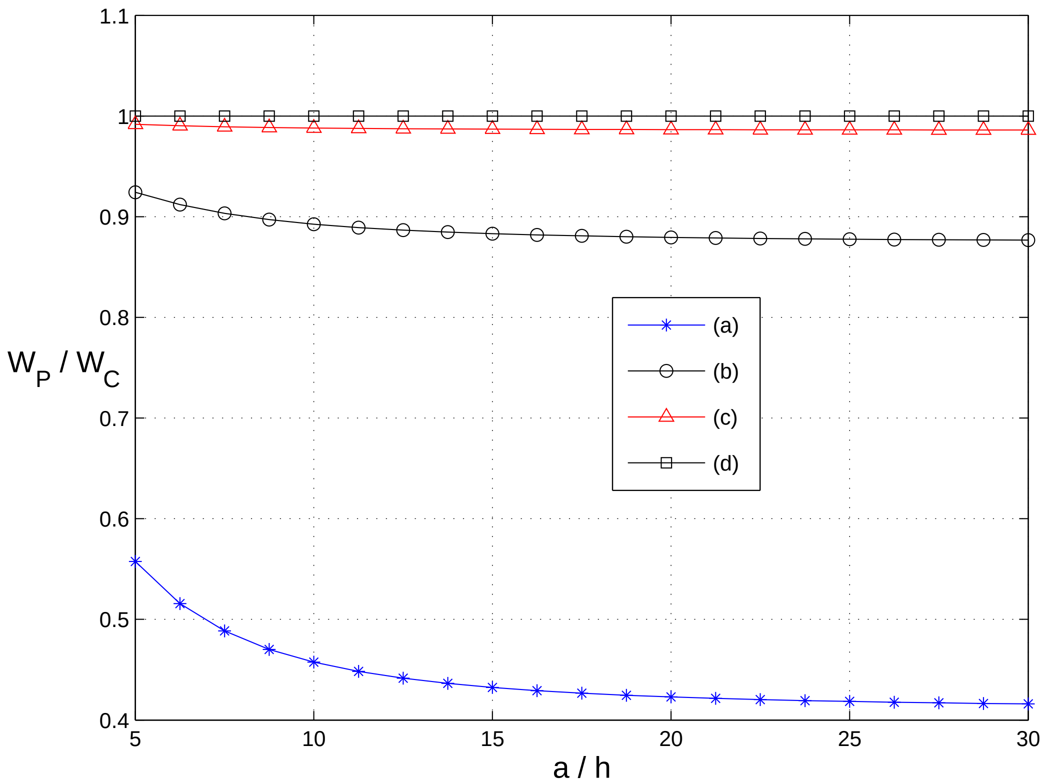

Figure 4–2: Classic case

Comments of Experiment 2

As we can appreciate, figure 4–2 shows 3 plots which are described as follows:

- Plot (a) compares W_P and W_C for different values of a/h when the effect of microstructure is considered at 100%. As we can see, the values of W_P and W_C tend to be farther away as h becomes alot smaller compared to size a. For all plate sizes, $W_P < W_C$, as expected. Notice that the best case when W_P is closer to W_C is when h is 5 times smaller than a. In this case W_P is 60% of W_C.

- Plot (b) compares W_P and W_C for different values of a/h when the effect of microstructure is considered at 10%. Notice that in this case W_P and W_C are closer than in (a), but the behaviour that W_P and W_C tend to be farther away still remains.

- Plot (c) compares W_P and W_E for different values of a/h when the effect of microstructure is considered at 1%. As we can see, the values of W_P and W_E are closer than in (b) and as h is alot smaller than a, W_P and W_C tend to be farther away.

4.2 Conclusions

1. The effect of microstructure plays a significant role in the calculation of deformations of elastic bodies. As shown in experiment 2, the bigger the values of the complementary constants, more difference is appreciated between classic results and Cosserat's theory results.
2. Experiment 2 shows that microstructure effect in a plate becomes significant when the thickness of the plate is alot smaller than its dimensions.
3. Experiment 1 shows that when microstructure is reduced, Eringen's model and the proposed model (3.47) become altmost the same. In cases when G is not so big, results may be far away. This can be seen by comparing (3.54) and (B.4).
4. In Eringen's theory the vertical microrotation of the middle plane is considered to be zero. According to (3.47) we can appreciate that in general this quantity is not zero. It's value becomes significant when the load μ^t is different than zero.
5. When the load $\mu^t = 0$, the assumption made by Eringen about zero vertical microrotation seems to be correct. In our experiments Ω_3^0 is practically zero compared to Ω_i^0.

APPENDICES

APPENDIX A
THE CONCEPT OF STRESS AND COUPLE STRESS IN ASYMMETRIC ELASTICITY

When an elastic body is exposed to some external forces every material point is in correspondence with a force and a momentum per unit area. In the classical theory of elasticity only the force per unit area vector is taken in consideration. The direction of the force and momentum depend on the transversal cut made to the body. Figure A–1 illustrates this situation. The unitary vector **n** is normal and describes the orientation of the transversal cut, σ illustrates the force per unit area and is known as the stress vector. The stress vector is responsible for the displacement of material points. The momentum per unit area is illustrated with μ and is known as the couple stress vector, this stress is responsible for the microrotation of a material point.

Given a transversal cut described by **n**, the stress or couple stress vectors at a point can be calculated by means of a linear transformation. Linear transformations associated to the stress and couple stress receive the name of the stress and couple stress tensors respectively. Usually components of both tensors at a material point are calculated along the cuts described by $\mathbf{e}_1 = (1, 0, 0)^T$, $\mathbf{e}_2 = (0, 1, 0)^T$ and $\mathbf{e}_3 = (0, 0, 1)^T$. Once known all stress and couple stress components associated to $\mathbf{e}_k$, the stress and couple stress vectors are claculated in the following way:

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix}^T \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}^T \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}.$$

where n_k are the components of the unitary normal vector.

In the asymmetric theory of elasticity the stress and couple stress tensors are in general asymmetric. When the couple stress effect is neglected then the stress tensor becomes symmetric [1].

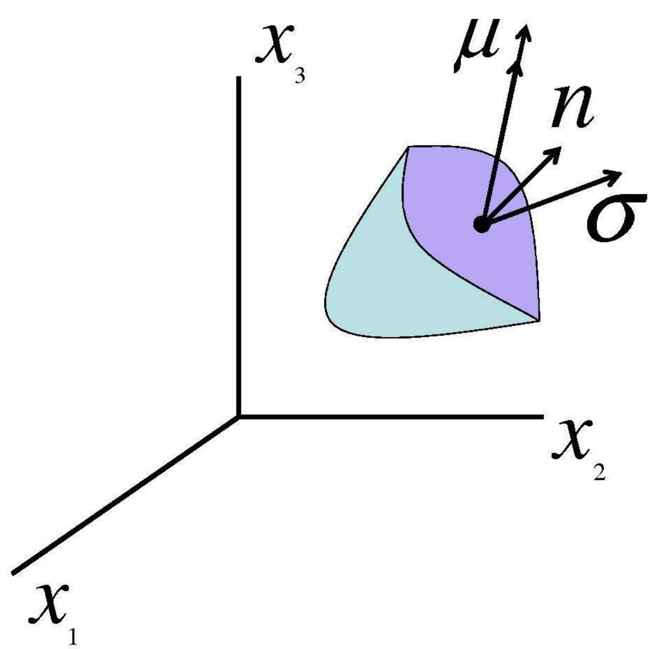

Figure A–1: The stress vector σ and the couple stress vector μ.

APPENDIX B
ERINGEN'S MODEL

Eringen in page 18 of [2] proposes a model for the case of the plate we consider in chapter 3. The bending system of equations Eringen derived has the following form:

$$\frac{I}{2}\left(\frac{E}{1-\nu}-\varkappa\right)v_{i,ji}+\frac{I}{2}\left(\frac{E}{1+\nu}+\varkappa\right)v_{j,i}-2H\left(G-\frac{\varkappa}{2}\right)w_j-$$

$$2H\left(G+\frac{\varkappa}{2}\right)v_j+2\varkappa H\epsilon_{ji}\varphi_k=0 \tag{B.1}$$

$$\left(G-\frac{\varkappa}{2}\right)v_{i,i}+\left(G+\frac{\varkappa}{2}\right)w_{i,i}+\varkappa\epsilon_{ij}\varphi_{j,i}+\frac{p}{2H}=0 \tag{B.2}$$

$$\left(\alpha^E+\beta^E\right)\varphi_{i,ji}+\gamma^E\varphi_{j,ii}+\varkappa\epsilon_{kl}\left(v_i-w_{,i}\right)-2\varkappa\varphi_j=0 \tag{B.3}$$

where $I=\frac{2}{3}H^3$, $\epsilon_{11}=\epsilon_{11}=0$, $\epsilon_{12}=-\epsilon_{21}=1$, $H=\frac{h}{2}$, and v_i plays the same role as Ψ_i. In [2], it's assumed that for thin plates, φ_1 and φ_2 are constant along x_3. The vertical microrotation φ_3 is assumed to be zero by the fact that a pure vertical load is applied to the plate.

If microstructure can be neglected and the value of $G\to\infty$, then after some manipulations of (B.1) - (B.3) the governing system for the vertical deflection of the plate becomes:

$$\Delta^2 w = \frac{p}{D}, \tag{B.4}$$

which is Kirchoff's model.

REFERENCE LIST

[1] Nowacki W. *Theory of Asymmetric Elasticity.* Pergamon Press, Oxford, New York, Toronto, Sydney, Paris, 1986.

[2] Eringen C. A. Theory of micropolar plates. *Journal of Applied Mathematics and Physics*, Vol 18:12–31, 1967.

[3] R.D.; Jahsman W.E. Gauthier. A quest for micropolar elastic constants. *J. Applied Mechanics*, Vol 42:369–374, 1975.

[4] Lakes R. Experimental methods for study of cosserat elastic solids and other generalized elastic continua. *In Mühlhaus H, Wiley J.(eds.) Continuum Models for Materials with Microstructures*, pages 1–22, New York, 1995.

[5] Love A.E.H. *A Treatise on the Mathematical Theory of Elasticity.* Dover, New York, 1986.

[6] Timoshenko S. and Woinowsky-Krieger S. *Theory of Plates and Shells.* McGraw-Hill, 1959.

[7] P. M. Naghdi. The theory of shells and plates. *in Handbuch der Physik, Springer-Verlag*, 6a.2:425–640, 1972.

[8] Kirchhoff G. Über das gleichgewicht and die bewegung einer elastischen scheibe. *Journal für reine and angewandte Mathematik*, 40:51–88, 1850.

[9] E. Reissner. The effect of transverse shear deformation on the bending of elastic plates. *Journal of applied Mechanics*, pages 69–77, 1945.

[10] Bathe K. J.; Brezzi F. On the convergence of a four mode plate bending element based on mindlin-reissner plate theory and a mixed interpolation. *The Mathematics of Finite Elements and Applications V, Academic Press, London*, pages 491–503, 1985.

[11] Rössle A.; Bischoff M.; Wendland W.; Ramm E. On the mathematical foundation of the (1,1,2)-plate model. *International Journal of Solids and Structures*, (36):2143–2168, 1999.

[12] E. Reissner. Reflections on the theory of elastic plates. *Applied Mechanics Reviews*, (38):1453–1464, 1986.

[13] Madrid Pedro Steinberg Lev. Bending of cosserat thin plates. *Proceedings of the* 23^{rd} *Southeastern Conference on Applied Mechanics, Mayagüez Puerto Rico*, may 21-23, 2006.

[14] Madrid Pedro Steinberg Lev. Bending of cosserat plates with transverse variation of microrotation. *submitted in the journal Mathematics and Mechanics of Solids (MMS)*, january 2007.

REISSNER'S PLATE THEORY IN THE FRAMEWORK OF ASYMMETRIC ELASTICITY

Pedro Joaquín Madrid
(787) XXX-XXXX
Department of Department of Mathematical Sciences
Chair: Dr. Lev Steinberg
Degree: Master of Science
Graduation Date: November 2007

This is the general Audience Abstract.