# CLOSURE OPERATORS, TORSION THEORIES, AND RADICALS IN A NON-ABELIAN ENVIRONMENT

by

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## Abstract

In the category of abelian groups, there exists a relationship between torsion theories and idempotent radicals where we can use one to find the other. In the process we can learn of different properties that apply to the associated subclasses. We can also use a pre-radical to obtain a closure operator and a closure operator to obtain a pre-radical. The properties of the pre-radical (whether it's idempotent or a radical) indicate corresponding properties of the closure operator and vice-versa.

In this thesis, we determine how much of the relationship that exists between closure operators, torsion theories and radicals in the category of abelian groups ($\underline{\mathbf{Ab}}$) can be extended to the category of all groups ($\underline{\mathbf{Grp}}$). Once it's clear which properties do not apply, our goal is to find, for each property, which modifications can be made so that the property still holds.

## Resumen

En la categoría de grupos abelianos, existe una relación entre las teorías de torsión y los radicales idempotentes donde podemos usar uno para encontrar el otro. En el proceso podemos aprender sobre diferentes propiedades que aplican a las subclases implicadas. Además podemos usar un pre-radical para obtener un operador de clausura y un operador de clausura para obtener un pre-radical. Las propiedades del pre-radical (si es idempotente o un radical) nos indican propiedades correspondientes en el operador de clausura y vice-versa.

En esta tesis determinamos qué aspectos de la relación que existe entre los operadores de clausura, teorías de torsión y radicales en la categoría de grupos abelianos (**Ab**) se pueden extender a la categoría de todos los grupos (**Grp**). Una vez tengamos claro cuales propiedades no aplican, nuestra meta es investigar, para cada propiedad, qué modificaciones se les puede hacer para que la propiedad todavía aplique.

*To my parents,*
*who have always been and always will be*
*an endless source of motivation and inspiration.*

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# Contents

# 1 Preliminary Notions

Before we begin with the subject matter, we present several definitions, lemmas, and theorems that will be of help later on with our proofs.

**Notation** *Let* $\underline{\mathbf{Grp}}$ *and* $\underline{\mathbf{Ab}}$ *denote the category of groups and abelian groups, respectively, with group homomorphisms.*

*If* $H$ *is a subgroup of* $G \in \underline{\mathbf{Grp}}$*, we denote it by* $H < G$*.*

*If* $N$ *is a normal subgroup of* $G \in \underline{\mathbf{Grp}}$*, we denote it by* $N \lhd G$*.*

*In the context of groups,* $e$ *will denote the identity element. In the context of group homomorphisms,* $e$ *will denote the trivial homomorphism that maps every element to the identity element.*

We begin with a classical correspondence between subclasses of groups.

**Definition 1** *If* $\underline{\mathbf{B}}$ *and* $\underline{\mathbf{C}}$ *are subclasses of* $\underline{\mathbf{Grp}}$ $(\underline{\mathbf{Ab}})$ *(i.e.* $\underline{\mathbf{B}}$*,* $\underline{\mathbf{C}} \subseteq \underline{\mathbf{Grp}}$ $(\underline{\mathbf{Ab}})$*),*

$$\underline{\mathbf{B}}^r = \{\, C \in \underline{\mathbf{Grp}} \mid \forall B \in \underline{\mathbf{B}}\ Hom(B, C) = e \,\}$$
$$\underline{\mathbf{C}}^l = \{\, B \in \underline{\mathbf{Grp}} \mid \forall C \in \underline{\mathbf{C}}\ Hom(B, C) = e \,\}$$

Claim: $\underline{\mathbf{C}} \subseteq \underline{\mathbf{B}}^r \iff \underline{\mathbf{B}} \subseteq \underline{\mathbf{C}}^l$ (Such a situation is called a *polarity*) [L].

*Proof.*

Assume $\underline{\mathbf{C}} \subseteq \underline{\mathbf{B}}^r$.

Let $B_1 \in \underline{\mathbf{B}}$ and $C_1 \in \underline{\mathbf{C}}$ be arbitrary. Then $C_1 \in \underline{\mathbf{B}}^r$ (hypothesis), so $\forall B \in \underline{\mathbf{B}}$ $\mathrm{Hom}(B, C_1) = e$. In particular, since $B_1 \in \underline{\mathbf{B}}$, $\mathrm{Hom}(B_1, C_1) = e$. But this argument holds for any $C \in \underline{\mathbf{C}}$. Hence, $\forall C \in \underline{\mathbf{C}}$ $\mathrm{Hom}(B_1, C) = e$. Thus, $B_1 \in \underline{\mathbf{C}}^l$. So $\underline{\mathbf{B}} \subseteq \underline{\mathbf{C}}^l$.

Now, let's assume $\underline{\mathbf{B}} \subseteq \underline{\mathbf{C}}^l$.

Let $C_2 \in \underline{\mathbf{C}}$ and $B_2 \in \underline{\mathbf{B}}$ be arbitrary. Then $B_2 \in \underline{\mathbf{C}}^l$ (hypothesis), so $\forall C \in \underline{\mathbf{C}}$ $\mathrm{Hom}(B_2, C) = e$. In particular, since $C_2 \in \underline{\mathbf{C}}$, $\mathrm{Hom}(B_2, C_2) = e$. But this argument holds for any $B \in \underline{\mathbf{B}}$. Hence, $\forall B \in \underline{\mathbf{B}}$ $\mathrm{Hom}(B, C_2) = e$. Thus, $C_2 \in \underline{\mathbf{B}}^r$. So $\underline{\mathbf{C}} \subseteq \underline{\mathbf{B}}^r$. ■

**Definition 2** *An object function* $T : \underline{\mathbf{Ab}} \to \underline{\mathbf{Ab}}$ *is called a* pre-radical *if* $T(M) < M$ *and* $f : M \to N \Rightarrow f(T(M)) \subseteq T(N)$*, for all abelian groups* $M$ *and* $N$ *and all homomorphisms* $f$*. If in addition,* $T(M/T(M)) = e$ *for all abelian groups* $M$*, then* $T$ *is called a* radical.

**Definition 3** *Let* $\{B_i \mid i \in I\}$ *be a family of groups. The cartesian product of those groups, coupled with the familiar component-wise multiplication as its binary operation, forms a group called the* direct product *of* $\{B_i \mid i \in I\}$*, denoted by* $\prod_{i\in I} B_i$*. An element of* $\prod_{i\in I} B_i$ *is denoted as* $\{b_i \mid i \in I\}$ *(or simply as* $\{b_i\}$ *when no confusion is likely) where* $b_j \in B_j$ *for all* $j \in I$*. If* $\{b_i\}, \{c_i\} \in \prod_{i\in I} B_i$*, then* $\{b_i\} \cdot \{c_i\} = \{b_i \cdot c_i\}$*.*

**Definition 4** *If* $\{G_i \mid i \in I\}$ *is a family of groups, then*

1. *For each* $k \in I$*, the epimorphism* $\pi_k : \prod_{i\in I} G_i \to G_k$ *given by* $\pi_k(\{g_i\}) = g_k$ *is called a* canonical projection [H].

2. *For each* $k \in I$*, the monomorphism* $\tau_k : G_k \to \prod_{i\in I} G_i$ *given by* $\tau_k(g) = \{g_i\}$ *where* $g_i = e_i$ *for* $i \neq k$ *and* $g_k = g$*, is called a* canonical injection [H].

**Definition 5** *Let* $\{B_i \mid i \in I\}$ *be a family of groups. Then the* direct sum *of* $\{B_i \mid i \in I\}$*, denoted by* $\sum_{i\in I} B_i$*, is the subgroup of the direct product where, for every element* $\{b_i\}$*,* $b_j = e$ *for all but a finite amount of* $j \in I$*.*

**Definition 6** *Let* $\underline{\mathbf{B}}$ *and* $\underline{\mathbf{C}}$ *be subclasses of* $\underline{\mathbf{Grp}}$*. Then the pair (*$\underline{\mathbf{B}}$*,* $\underline{\mathbf{C}}$*) is called a* torsion theory *if the following conditions hold:*

$B \in \underline{\mathbf{B}} \iff \forall C \in \underline{\mathbf{C}}\ Hom(B, C) = e$ *(i.e.* $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$*)*

$C \in \underline{\mathbf{C}} \iff \forall B \in \underline{\mathbf{B}}\ Hom(B, C) = e$ *(i.e.* $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$*)*

The starting point of our investigation is the following well-known theorem in the category of abelian groups:

**Theorem 1** *If* $\underline{\mathbf{B}}$ *and* $\underline{\mathbf{C}}$ *are subclasses of* $\underline{\mathbf{Ab}}$*, the following statements are equivalent:* [L]

1. $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$ *and* $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$*.*

2. $\underline{\mathbf{B}}$ *is closed under isomorphic images, quotient groups, group extensions, and direct sums; moreover,* $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$*.*

3. $\underline{\mathbf{C}}$ *is closed under isomorphic images, subgroups, group extensions, and direct products; moreover,* $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$.

4. *There is a radical* $T$ *on* $\underline{\mathbf{Ab}}$ *such that* $T(T(M)) = T(M)$ *for all groups* $M$*; moreover,* $\underline{\mathbf{B}} = \{\, M \mid T(M) = M \,\}$, $\underline{\mathbf{C}} = \{\, M \mid T(M) = e \,\}$.

By saying that $\underline{\mathbf{B}}$ is *closed under group extensions*, we mean that, whenever $B$ is a subgroup of $M$, and $B$ and $M/B$ are in $\underline{\mathbf{B}}$, then so is $M$.

Similar to the abelian case, we define a pre-radical in $\underline{\mathbf{Grp}}$ as follows:

**Definition 7** *An object function* $T : \underline{\mathbf{Grp}} \to \underline{\mathbf{Grp}}$ *is called a* pre-radical *if* $T(M) < M$ *and* $f : M \to N \Rightarrow f(T(M)) \subseteq T(N)$*, for all groups* $M$ *and* $N$ *and all homomorphisms* $f$.

Before we present our first Lemma, we will need another definition:

**Definition 8** *A subgroup* $H$ *of a group* $G$ *is said to be* characteristic (fully invariant) *if* $f(H) < H$ *for every automorphism (endomorphism)* $f : G \to G$ [H].

In our attempt to export properties and theorems from $\underline{\mathbf{Ab}}$ to $\underline{\mathbf{Grp}}$, one of the main difficulties lies with quotient groups, since to create one we need a normal subgroup. To that end, we introduce the following lemma:

**Lemma 1** *If* $T : \underline{\mathbf{Grp}} \to \underline{\mathbf{Grp}}$ *is a pre-radical, then* $\forall M \in \underline{\mathbf{Grp}}$ $T(M)$ *is fully invariant. As a result,* $\forall M \in \underline{\mathbf{Grp}}$ $T(M) \triangleleft M$.

*Proof.*

Assume $T$ is a pre-radical in $\underline{\mathbf{Grp}}$.

Let $M \in \underline{\mathbf{Grp}}$ be arbitrary, and let $\psi : M \to M$ be an arbitrary endomorphism. Then, by definition of pre-radical, we have $\psi(T(M)) \subseteq T(M)$. Since $\psi$ was arbitrary, this holds for every endomorphism, and thus $T(M)$ is fully invariant. But $M$ was also arbitrary, so $\forall M \in \underline{\mathbf{Grp}}$ $T(M)$ is fully invariant.

However, every fully invariant subgroup is characteristic and every characteristic subgroup is normal (since conjugation is an automorphism) [H]. Thus, $\forall M \in \underline{\mathbf{Grp}}$ $T(M) \triangleleft M$. ■

As a consequence of Lemma 1, if $T$ is a pre-radical such that $T(M/T(M)) = e$ for all groups $M$, then $T$ is called a radical.

**Lemma 2** *Let* $r : \underline{\mathbf{Grp}} \to \underline{\mathbf{Grp}}$ *be an object function. Then* $\forall\, X, Y \in \underline{\mathbf{Grp}}$ *and* $\forall\, f \in$ *Hom*$(X,Y)$ $f(r(X)) \subseteq r(Y)$ *if and only if*

1. $\forall\, Y \in \underline{\mathbf{Grp}}, H < Y \Rightarrow r(H) \subseteq r(Y)$. *(Monotonicity)*

2. $\forall\, X, Y \in \underline{\mathbf{Grp}}$ *and* $\forall f \in$ *Hom*$(X,Y)$*,* $f(r(X)) \subseteq r(f(X))$. *(Continuity)*

*Proof.*

Assume $\forall\, X, Y \in \underline{\mathbf{Grp}}$ and $\forall\, f \in \mathrm{Hom}(X,Y)$ $f(r(X)) \subseteq r(Y)$.

Let $Y \in \underline{\mathbf{Grp}}$ be arbitrary, and let $H < Y$. Then $i \in \mathrm{Hom}(H,Y)$ where $i$ is the inclusion map. By our hypothesis, we have $i(r(H)) \subseteq r(Y) \Rightarrow r(H) \subseteq r(Y)$. Since $Y$ was arbitrary, we have $\forall\, Y \in \underline{\mathbf{Grp}}, H < Y \Rightarrow r(H) \subseteq r(Y)$.

Let $X, Y \in \underline{\mathbf{Grp}}$ and $f \in \mathrm{Hom}(X,Y)$ be arbitrary. Then $f(X) < Y$ and $f(X) \in \underline{\mathbf{Grp}}$. Thus, $f \in \mathrm{Hom}(X, f(X))$. By our hypothesis, we have $f(r(X)) \subseteq r(f(X))$. Since $X$, $Y$, and $f$ were arbitrary, we have $\forall\, X, Y \in \underline{\mathbf{Grp}}$ and $\forall f \in \mathrm{Hom}(X,Y)$, $f(r(X)) \subseteq r(f(X))$.

Now, assume the Monotonicity and Continuity conditions are true.

Let $X, Y \in \underline{\mathbf{Grp}}$ and $f \in \mathrm{Hom}(X,Y)$ be arbitrary. Then, by Continuity we have $f(r(X)) \subseteq r(f(X))$. Also, by Monotonicity, since $f(X) < Y$, we have $r(f(X)) \subseteq r(Y)$. Thus, by transitivity, $f(r(X)) \subseteq r(Y)$. Since $X$, $Y$, and $f$ were arbitrary, we have $\forall\, X, Y \in \underline{\mathbf{Grp}}$ and $\forall\, f \in \mathrm{Hom}(X,Y)$ $f(r(X)) \subseteq r(Y)$. ■

**Lemma 3** *Let* $f \in Hom(X,Y)$*,* $g \in Hom(Y,Z)$*, where* $X, Y, Z \in \underline{\mathbf{Grp}}$*. Suppose* $g \circ f = e$*.*

1. *If f is an epimorphism, then* $g = e$*.*

2. *If* $g$ *is a monomorphism, then* $f = e$*.*

*Proof.*

(1) Assume f is an epimorphism.

Suppose that $g \neq e$. Then $\exists y \in Y$ such that $g(y) = z \in Z$ where $z \neq e$. But $f$ is an epimorphism, so $\exists x \in X$ such that $f(x) = y$. Hence, $e \neq z = g(y) = g(f(x)) = (g \circ f)(x) = e$ because $g \circ f = e$. $\Rightarrow\Leftarrow$

Thus, $g = e$.

(2) Assume $g$ is a monomorphism.

Suppose that $f \neq e$. Then $\exists x \in X$ such that $f(x) = y \in Y$ where $y \neq e$. But $g$ is a monomorphism, so $y \neq e \Rightarrow g(y) \neq e$. Hence, $e \neq g(y) = g(f(x)) = (g \circ f)(x) = e$ because $g \circ f = e$. $\Rightarrow\Leftarrow$

Thus, $f = e$. ■

**Definition 9** *Let $G$ be a group and $X$ a subset of $G$. Let $\{ H_i \mid i \in I \}$ be the family of all (normal) subgroups of $G$ which contain $X$. Then $\bigcap_{i \in I} H_i$ is called the* (normal) subgroup *of $G$* generated by the set X *and denoted by* $(<X>_{\triangleleft})$ $<X>$.

**Definition 10** *If $H$ and $K$ are subgroups of $G$, the subgroup $<H \cup K>$ generated by $H$ and $K$ is called the* join *of $H$ and $K$ and is denoted $H \vee K$. Furthermore, if $\{ H_i \mid i \in I \}$ is a family of subgroups of $G$, the subgroup $\langle \bigcup_{i \in I} H_i \rangle$ generated by the set $\bigcup_{i \in I} H_i$ is denoted $\bigvee_{i \in I} H_i$.*

**Theorem 2** *If $G$ is a group and $X$ is a nonempty subset of $G$, then the subgroup $<X>$ generated by $X$ consists of all finite products $a_1^{n_1} a_2^{n_2} \ldots a_t^{n_t}$ ($a_i \in X; n_i$ is an integer). In particular for every $a \in G$, $<a> = \{ a^n \mid n \text{ is an integer} \}$* [H].

**Theorem 3** *If $f : G \to H$ is a homomorphism of groups, $N \triangleleft G$, $M \triangleleft H$, and $f(N) < M$, then $f$ induces a homomorphism $\bar{f} : G/N \to H/M$, given by $\bar{f}(aN) = f(a)M$* [H].

**Theorem 4** *If $N$ is a normal subgroup of a group $G$, then every subgroup of $G/N$ is of the form $K/N$, where $K$ is a subgroup of $G$ that contains $N$. Furthermore, $K/N$ is normal in $G/N$ if and only if $K$ is normal in $G$* [H].

**Theorem 5** *Let $\{G_i \mid i \in I\}$ be a family of groups and $\{\varphi_i : H \to G_i \mid i \in I\}$ a family of group homomorphisms. Then there is a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \circ \varphi = \varphi_i$ for all $i \in I$ and this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism* [H].

**Lemma 4** *Let $\{G_i \mid i \in I\}$ be a family of groups, $\{\varphi_i : H \to G_i \mid i \in I\}$ a family of homomorphisms and $\varphi : H \to \prod_{i \in I} G_i$ the unique homomorphism such that $\pi_i \circ \varphi = \varphi_i$ for all $i \in I$. Then $Ker\ \varphi = \bigcap_{i \in I} Ker\ \varphi_i$.*

*Proof.*

Let $x \in \text{Ker } \varphi$. For $j \in I$, $\varphi(x) = e \Rightarrow (\pi_j \circ \varphi)(x) = e \Rightarrow \varphi_j(x) = e \Rightarrow x \in \text{Ker } \varphi_j$. But this argument holds for all $j \in I$, so $x \in \bigcap_{i \in I} \text{Ker } \varphi_i$.

Let $y \in \bigcap_{i \in I} \text{Ker } \varphi_i$. For $j \in I$, $\varphi_j(y) = e \Rightarrow (\pi_j \circ \varphi)(y) = e \Rightarrow \pi_j(\varphi(y)) = e \Rightarrow \varphi(y) \in \text{Ker } \pi_j$. But this argument holds for all $j \in I$, so $\varphi(y) \in \bigcap_{i \in I} \text{Ker } \pi_i = e$. Hence, $\varphi(y) = e \Rightarrow y \in \text{Ker } \varphi$.

Therefore, $\text{Ker } \varphi = \bigcap_{i \in I} \text{Ker } \varphi_i$. ■

**Lemma 5** *Let $N$ be a normal subgroup of a group $G$ and $H$ any subgroup of $G$. If $H$ is a characteristic subgroup of $N$, then $H$ is normal in $G$* [H][1]

*Proof.*

Since $aNa^{-1} = N$ for all $a \in G$, conjugation by $a$ is an automorphism of $N$. Since $H$ is characteristic in $N$, $aHa^{-1} < H$ for all $a \in G$. Hence, $H$ is normal in $G$. ■

[1]Hungerford actually states this as the first part of a three-part lemma for a *finite* group $G$. However, the finiteness of $G$ is needed only for the other two parts of the lemma, so the proof we use here is exactly the one used by Hungerford.

# 2 Torsion Theories and Radicals in $\underline{\mathbf{Grp}}$

## 2.1 Equivalence Theorem

**Theorem 6** *If* $\underline{\mathbf{B}}$ *and* $\underline{\mathbf{C}}$ *are subclasses of* $\underline{\mathbf{Grp}}$*, the following statements are equivalent:*

1. $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$ *and* $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$.

2. $\underline{\mathbf{B}}$ *is closed under isomorphic images, quotient groups, group extensions, direct sums, and joins of subgroups; moreover,* $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$.

3. $\underline{\mathbf{C}}$ *is closed under isomorphic images, subgroups, group extensions, and direct products; moreover,* $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$.

4. *There is a radical* $T$ *on* $\underline{\mathbf{Grp}}$ *such that* $T(T(M)) = T(M)$ *for all groups* $M$*; moreover,* $\underline{\mathbf{B}} = \{\, M \mid T(M) = M \,\}$, $\underline{\mathbf{C}} = \{\, M \mid T(M) = e \,\}$.

We'd like to point out a slight difference in the second statement between this theorem and its abelian version (Theorem 1). In $\underline{\mathbf{Ab}}$, for any family $\{H_i \mid i \in I\}$ of subgroups of a given group $G$, where $H_i \in \underline{\mathbf{B}}$ for every $i \in I$, there is an epimorphism from the direct sum ($\sum_{i\in I} H_i$) to their join ($\bigvee_{i\in I} H_i$). Hence, using the kernel of that homomorphism, we can create a quotient group that is isomorphic to the join (by the First Theorem of Isomorphisms). Since $\underline{\mathbf{B}}$ is closed under direct sums, quotient groups and isomorphisms, we have that the join is also in $\underline{\mathbf{B}}$. Therefore, in the abelian case, the second statement implicitly includes joins because it follows from the other attributes.

However, in $\underline{\mathbf{Grp}}$ we cannot use the same argument, since that epimorphism does not hold without the commutativity of the elements. Thus, joins are not implicitly included as in the abelian case, so we explicitly include them in the second statement.

*Proof.*

**4 ⇒ 1**

Assume there is a radical $T$ on $\underline{\mathbf{Grp}}$ such that $T(T(M)) = T(M)$ for all groups $M$; moreover, let $\underline{\mathbf{B}} = \{\, M \mid T(M) = M \,\}$, $\underline{\mathbf{C}} = \{\, M \mid T(M) = e \,\}$.

First, note that neither $\underline{\mathbf{B}}$ nor $\underline{\mathbf{C}}$ are empty, since they both contain the group that consists solely of the identity element.

Let $B_1 \in \underline{\mathbf{B}}$ and $C_1 \in \underline{\mathbf{C}}$, and let $f \in \text{Hom}(B_1, C_1)$. Then $f(T(B_1)) \subseteq T(C_1)$ because $T$ is a radical on $\underline{\mathbf{Grp}}$. Since $B_1 \in \underline{\mathbf{B}}$ and $C_1 \in \underline{\mathbf{C}}$, we have that $T(B_1) = B_1$ and $T(C_1) = e$. Hence, $f(T(B_1)) \subseteq T(C_1) \Rightarrow f(B_1) \subseteq e \Rightarrow f(B_1) = e \Rightarrow f = e$. But this argument holds for any $C \in \underline{\mathbf{C}}$ and any homomorphism $f : B_1 \to C$. Thus, $\forall C \in \underline{\mathbf{C}}$ $\text{Hom}(B_1, C) = e$. This means that $B_1 \in \underline{\mathbf{C}}^l$. Therefore, $\underline{\mathbf{B}} \subseteq \underline{\mathbf{C}}^l$. Since $\underline{\mathbf{B}} \subseteq \underline{\mathbf{C}}^l \iff \underline{\mathbf{C}} \subseteq \underline{\mathbf{B}}^r$, we have $\underline{\mathbf{C}} \subseteq \underline{\mathbf{B}}^r$.

Let $B_2 \in \underline{\mathbf{C}}^l$. Since $T$ is a radical on $\underline{\mathbf{Grp}}$, we have $T(B_2) \triangleleft B_2$ (See Lemma 1). Hence, we can form the quotient group $B_2/T(B_2)$. But $T(B_2/T(B_2)) = e$, because $T$ is a radical, so $B_2/T(B_2) \in \underline{\mathbf{C}}$.

Let $\pi : B_2 \to B_2/T(B_2)$ be the canonical epimorphism. Since $B_2 \in \underline{\mathbf{C}}^l$, then $\forall C \in \underline{\mathbf{C}}$ $\text{Hom}(B_2, C) = e$. In particular, $B_2/T(B_2) \in \underline{\mathbf{C}}$, so $\text{Hom}(B_2, B_2/T(B_2)) = e$. Hence, $\pi = e \Rightarrow \text{Ker } \pi = B_2$. But by definition of canonical epimorphism, it must be that $\text{Ker } \pi = T(B_2)$. Thus, $T(B_2) = B_2 \Rightarrow B_2 \in \underline{\mathbf{B}}$. Therefore, $\underline{\mathbf{C}}^l \subseteq \underline{\mathbf{B}}$. But $\underline{\mathbf{B}} \subseteq \underline{\mathbf{C}}^l$, so $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$.

Let $C_2 \in \underline{\mathbf{B}}^r$. We have that $T(T(C_2)) = T(C_2)$ (by hypothesis). Hence, $T(C_2) \in \underline{\mathbf{B}}$. We also know that $T(C_2) \triangleleft C_2$. Let $i \in \text{Hom}(T(C_2), C_2)$ be the inclusion map. But $C_2 \in \underline{\mathbf{B}}^r$, so $\forall B \in \underline{\mathbf{B}}$ Hom(B,$C_2$) $= e$. In particular, since $T(C_2) \in \underline{\mathbf{B}}$, we have $\text{Hom}(T(C_2), C_2) = e$. Thus, $i = e$. This means that $\forall y \in T(C_2)$ $i(y) = e$. But $i$ is the inclusion map, so this implies that $T(C_2) = e$. Then, $C_2 \in \underline{\mathbf{C}}$. Therefore $\underline{\mathbf{B}}^r \subseteq \underline{\mathbf{C}}$. But $\underline{\mathbf{C}} \subseteq \underline{\mathbf{B}}^r$, so $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$.

**1 ⇒ 2**

Assume $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$ and $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$.

Note that $\underline{\mathbf{B}} \neq \emptyset \iff \underline{\mathbf{C}} \neq \emptyset$, because if $\underline{\mathbf{B}} \neq \emptyset$, then $\underline{\mathbf{B}}^r$ contains the group that consists solely of the identity element (Similarly with $\underline{\mathbf{C}}$ and $\underline{\mathbf{C}}^l$).

*ISOMORPHIC IMAGES*

Let $X \in \underline{\mathbf{B}}$, and let $Y \in \underline{\mathbf{Grp}}$ be an isomorphic image of $X$. Then there exists $f_1 : X \to Y$ such that $f_1$ is an isomorphism.

Let $C_1 \in \underline{\mathbf{C}}$ and $g_1 \in \text{Hom}(Y, C_1)$. Now, by function composition, $g_1 \circ f_1 \in \text{Hom}(X, C_1)$. But $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$ (by hypothesis), and $X \in \underline{\mathbf{B}}$, so $X \in \underline{\mathbf{C}}^l$. This means that $\forall C \in \underline{\mathbf{C}}$ $\text{Hom}(X, C) = e$. In particular, since $C_1 \in \underline{\mathbf{C}}$, $\text{Hom}(X, C_1) = e$. Hence, $g_1 \circ f_1 = e$. Since $f_1$ is an isomorphism, $g_1 = e$ (See Lemma 3). However, $g_1$ was arbitrary, so $\text{Hom}(Y, C_1) = e$. But this argument holds for all $C \in \underline{\mathbf{C}}$, so $\forall C \in \underline{\mathbf{C}}$ $\text{Hom}(Y, C) = e$. Thus, $Y \in \underline{\mathbf{C}}^l = \underline{\mathbf{B}}$.

Therefore, $\underline{\mathbf{B}}$ is closed under isomorphic images.

*QUOTIENT GROUPS*

Let $H \triangleleft X \in \underline{\mathbf{B}}$. Then $X/H \in \underline{\mathbf{Grp}}$.

Let $C_2 \in \underline{\mathbf{C}}$ and $g_2 \in \mathrm{Hom}(X/H, C_2)$. Also, let $\pi \in \mathrm{Hom}(X, X/H)$ be the canonical epimorphism. Now, by function composition, $g_2 \circ \pi \in Hom(X, C_2)$. But $X \in \underline{\mathbf{C}}^l$, so $\forall C \in \underline{\mathbf{C}}$ $\mathrm{Hom}(X, C) = e$. In particular, since $C_2 \in \underline{\mathbf{C}}$, $\mathrm{Hom}(X, C_2) = e$. Hence, $g_2 \circ \pi = e \Rightarrow g_2 = e$ (Lemma 3). However, $g_2$ was arbitrary, so we have $\mathrm{Hom}(X/H, C_2) = e$. But this argument holds for any $C \in \underline{\mathbf{C}}$, so $\forall C \in \underline{\mathbf{C}}$ $\mathrm{Hom}(X/H, C) = e$. Thus, $X/H \in \underline{\mathbf{C}}^l = \underline{\mathbf{B}}$.

Therefore, $\underline{\mathbf{B}}$ is closed under quotient groups.

*GROUP EXTENSIONS*

Let $B \triangleleft M \in \underline{\mathbf{Grp}}$, such that $B, M/B \in \underline{\mathbf{B}}$.

Let $C_3 \in \underline{\mathbf{C}}$ and let $f_3 \in \mathrm{Hom}(M, C_3)$. Let $i \in \mathrm{Hom}(B, M)$ be the inclusion map. Now, by function composition, $f_3 \circ i \in \mathrm{Hom}(B, C_3)$. But $B \in \underline{\mathbf{C}}^l$, so $\forall C \in \underline{\mathbf{C}}$ $\mathrm{Hom}(B, C) = e$. In particular, since $C_3 \in \underline{\mathbf{C}}$, $\mathrm{Hom}(B, C_3) = e$. Hence, $f_3 \circ i = e \Rightarrow i(B) = B \subseteq \mathrm{Ker}\ f_3$. Since $B \triangleleft M$, $f_3 \in \mathrm{Hom}(M, C)$ and $B \subseteq \mathrm{Ker}\ f_3$, there exists a homomorphism $\bar{f}_3 : M/B \to C_3$ such that $\mathrm{Im}\ f_3 = \mathrm{Im}\ \bar{f}_3$ [H]. But $M/B \in \underline{\mathbf{C}}^l$, so $\forall C \in \underline{\mathbf{C}}$ $\mathrm{Hom}(M/B, C) = e$. In particular, since $C_3 \in \underline{\mathbf{C}}$, $\mathrm{Hom}(M/B, C_3) = e$. Hence, $\bar{f}_3 = e \Rightarrow f_3 = e$. However, $f_3$ was arbitrary, so $\mathrm{Hom}(M, C_3) = e$. But this argument holds for any $C \in \underline{\mathbf{C}}$, so $\forall C \in \underline{\mathbf{C}}$ $\mathrm{Hom}(M, C) = e$. Thus, $M \in \underline{\mathbf{C}}^l = \underline{\mathbf{B}}$.

Therefore, $\underline{\mathbf{B}}$ is closed under group extensions.

*DIRECT SUMS*

Let $\{B_i \mid i \in I\} \subseteq \underline{\mathbf{B}}$. Clearly, $\sum_{i \in I} B_i \in \underline{\mathbf{Grp}}$.

Let $C_4 \in \underline{\mathbf{C}}$ and $f_4 \in \mathrm{Hom}(\sum_{i \in I} B_i, C_4)$. For some $k \in I$, let $\tau_k \in \mathrm{Hom}(B_k, \sum_{i \in I} B_i)$ be the canonical injection. Now, by function composition, $f_4 \circ \tau_k \in \mathrm{Hom}(B_k, C_4)$. But $B_k \in \underline{\mathbf{C}}^l$, so $\forall C \in \underline{\mathbf{C}}$ $\mathrm{Hom}(B_k, C) = e$. In particular, since $C_4 \in \underline{\mathbf{C}}$, $\mathrm{Hom}(B_k, C_4) = e$. Hence, $f_4 \circ \tau_k = e \Rightarrow \tau_k(B_k) \subseteq \mathrm{Ker}\ f_4$. Since $k$ was arbitrary, this holds true for all $k \in I$.

Let $b \in \sum_{i \in I} B_i$ be arbitrary, then by definition of direct sum (Definition 5) we have $b = \{b_i\}$ where $b_j = e$ for all but a finite amount of $j \in I$. Let $J = \{\, j \in I \mid b_j \neq e \,\}$. Since $J$ is finite, for the convenience of notation suppose $J = \{\, j_1,\ j_2,\ \ldots,\ j_n \,\}$ where $n$ is an integer. We observe that for $1 \leq m \leq n$, $\tau_{j_m}(b_{j_m}) = \{b_i\}$ where $b_j = e$ for all $j \in I$ except for $j = j_m$ (in which case $b_j = b_{j_m}$). Hence, since $b = \{b_i\}$ where $b_j = e$ except for $j \in J$, we can write

$b$ the following way: $b = \tau_{j_1}(b_{j_1}) \cdot \tau_{j_2}(b_{j_2}) \cdot \ldots \cdot \tau_{j_n}(b_{j_n})$. But $f_4$ is a homomorphism, so we have $f_4(b) = f_4(\tau_{j_1}(b_{j_1}) \cdot \tau_{j_2}(b_{j_2}) \cdot \ldots \cdot \tau_{j_n}(b_{j_n})) = f_4(\tau_{j_1}(b_{j_1})) \cdot f_4(\tau_{j_2}(b_{j_2})) \cdot \ldots \cdot f_4(\tau_{j_n}(b_{j_n})) = e \cdot e \cdot \ldots \cdot e = e$, since $\forall k \in I$ we have $\tau_k(B_k) \subseteq \text{Ker } f_4$. However, $b$ was arbitrary, so this argument holds for all elements of $\sum_{i\in I} B_i$. Therefore, $f_4 = e$. Since $f_4$ was also arbitrary, we have $\text{Hom}(\sum_{i\in I} B_i, C_4) = e$. But this argument holds for any $C \in \underline{\mathbf{C}}$, so $\forall C \in \underline{\mathbf{C}}$ $\text{Hom}(\sum_{i\in I} B_i, C) = e$. Thus, $\sum_{i\in I} B_i \in \underline{\mathbf{C}}^l = \underline{\mathbf{B}}$.

Therefore, $\underline{\mathbf{B}}$ is closed under direct sums.

*JOINS*

Let $\{B_i \mid i \in I\}$ be a family of subgroups of a group $G$, such that $B_i \in \underline{\mathbf{B}}$ for every $i \in I$. Clearly, $\bigvee_{i\in I} B_i \in \underline{\mathbf{Grp}}$.

Let $C_5 \in \underline{\mathbf{C}}$ and $f_5 \in \text{Hom}(\bigvee_{i\in I} B_i, C_5)$. For some $k \in I$, let $i_k \in \text{Hom}(B_k, \bigvee_{i\in I} B_i)$ be the inclusion homomorphism. Now, by function composition, $f_5 \circ i_k \in \text{Hom}(B_k, C_5)$. But $B_k \in \underline{\mathbf{C}}^l$, so $\forall C \in \underline{\mathbf{C}}$ $\text{Hom}(B_k, C) = e$. In particular, since $C_5 \in \underline{\mathbf{C}}$, $\text{Hom}(B_k, C_5) = e$. Hence, $f_5 \circ i_k = e \Rightarrow i_k(B_k) = B_k \subseteq \text{Ker } f_5$. Since $k$ was arbitrary, this holds true for all $k \in I$. This means that $\bigcup_{i\in I} B_i \subseteq \text{Ker } f_5$.

Let $b \in \bigvee_{i\in I} B_i$ be arbitrary. By definition of join (Definition 10) we have $b = a_1^{n_1} a_2^{n_2} \ldots a_t^{n_t}$ where $a_j \in \bigcup_{i\in I} B_i \subseteq \text{Ker } f_5$. In particular, $f_5(a_j) = e$ for $1 \leq j \leq t$. Hence, $f_5(b) = f_5(a_1^{n_1} a_2^{n_2} \ldots a_t^{n_t}) = f_5(a_1)^{n_1} f_5(a_2)^{n_2} \ldots f_5(a_t)^{n_t} = e^{n_1} e^{n_2} \ldots e^{n_t} = e$. However, $b$ was arbitrary, so this argument holds for all elements of $\bigvee_{i\in I} B_i$. Therefore, $f_5 = e$. Since $f_5$ was also arbitrary, we have $\text{Hom}(\bigvee_{i\in I} B_i, C_5) = e$. But this argument holds for any $C \in \underline{\mathbf{C}}$, so $\forall C \in \underline{\mathbf{C}}$ $\text{Hom}(\bigvee_{i\in I} B_i, C) = e$. Thus, $\bigvee_{i\in I} B_i \in \underline{\mathbf{C}}^l = \underline{\mathbf{B}}$.

Therefore, $\underline{\mathbf{B}}$ is closed under joins.

**1 ⇒ 3**

Assume $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$ and $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$.

*ISOMORPHIC IMAGES*

Let $X \in \underline{\mathbf{C}}$, and let $Y \in \underline{\mathbf{Grp}}$ be an isomorphic image of $X$. Then there exists $f_1 : X \to Y$ such that $f_1$ is an isomorphism.

Let $B_1 \in \underline{\mathbf{B}}$ and $g_1 \in \text{Hom}(B_1, Y)$. By function composition, $f_1^{-1} \circ g_1 \in \text{Hom}(B_1, X)$. But $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$ (by hypothesis), and $X \in \underline{\mathbf{C}}$, so $X \in \underline{\mathbf{B}}^r$. This means that $\forall B \in \underline{\mathbf{B}}$, $\text{Hom}(B, X) = e$. In particular, since $B_1 \in \underline{\mathbf{B}}$, $\text{Hom}(B_1, X) = e$. Hence, $f_1^{-1} \circ g_1 = e$. Since $f$ is an isomorphism, $g_1 = e$ (Lemma 3). However, $g_1$ was arbitrary, so we have $\text{Hom}(B_1, Y) = e$. But this argument holds for all $B \in \underline{\mathbf{B}}$, so $\forall B \in \underline{\mathbf{B}}\ \text{Hom}(B, Y) = e$. Thus, $Y \in \underline{\mathbf{B}}^r = \underline{\mathbf{C}}$.

Therefore, $\underline{\mathbf{C}}$ is closed under isomorphic images.

*SUBGROUPS*

Let $H < X \in \underline{\mathbf{C}}$, and let $i \in \text{Hom}(H, X)$ be the inclusion map.

Let $B_2 \in \underline{\mathbf{B}}$ and let $f_2 \in \text{Hom}(B_2, H)$. By function composition, $i \circ f_2 \in \text{Hom}(B_2, X)$. But $X \in \underline{\mathbf{C}} = \underline{\mathbf{B}}^r$, so $\forall B \in \underline{\mathbf{B}}\ \text{Hom}(B, X) = e$. In particular, since $B_2 \in \underline{\mathbf{B}}$, $\text{Hom}(B_2, X) = e$. Hence, $i \circ f_2 = e \Rightarrow f_2 = e$ (Lemma 3). However, $f_2$ was arbitrary, so $\text{Hom}(B_2, H) = e$. But this argument holds for all $B \in \underline{\mathbf{B}}$, so $\forall B \in \underline{\mathbf{B}}\ \text{Hom}(B, H) = e$. Thus, $H \in \underline{\mathbf{B}}^r = \underline{\mathbf{C}}$.

Therefore, $\underline{\mathbf{C}}$ is closed under subgroups.

*GROUP EXTENSIONS*

Let $C \triangleleft M$, where $M \in \underline{\mathbf{Grp}}$, such that $C, M/C \in \underline{\mathbf{C}}$.

Let $\pi \in \text{Hom}(M, M/C)$ be the canonical epimorphism. Let $B_3 \in \underline{\mathbf{B}}$ be arbitrary and $f_3 \in \text{Hom}(B_3, M)$. By function composition, $\pi \circ f_3 \in \text{Hom}(B_3, M/C)$. But $M/C \in \underline{\mathbf{C}} = \underline{\mathbf{B}}^r$, so $\forall B \in \underline{\mathbf{B}}$, $\text{Hom}(B, M/C) = e$. In particular, since $B_3 \in \underline{\mathbf{B}}$, $\text{Hom}(B_3, M/C) = e$. Hence, $\pi \circ f_3 = e \Rightarrow f_3(B_3) \subseteq \text{Ker } \pi = C$. Thus, $f_3 \in \text{Hom}(B_3, C)$. But $C \in \underline{\mathbf{B}}^r$, so $\forall B \in \underline{\mathbf{B}}$, $\text{Hom}(B, C) = e$. In particular, $B_3 \in \underline{\mathbf{B}}$, so $\text{Hom}(B_3, C) = e$. This means that $f_3 = e$. However, $f_3$ was arbitrarily chosen from $\text{Hom}(B_3, M)$, so $\text{Hom}(B_3, M) = e$. But $B_3$ was arbitrary, so $\forall B \in \underline{\mathbf{B}}\ \text{Hom}(B, M) = e$. Thus, $M \in \underline{\mathbf{B}}^r = \underline{\mathbf{C}}$.

Therefore, $\underline{\mathbf{C}}$ is closed under group extensions.

*DIRECT PRODUCTS*

Let $\{C_i\} \subseteq \underline{\mathbf{C}}$ where $i \in I$. Clearly, $\prod_{i \in I} C_i \in \underline{\mathbf{Grp}}$.

Let $B_4 \in \underline{\mathbf{B}}$ and $f_4 \in \text{Hom}(B_4, \prod_{i \in I} C_i)$. For some $j \in I$, let $\pi_j \in \text{Hom}(\prod_{i \in I} C_i, C_j)$ be the canonical projection. By function composition, $\pi_j \circ f_4 \in \text{Hom}(B_4, C_j)$. But $C_j \in \underline{\mathbf{C}} = \underline{\mathbf{B}}^r$, so $\forall B \in \underline{\mathbf{B}}\ \text{Hom}(B, C_j) = e$. In particular, since $B_4 \in \underline{\mathbf{B}}$, $\text{Hom}(B_4, C_j) = e$. Hence, $\pi_j \circ f_4 = e \Rightarrow f_4(B_4) \subseteq \text{Ker } \pi_j$. However, this holds true for all $j \in I$, so $f_4(B_4) \subseteq \bigcap_{i \in I} \text{Ker } \pi_i = e$. Hence, $f_4 = e$. Since $f_4$ was arbitrary, $\text{Hom}(B_4, \prod_{i \in I} C_i) = e$. But this argument holds for any $B \in \underline{\mathbf{B}}$, so $\forall B \in \underline{\mathbf{B}}\ \text{Hom}(B, \prod_{i \in I} C_i) = e$. Thus, $\prod_{i \in I} C_i \in \underline{\mathbf{B}}^r = \underline{\mathbf{C}}$.

Therefore, $\underline{\mathbf{C}}$ is closed under direct products.

**2 ⇒ 4**

Assume $\underline{\mathbf{B}}$ is closed under isomorphic images, quotient groups, group extensions, direct sums, and joins; moreover, $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$.

For any $G \in \underline{\mathbf{Grp}}$, let $T(G)$ be the join of all subgroups of $G$ that are in $\underline{\mathbf{B}}$. Let $M$, $N \in \underline{\mathbf{Grp}}$ be arbitrary. By definition, $T(M) = \bigvee_{i\in I} B_i$, where $B_i < M$ and $B_i \in \underline{\mathbf{B}}$ for all $i \in I$. Note that since $\underline{\mathbf{B}}$ is closed under joins, we have that $T(G) \in \underline{\mathbf{B}}$ for all groups $G$.

First we show that $T(M)$ is a radical.

*T(M) < M*

Clearly, $T(M) < M$ since the join of subgroups is a subgroup.

*$f : M \to N \Rightarrow f(T(M)) \subseteq T(N)$*

Let $f \in \mathrm{Hom}(M, N)$ and $y \in f(T(M))$ be arbitrary. Then $\exists x \in T(M)$ such that $f(x) = y$. Since $x \in T(M) = \bigvee_{i\in I} B_i$, we have $x = a_1^{n_1} a_2^{n_2} \dots a_t^{n_t}$ where $a_j \in \bigcup_{i\in I} B_i$. Let $fi : B_i \to N$ be the limitation of $f$ to $B_i$ (i.e. $\forall b \in B_i, f_i(b) = f(b)$). Since $\underline{\mathbf{B}}$ is closed under quotient groups, $B_i/Kerf_i \in \underline{\mathbf{B}}$. But $\underline{\mathbf{B}}$ is also closed under isomorphic images, and $B_i/Kerf_i \cong f_i(B_i)$, so $f_i(B_i) \in \underline{\mathbf{B}}$. However, $f_i(B_i) = f(B_i)$, which means $f(B_i) \in \underline{\mathbf{B}}$. Thus, every $f(B_i)$ is a subgroup of $N$ which is in $\underline{\mathbf{B}}$. Therefore, by definition of $T(N)$, $\bigvee_{i\in I} f(B_i) \subseteq T(N)$.

Since $x = a_1^{n_1} a_2^{n_2} \dots a_t^{n_t}$ where $a_j \in \bigcup_{i\in I} B_i$, then $f(a_j) \in f\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f(B_i)$. Hence, $y = f(x) = f(a_1^{n_1} a_2^{n_2} \dots a_t^{n_t}) = f(a_1)^{n_1} f(a_2)^{n_2} \dots f(a_t)^{n_t} \in \bigvee_{i\in I} f(B_i) \subseteq T(N)$. So $y \in T(N)$.

Therefore, $f(T(M)) \subseteq T(N)$.

*$T(M/T(M)) = e$*

By definition, $T(M/T(M))$ is the join of all subgroups of $M/T(M)$ that are in $\underline{\mathbf{B}}$. Each of these subgroups is of the form $K/T(M)$, where $T(M) < K < M$ (Theorem 4). Hence, $T(M/T(M)) = \bigvee_{i\in I} K_i/T(M)$, where $K_i/T(M) \in \underline{\mathbf{B}}$ and $T(M) < K_i < M$ for all $i \in I$.

For an arbitrary $j \in I$, let $K_j/T(M)$ be such a subgroup. Then, since both $T(M)$ and $K_j/T(M)$ are in $\underline{\mathbf{B}}$, and $\underline{\mathbf{B}}$ is closed under group extensions, we have $K_j \in \underline{\mathbf{B}}$. Hence, $K_j$ is a subgroup of $M$ such that $K_j \in \underline{\mathbf{B}}$. By definition of $T(M)$, this means $K_j \subseteq T(M)$. But for $K_j/T(M)$ to be a subgroup of $M/T(M)$ it must be that $T(M) \subseteq K_j$. Thus, $K_j = T(M)$. Therefore, $K_j/T(M) = T(M)/T(M) = e$. However, $j$ was arbitrary, which means that $K_i/T(M) = e$ for all $i \in I$. Then $T(M/T(M)) = \bigvee_{i \in I} K_i/T(M) = e$.

Therefore, $T(M/T(M)) = e$.

Hence, T is a radical.

$\underline{\mathbf{B}} = \{\, M \mid T(M) = M \,\}$

Using the fact that $T(N) \in \underline{\mathbf{B}}$ and the definition of $T(N)$, we have $N \in \{\, M \mid T(M) = M \,\} \iff T(N) = N \iff N \in \underline{\mathbf{B}}$.

Therefore $\underline{\mathbf{B}} = \{\, M \mid T(M) = M \,\}$.

$T(T(N)) = T(N)$

Since $T(N) \in \underline{\mathbf{B}} = \{\, M \mid T(M) = M \,\}$, it follows that $T(T(N)) = T(N)$.

$\underline{\mathbf{C}} = \{\, M \mid T(M) = e \,\}$

Let $C_1 \in \underline{\mathbf{C}} = \underline{\mathbf{B}}^r$. Then $\forall B \in \underline{\mathbf{B}}$, $\mathrm{Hom}(B, C_1) = e$. In particular, since $T(C_1) \in \underline{\mathbf{B}}$, we have $\mathrm{Hom}(T(C_1), C_1) = e$. However, $i \in \mathrm{Hom}(T(C_1), C_1)$, where $i$ is the inclusion map, so $i = e$. But by definition of the inclusion map, $i(T(C_1)) = T(C_1)$. So it must be that $T(C_1) = e \Rightarrow C_1 \in \{\, M \mid T(M) = e \,\}$. Thus, $\underline{\mathbf{C}} \subseteq \{\, M \mid T(M) = e \,\}$.

Let $C_2 \in \{\, M \mid T(M) = e \,\}$. Then $T(C_2) = e$. Let $B_2 \in \underline{\mathbf{B}}$ and $f_2 \in \mathrm{Hom}(B_2, C_2)$ be arbitrary. Then $T(B_2) = B_2$. Hence, since $T$ is a radical, $f_2(T(B_2)) \subseteq T(C_2) \Rightarrow f_2(B_2) \subseteq e \Rightarrow f_2 = e$. But $f_2$ was arbitrary so $\mathrm{Hom}(B_2, C_2) = e$. However, $B_2$ was also arbitrary, so $\forall B \in \underline{\mathbf{B}}$, we have $\mathrm{Hom}(B, C_2) = e$. This means that $C_2 \in \underline{\mathbf{B}}^r = \underline{\mathbf{C}}$. Thus, $\{\, M \mid T(M) = e \,\} \subseteq \underline{\mathbf{C}}$.

Therefore, $\underline{\mathbf{C}} = \{\, M \mid T(M) = e \,\}$.

**3 ⇒ 4**

Assume $\underline{\mathbf{C}}$ is closed under isomorphic images, subgroups, group extensions, and direct products; moreover, $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$.

For $M \in \underline{\mathbf{Grp}}$, let $T(M)$ be the intersection of all normal subgroups $K$ of $M$ for which $M/K \in \underline{\mathbf{C}}$. Let $M, N \in \underline{\mathbf{Grp}}$ be arbitrary. By definition, $T(M) = \bigcap\{K \mid K \triangleleft M, M/K \in \underline{\mathbf{C}}\}$ and $T(N) = \bigcap\{H \mid H \triangleleft N, N/H \in \underline{\mathbf{C}}\}$.

First, we will prove that $M/T(M) \in \underline{\mathbf{C}}$.

Let $\{K_i \mid i \in I\}$ be the family of all $K \triangleleft M$ such that $M/K \in \underline{\mathbf{C}}$. Also, let $\{\varphi_i : M \to M/K_i\}$ be the family of canonical epimorphisms and $\varphi : M \to \prod_{i\in I} M/K_i$ the unique homomorphism such that $\pi_i \circ \varphi = \varphi_i$ for all $i \in I$ (Theorem 5). By Lemma 4, $\text{Ker } \varphi = \bigcap_{i\in I} \text{Ker } \varphi_i = \bigcap_{i\in I} K_i = T(M)$. Then, by the First Theorem of Isomorphisms, $M/T(M) \cong \varphi(M) < \prod_{i\in I} M/K_i$. But $M/K_i \in \underline{\mathbf{C}}$ for all $i \in I$ and $\underline{\mathbf{C}}$ is closed under direct products, subgroups, and isomorphisms, so $M/T(M) \in \underline{\mathbf{C}}$.

Notice that this holds for all groups (i.e. $\forall\, G \in \underline{\mathbf{Grp}}, G/T(G) \in \underline{\mathbf{C}}$).

$\underline{\mathbf{C}} = \{\, M \mid T(M) = e \,\}$

Using the fact that $N/T(N) \in \underline{\mathbf{C}}$ and $\underline{\mathbf{C}}$ is closed under isomorphisms, we have $N \in \{\, M \mid T(M) = e \,\} \iff T(N) = e \iff N/\{e\} \in \underline{\mathbf{C}} \iff N \in \underline{\mathbf{C}}$.

Therefore, $\underline{\mathbf{C}} = \{\, M \mid T(M) = e \,\}$.

Now, we proceed to show that T is a radical.

*$T(M) < M$*

Clearly, $T(M) < M$ since the intersections of normal subgroups is a normal subgroup.

*$f : M \to N \Rightarrow f(T(M)) \subseteq T(N)$*

Let $f \in \text{Hom}(M, N)$ be arbitrary, and let $H \triangleleft N$ be such that $N/H \in \underline{\mathbf{C}}$. Note that $H \triangleleft N \Rightarrow f^{-1}(H) \triangleleft M$. Since $f(f^{-1}(H)) < H$, Theorem 3 tells us that $f$ induces a homomorphism $\bar{f} : M/f^{-1}(H) \to N/H$ given by $\bar{f}(mf^{-1}(H)) = f(m)H$. Suppose $\bar{f}(mf^{-1}(H)) = H$ for some $m \in M$. Since $\bar{f}(mf^{-1}(H)) = f(m)H$, then $f(m)H = H \Rightarrow f(m) \in H \Rightarrow m \in f^{-1}(H) \Rightarrow mf^{-1}(H) = f^{-1}(H)$. Hence, $\bar{f}(mf^{-1}(H)) = H \Rightarrow mf^{-1}(H) = f^{-1}(H)$. Thus, $\bar{f}$ is injective, so $M/f^{-1}(H) \cong \bar{f}(M/f^{-1}(H)) < N/H \in \underline{\mathbf{C}}$. Since $\underline{\mathbf{C}}$ is closed under subgroups and isomorphisms, $M/f^{-1}(H) \in \underline{\mathbf{C}}$.

Now we have that $\forall H \triangleleft N$ such that $N/H \in \underline{\mathbf{C}}, \exists K \triangleleft M$ such that $M/K \in \underline{\mathbf{C}}$ and $f(K) \subseteq H$. Thus, since $f(f^{-1}(H)) \subseteq H$, $f(T(M)) = f(\bigcap\{K \mid K \triangleleft M, M/K \in \underline{\mathbf{C}}\}) \subseteq \bigcap\{f(K) | K \triangleleft M, M/K \in \underline{\mathbf{C}}\} \subseteq \bigcap\{f(f^{-1}(H)) | H \triangleleft N, N/H \in \underline{\mathbf{C}}\} \subseteq \bigcap\{H | H \triangleleft N, N/H \in \underline{\mathbf{C}}\} = T(N)$. Therefore, $f(T(M)) \subseteq T(N)$.

$T(M/T(M)) = e$

Since $M/T(M) \in \underline{\mathbf{C}} = \{\, M \mid T(M) = e \,\}$, it follows that $T(M/T(M)) = e$.

Hence, T is a radical.

$T(T(M)) = T(M)$

Since $T(M)$ is a normal subgroup of $M$ and $T(T(M))$ is a characteristic subgroup of $T(M)$ (Lemma 1), then $T(T(M))$ is normal in $M$ (Lemma 5).

Since $M/T(M) \in \underline{\mathbf{C}}$, $T(M)/T(T(M)) \in \underline{\mathbf{C}}$, $\underline{\mathbf{C}}$ is closed under isomorphisms and group extensions, and $M/T(M) \cong (M/T(T(M)))/(T(M)/T(T(M)))$, we have $M/T(T(M)) \in \underline{\mathbf{C}}$. Hence, $T(T(M))$ is a subgroup of $M$ such that $M/T(T(M)) \in \underline{\mathbf{C}}$. By definition, $T(M)$ is the intersection of all such subgroups, so $T(M) \subseteq T(T(M))$. However, $T(T(M))$ is a subgroup of $T(M)$, so $T(T(M)) \subseteq T(M)$. Thus, $T(T(M)) = T(M)$.

$\underline{\mathbf{B}} = \{\, M \mid T(M) = M \,\}$

Let $M_3 \in \underline{\mathbf{B}}$ and let $\pi \in \mathrm{Hom}(M_3, M_3/T(M_3))$ be the canonical epimorphism. Since $M_3 \in \underline{\mathbf{C}}^l$, it must be that $\forall C \in \underline{\mathbf{C}}\ \mathrm{Hom}(M_3, C) = e$. In particular, since we know $M_3/T(M_3) \in \underline{\mathbf{C}}$, $\mathrm{Hom}(M_3, M_3/T(M_3)) = e$. Hence, $\pi = e \Rightarrow \mathrm{Ker}\ \pi = M_3$. But by definition of the canonical epimorphism, $\mathrm{Ker}\ \pi = T(M_3)$. This means that $T(M_3) = M_3 \Rightarrow M_3 \in \{\, M \mid T(M) = M \,\}$. Thus, $\underline{\mathbf{B}} \subseteq \{\, M \mid T(M) = M \,\}$.

Let $M_4 \in \{\, M \mid T(M) = M \,\}$. Then $T(M_4) = M_4$. Let $C_4 \in \underline{\mathbf{C}}$ and $g \in \mathrm{Hom}(M_4, C_4)$. Knowing that $\underline{\mathbf{C}} = \{\, M \mid T(M) = e \,\}$, we have $T(C_4) = e$. But T is a radical, so we have $g(T(M_4)) \subseteq T(C_4) \Rightarrow g(M_4) \subseteq e \Rightarrow g = e$. Since $g$ was arbitrary, $\mathrm{Hom}(M_4, C_4) = e$. But this argument holds for any $C \in \underline{\mathbf{C}}$. So $\forall C \in \underline{\mathbf{C}}\ \mathrm{Hom}(M_4, C) = e \Rightarrow M_4 \in \underline{\mathbf{C}}^l = \underline{\mathbf{B}}$. Thus, $\{\, M \mid T(M) = M \,\} \subseteq \underline{\mathbf{B}}$.

Therefore, $\underline{\mathbf{B}} = \{\, M \mid T(M) = M \,\}$. ■

For any subclasses $\underline{\mathbf{B}}$ and $\underline{\mathbf{C}}$ of $\underline{\mathbf{Grp}}$, let $T_B$ and $T_C$ denote the object functions defined in 2⇒4 and 3⇒4, respectively.

## 2.2 Corollaries of the Equivalence Theorem

**Corollary 6.1** *Let $T$ be a pre-radical. Furthermore, let $\underline{\mathbf{B}} = \{M \mid T(M) = M\}$ and $\underline{\mathbf{C}} = \{M \mid T(M) = e\}$ for all groups $M$. Consider the following statements:*

1. *$T$ is a radical.*
2. *$\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$.*
3. *$\underline{\mathbf{B}}$ is closed under isomorphic images, quotient groups, group extensions, direct sums, and joins of subgroups.*
4. *$T_B$ is an idempotent radical such that $\underline{\mathbf{B}} = \{M \mid T_B(M) = M\}$.*

*Then $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.*

**Corollary 6.2** *Let $T$ be a pre-radical. Furthermore, let $\underline{\mathbf{B}} = \{M \mid T(M) = M\}$ and $\underline{\mathbf{C}} = \{M \mid T(M) = e\}$ for all groups $M$. Consider the following statements:*

1. *$T$ is idempotent.*
2. *$\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$.*
3. *$\underline{\mathbf{C}}$ is closed under isomorphic images, subgroups, group extensions, and direct products.*
4. *$T_C$ is an idempotent radical such that $\underline{\mathbf{C}} = \{M \mid T_C(M) = e\}$.*

*Then $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.*

*Proof.*

Corollary 6.1 can be proven by using the applicable parts of the proof of Theorem 6, following the path $4 \Rightarrow 1 \Rightarrow 2 \Rightarrow 4$ in said Theorem.

Similarly, Corollary 6.2 can be proven by using the applicable parts of the proof of Theorem 6, following the path $4 \Rightarrow 1 \Rightarrow 3 \Rightarrow 4$ in said Theorem. ■

**Corollary 6.3** *Let $T$ be a pre-radical. Furthermore, let $\underline{\mathbf{B}} = \{M \mid T(M) = M\}$ and $\underline{\mathbf{C}} = \{M \mid T(M) = e\}$ for all groups $M$. Then:*

a. *If $T$ is a radical, then $\forall M \in \underline{\mathbf{Grp}}$ $T_B(M) \subseteq T(M)$ and $T_C(M) \subseteq T(M)$.*

b. *If $T$ is idempotent, then* $\forall M \in \underline{\mathbf{Grp}}$ $T(M) \subseteq T_B(M)$ *and* $T(M) \subseteq T_C(M)$.

c. *If $T$ is an idempotent radical, then* $\forall M \in \underline{\mathbf{Grp}}$ $T_B(M) = T(M) = T_C(M)$.

*Proof.*
(a) Assume $T$ is a radical. By Corollary 6.1, $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$ and $T_B$ is an idempotent radical such that $\underline{\mathbf{B}} = \{M \mid T_B(M) = M\}$. Hence, $\forall G \in \underline{\mathbf{Grp}}$ $T_B(G) \in \underline{\mathbf{B}}$.

Let $N \in \underline{\mathbf{Grp}}$ be arbitrary. Since $T$ is a radical, then $T(N/T(N)) = e$. But $\underline{\mathbf{C}} = \{M \mid T(M) = e\}$, so $N/T(N) \in \underline{\mathbf{C}}$. Let $i : T_B(N) \to N$ be the inclusion homomorphism and $\pi : N \to N/T(N)$ be the canonical epimorphism. Then, by function composition, $\pi \circ i \in \text{Hom}(T_B(N), N/T(N))$. But $N/T(N) \in \underline{\mathbf{C}}$ and $T_B(N) \in \underline{\mathbf{B}} = \underline{\mathbf{C}}^l$, so $\pi \circ i = e \Rightarrow i(T_B(N)) \subseteq \text{Ker } \pi \Rightarrow T_B(N) \subseteq T(N)$.

Since $N/T(N) \in \underline{\mathbf{C}}$ and $T_C(N)$ is the intersection of all subgroups $K$ of $N$ such that $N/K \in \underline{\mathbf{C}}$, then $T_C(N) \subseteq T(N)$.

Since $N$ was arbitrary, it follows that $\forall M \in \underline{\mathbf{Grp}}$ $T_B(M) \subseteq T(M)$ and $T_C(M) \subseteq T(M)$.

(b) Assume $T$ is idempotent. By Corollary 6.2, $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$ and $T_C$ is an idempotent radical such that $\underline{\mathbf{C}} = \{M \mid T_C(M) = e\}$. Hence, $\forall G \in \underline{\mathbf{Grp}}$ $G/T_C(G) \in \underline{\mathbf{C}}$.

Let $N \in \underline{\mathbf{Grp}}$ be arbitrary. Since $T$ is idempotent, then $T(T(N)) = T(N)$. But $\underline{\mathbf{B}} = \{M \mid T(M) = M\}$, so $T(N) \in \underline{\mathbf{B}}$. Let $i : T(N) \to N$ be the inclusion homomorphism and $\pi : N \to N/T_C(N)$ be the canonical epimorphism. Then, by function composition, $\pi \circ i \in \text{Hom}(T(N), N/T_C(N))$. But $T(N) \in \underline{\mathbf{B}}$ and $N/T_C(N) \in \underline{\mathbf{C}} = \underline{\mathbf{B}}^r$, so $\pi \circ i = e \Rightarrow i(T(N)) \subseteq \text{Ker } \pi \Rightarrow T(N) \subseteq T_C(N)$.

Since $T(N) \in \underline{\mathbf{B}}$ and $T_B(N)$ is the join of all subgroups of $N$ that are in $\underline{\mathbf{B}}$, then $T(N) \subseteq T_B(N)$.

Since $N$ was arbitrary, it follows that $\forall M \in \underline{\mathbf{Grp}}$ $T(M) \subseteq T_B(M)$ and $T(M) \subseteq T_C(M)$.

(c) Assume $T$ is an idempotent radical.

By (a) and (b), it follows that $\forall M \in \underline{\mathbf{Grp}}$ $T_B(M) = T(M) = T_C(M)$. ■

We'd like to emphasize an important result. Corollary 6.3c tells us that, for any given torsion theory, the idempotent radical associated with it is unique!

**Corollary 6.4** *Let $T$ be a pre-radical. Furthermore, let $\underline{\mathbf{B}} = \{M \mid T(M) = M\}$ and $\underline{\mathbf{C}} = \{M \mid T(M) = e\}$. Then ($\underline{\mathbf{B}}$, $\underline{\mathbf{C}}$) is a torsion theory $\iff$ $T$ is an idempotent radical.*

*Proof.*
($\Rightarrow$) Assume ($\underline{\mathbf{B}}$, $\underline{\mathbf{C}}$) is a torsion theory.

By Theorem 6, $T_B$ is an idempotent radical where $\underline{\mathbf{B}} = \{M \mid T_B(M) = M\}$ and $\underline{\mathbf{C}} = \{M \mid T_B(M) = e\}$. Additionally, by Corollary 6.3c, $T_B(G) = T_C(G)$ for all groups $G$.

Let $N \in \underline{\mathbf{Grp}}$ be arbitrary. Since $T_B$ is idempotent, then $T_B(T_B(N)) = T_B(N)$, so $T_B(N) \in \underline{\mathbf{B}} = \{M \mid T(M) = M\}$, so $T(T_B(N)) = T_B(N)$. Let $i : T_B(N) \to N$ be the inclusion homomorphism. By definition of pre-radical, we have $i(T(T_B(N))) \subseteq T(N) \Rightarrow T(T_B(N)) \subseteq T(N) \Rightarrow T_B(N) \subseteq T(N)$.

Since $T_C$ is a radical, then $N/T_C(N) \in \underline{\mathbf{C}} = \{M \mid T(M) = e\}$, so $T(N/T_C(N)) = e$. Let $\pi : N \to N/T_C(N)$ be the canonical epimorphism. By definition of pre-radical, we have $\pi(T(N)) \subseteq T(N/T_C(N)) \Rightarrow \pi(T(N)) \subseteq e \Rightarrow T(N) \subseteq \text{Ker } \pi \Rightarrow T(N) \subseteq T_C(N)$.

Hence, $T_B(N) \subseteq T(N) \subseteq T_C(N) = T_B(N)$, so $T_B(N) = T(N) = T_C(N)$. Since $N$ was arbitrary, we have $\forall M \in \underline{\mathbf{Grp}}\ T_B(M) = T(M) = T_C(M)$. But $T_B$ and $T_C$ are idempotent radicals, so $T$ must also be an idempotent radical.

($\Leftarrow$) Assume $T$ is an idempotent radical.

Then by Theorem 6, it follows that ($\underline{\mathbf{B}}$, $\underline{\mathbf{C}}$) is a torsion theory. ■

Before we put Theorem 6 and its Corollaries to good use, we present some necessary defintions, examples, and lemmas, along with some preliminary claims.

**Definition 11** *Let $G$ be a group. The subgroup of $G$ generated by the set $\{\, aba^{-1}b^{-1} \mid a, b \in G\,\}$ is called the* commutator subgroup *of $G$ and denoted $G'$. The elements $aba^{-1}b^{-1}$ $(a, b \in G)$ are called* commutators.

**Theorem 7** *If $G$ is a group, then $G'$ is a normal subgroup of $G$ and $G/G'$ is abelian. If $N$ is a normal subgroup of $G$, then $G/N$ is abelian if and only if $N$ contains $G'$* [H].

**Definition 12** *A group that is equal to its commutator subgroup is called a* perfect group.

**Definition 13** *A group that has no non-trivial perfect subgroup is called a* hypoabelian group.

**Definition 14** *A group whose only normal subgroups are itself and the trivial group is called a* simple group*.*

## 2.3 Examples of simple groups, perfect groups, and hypoabelian groups

Since every subgroup of an abelian group is normal, the only abelian simple groups are the $Z_p$ (additive group of integers modulo $p$) with $p$ prime [H]. The commutator subgroup is normal, so every non-abelian simple group is also a perfect group. There are many non-abelian simple groups; in particular, the alternating group $A_n$ is simple whenever $n \neq 4$ [H].

Every abelian group has a trivial commutator subgroup, so the only abelian perfect group is the trivial group. Hence, every abelian group is hypoabelian. The *symmetric group* on 3 symbols, $S_3$, is an example of a non-abelian hypoabelian group. Its commutator subgroup is $A_3$, so $S_3$ is not a perfect group [S3]. Since the order of $S_3$ is 6, by Lagrange's Theorem any proper subgroup must have order 1, 2, or 3. But every group of order 1, 2, or 3 is abelian, so the only perfect subgroup of $S_3$ is of order 1, the trivial subgroup. Thus, $S_3$ is hypoabelian.

**Lemma 6** *If $H \triangleleft M \in \underline{\mathbf{Grp}}$ and $H \subseteq M'$, then $(M/H)' = M'/H$.*

*Proof.*

Assume $H \triangleleft M \in \underline{\mathbf{Grp}}$ and $H \subseteq M'$.

Let $x \in M$ be a commutator, then $x = a_1 b_1 a_1^{-1} b_1^{-1}$ for some $a_1, b_1 \in M$. Hence, $xH = (a_1 b_1 a_1^{-1} b_1^{-1})H = (a_1 H)(b_1 H)(a_1^{-1} H)(b_1^{-1} H)$. So $xH$ is a commutator element of $M/H$. Hence, if $x$ is a commutator of $M$ then $xH$ is a commutator of $M/H$. (In general, commutators are closed under homomorphic images, as we shall later prove.)

Let $mH \in M'/H$ be arbitrary. Then $m \in M'$ so $m = m_1^{n_1} m_2^{n_2} \ldots m_t^{n_t}$ where every $m_i$ is a commutator of $M$ (Definition 11 & Theorem 2). This means that every $m_i H$ is a commutator of $M/H$, so $mH = (m_1^{n_1} m_2^{n_2} \ldots m_t^{n_t})H = (m_1 H)^{n_1} (m_2 H)^{n_2} \ldots (m_t H)^{n_t} \in (M/H)'$. Hence, $M'/H \subseteq (M/H)'$.

Let $cH \in M/H$ be an arbitrary commutator element. Then by definition, $cH = (aH)(bH)(aH)^{-1}(bH)^{-1} = (aba^{-1}b^{-1})H$ for some $a, b \in M$. Since $aba^{-1}b^{-1}$ is a commutator of $M$, it must be that $aba^{-1}b^{-1} \in M'$. So $cH = (aba^{-1}b^{-1})H \in M'/H$. But $cH$ was just an arbitrary commutator of $M/H$, which means the group generated by these commutators is contained in $M'/H$. Thus, $(M/H)' \subseteq M'/H$.

Therefore, $(M/H)' = M'/H$. ■

## 2.4 Commutator Radical

Claim: Let $k : \underline{\mathbf{Grp}} \to \underline{\mathbf{Grp}}$ be the object function defined by $k(G) = G'$ for all groups $G$. Then $k$ is a radical, which we will call the commutator radical.

*Proof.*

Let $M, N \in \underline{\mathbf{Grp}}$ be arbitrary.

$k(M) < M$

Clearly, $k(M) = M' < M$.

$f : M \to N \Rightarrow f(k(M)) \subseteq k(N)$

Let $f \in \text{Hom}(M, N)$ be arbitrary. First, we show that commutator elements are closed under homomorphic images. Let $m \in M$ be a commutator, then $m = aba^{-1}b^{-1}$ for some $a, b \in M$. Hence, $f(m) = f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} \in N$. Thus, $f(m)$ is a commutator of $N$.

Let $y \in f(k(M))$, then $\exists x \in k(M) = M'$ such that $f(x) = y$. Now, we have $x = m_1^{n_1}m_2^{n_2}\dots m_t^{n_t}$ where every $m_i$ is a commutator of $M$. This means that every $f(m_i)$ is a commutator of $N$, so every $f(m_i)$ is in $N'$. Hence, $y = f(x) = f(m_1^{n_1}m_2^{n_2}\dots m_t^{n_t}) = f(m_1)^{n_1}f(m_2)^{n_2}\dots f(m_t)^{n_t} \in N' = k(N)$. Thus, $f(k(M)) \subseteq k(N)$.

$k(M/k(M)) = e$.

Since $M/M'$ is abelian (Theorem 7), it must have a trivial commutator subgroup. Hence, $k(M/k(M)) = (M/M')' = e$.

Therefore, $k$ is a radical. ■

## 2.5 Perfect Groups and the Perfect Radical

Claim: Perfect groups are closed under homomorphic images.

*Proof.*

Let $H, K \in \underline{\mathbf{Grp}}$ and $f \in \text{Hom}(H, K)$ be arbitrary. Suppose $H$ is a perfect group, then $H' = H$. Consider the commutator radical $k$. By Lemma 2, $k$ must be monotone and continuous. Using the continuity of $k$, we have $f(k(H)) \subseteq k(f(H)) \Rightarrow f(H') \subseteq f(H)' \Rightarrow$

$f(H) \subseteq f(H)'$. But by definition, $f(H)' \subseteq f(H)$, so $f(H)' = f(H) \Rightarrow f(H)$ is a perfect group. ■

Claim: Perfect subgroups are closed under joins.

*Proof.*

Let $\{ H_i \mid i \in I \}$ be a family of perfect subgroups of a group $M$. Then by definition of perfect groups (Definition 12), $k(H_i) = H_i' = H_i$ for all $i \in I$.

Let $j \in I$, then $k(H_j) = H_j$. Since $k$ is monotone, $H_j < \bigvee_{i\in I} H_i \Rightarrow k(H_j) < k\left(\bigvee_{i\in I} H_i\right) \Rightarrow H_j < \left(\bigvee_{i\in I} H_i\right)'$. But this holds for every $j \in I$, so it must be that $\bigvee_{i\in I} H_i < \left(\bigvee_{i\in I} H_i\right)'$. However, by definition of the commutator subgroup (Definition 11), $\left(\bigvee_{i\in I} H_i\right)' < \bigvee_{i\in I} H_i$. Thus, $\left(\bigvee_{i\in I} H_i\right)' = \bigvee_{i\in I} H_i$. Therefore, $\bigvee_{i\in I} H_i$ is a perfect group. ■

This means that for any group, we can define a *largest* perfect subgroup, which consists of the join of all perfect subgroups. This allows us to define an object function $p : \underline{\mathbf{Grp}} \to \underline{\mathbf{Grp}}$, $p(G) = \bigvee_{i\in I} H_i$, where $\{H_i \mid i \in I\}$ is the family of perfect subgroups of $G$, for any $G \in \underline{\mathbf{Grp}}$.

Claim: $p$ is an idempotent radical.

*Proof.*

Let $M, N \in \underline{\mathbf{Grp}}$ be arbitrary.

$p(M) < M$

By definition, $p(M)$ is a subgroup of $M$.

$f : M \to N \Rightarrow f(p(M)) \subseteq p(N)$

Let $f \in \mathrm{Hom}(M, N)$ be arbitrary. Since $p(M)$ is a perfect subgroup of $M$, and perfect groups are closed under homomorphic images, $f(p(M))$ is a perfect subgroup of $N$. But $p(N)$ is the largest perfect subgroup of $N$, so $f(p(M)) \subseteq p(N)$.

So far we've proved that $p$ is a pre-radical. Hence, by Lemma 1, $p(M) \triangleleft M$.

$p(M/p(M)) = e$

Let $H/p(M)$ be an arbitrary perfect subgroup of $M/p(M)$. Then $p(M) < H < M$ with $(H/p(M))' = H/p(M)$. But $(H/p(M))' = H'/p(M)$ (Lemma 6), so $H'/p(M) = H/p(M) \Rightarrow H/p(M) < H'/p(M) \Rightarrow H < H'$ (Theorem 4) $\Rightarrow H' = H \Rightarrow H$ is a perfect subgroup of M. But $p(M)$ is the largest perfect subgroup of $M$, so $H \subseteq p(M)$. However, $H/p(M)$ is a group, so it must be that $p(M) \subseteq H$. Thus, $H = p(M) \Rightarrow H/p(M) = e$. Since we picked $H/p(M)$ to be an arbitrary perfect subgroup of $M/p(M)$, it must be that $e$ is the only perfect subgroup of $M/p(M)$ (i.e. $M/p(M)$ is hypoabelian). Since, $p(M/p(M))$ denotes the largest perfect subgroup of $M/p(M)$, we have $p(M/p(M)) = e$ [PM].

$p(p(M)) = p(M)$

By definition, $p(p(M))$ is the largest perfect subgroup of $p(M)$. But $p(M)$ is a perfect group, so it follows that $p(p(M)) = p(M)$.

Therefore, $p$ is an idempotent radical, which we will call the perfect radical. ■

## 2.6 Applications of the Equivalence Theorem

Since in Theorem 6 we have proven $4 \Rightarrow 1$, the fact that $p$ is an idempotent radical yields a torsion theory by letting $\underline{\mathbf{B}} = \{ M \mid p(M) = M \}$ and $\underline{\mathbf{C}} = \{ M \mid p(M) = e \}$. In this case, $\underline{\mathbf{B}}$ consists of all perfect groups and $\underline{\mathbf{C}}$ consists of all hypoabelian groups. Corollary 6.3c tells us that for any group $G$, $p(G)$ is also equal to the intersection of all the hypoabelian subgroups of $G$.

Theorem 6 has enabled us to use an idempotent radical to find a torsion theory. Now, we will use a torsion theory to find an idempotent radical.

Let $\underline{\mathbf{Simfree}}$ denote the sub class of all groups that have no simple subgroups different from the trivial subgroup. Additionally, let $\underline{\mathbf{Simquo}}$ denote the sub class of all groups $X$ such that if $K$ is a proper normal subgroup of $X$, then $X/K$ has a simple subgroup different from the trivial subgroup.

Claim: ($\underline{\mathbf{Simquo}}$, $\underline{\mathbf{Simfree}}$) is a torsion theory. [C][2]

[2] Castellini proves these two subclasses form a *Galois Connection*, but the similarity allows us to use his proof almost identically. See [C] for more details on Galois connections.

*Proof.*

Let $X \in \underline{\textbf{Simquo}}$ and $Y \in \underline{\textbf{Simfree}}$. Suppose that there exists a non-constant homomorphism $f \in \text{Hom}(X, Y)$. Then $\text{Ker}\, f \neq X$ and so, by definition of $\underline{\textbf{Simquo}}$, $X/\text{Ker}f$ has a simple subgroup different from the trivial subgroup. Since $X/\text{Ker}f \cong f(X) < Y$, then $Y$ has a simple subgroup different from the trivial subgroup. This is a contradiction by definition of $\underline{\textbf{Simfree}}$, so $f$ must be constant. Thus, $\forall C \in \underline{\textbf{Simfree}}\ \text{Hom}(X, C) = e \Rightarrow X \in \underline{\textbf{Simfree}}^l$, and $\forall B \in \underline{\textbf{Simquo}}\ \text{Hom}(B, Y) = e \Rightarrow Y \in \underline{\textbf{Simquo}}^r$.

Therefore, $\underline{\textbf{Simquo}} \subseteq \underline{\textbf{Simfree}}^l$ and $\underline{\textbf{Simfree}} \subseteq \underline{\textbf{Simquo}}^r$.

Let $X \in \underline{\textbf{Simfree}}^l$ and suppose $X \notin \underline{\textbf{Simquo}}$. Then there exists a proper normal subgroup $K$ of $X$ such that $X/K$ has no simple subgroup different from the trivial subgroup. That is, $X/K \in \underline{\textbf{Simfree}}$. Since $X \in \underline{\textbf{Simfree}}^l$ and $X/K \in \underline{\textbf{Simfree}}$, then it must be that $\text{Hom}(X, X/K) = e$. But clearly, the quotient morphism $\pi : X \to X/K$ is not constant. This yields a contradiction. Hence, $X \in \underline{\textbf{Simquo}}$. Thus, $\underline{\textbf{Simfree}}^l \subseteq \underline{\textbf{Simquo}}$.

Therefore, $\underline{\textbf{Simquo}} = \underline{\textbf{Simfree}}^l$.

Let $X \in \underline{\textbf{Simquo}}^r$ and suppose $X \notin \underline{\textbf{Simfree}}$. Then there exists a subgroup $K$ of $X$ such that $K$ is simple and not trivial. That is, the only proper normal subgroup of $K$ is the trivial subgroup, so by definition it follows that $K \in \underline{\textbf{Simquo}}$. Since $K \in \underline{\textbf{Simquo}}$ and $X \in \underline{\textbf{Simquo}}^r$, then it must be that $\text{Hom}(K, X) = e$. But clearly, the inclusion homomorphism $i : K \to X$ is not constant because $K$ is not trivial. This yields a contradiction. Hence, $X \in \underline{\textbf{Simfree}}$. Thus, $\underline{\textbf{Simquo}}^r \subseteq \underline{\textbf{Simfree}}$.

Therefore, $\underline{\textbf{Simfree}} = \underline{\textbf{Simquo}}^r$.

Since $\underline{\textbf{Simquo}} = \underline{\textbf{Simfree}}^l$ and $\underline{\textbf{Simfree}} = \underline{\textbf{Simquo}}^r$, we have that $(\underline{\textbf{Simquo}}, \underline{\textbf{Simfree}})$ forms a torsion theory. ■

For any $G \in \underline{\textbf{Grp}}$, let $s_q(G)$ be the join of all subgroups of $G$ that are in $\underline{\textbf{Simquo}}$. Since in Theorem 6 we have proven $1 \Rightarrow 2 \Rightarrow 4$, we have that $s_q$ is an idempotent radical.

Similarly, for any $G \in \underline{\textbf{Grp}}$, let $s_f(G)$ be the intersection of all normal subgroups $H$ of $G$ such that $G/H \in \underline{\textbf{Simfree}}$. Since in Theorem 6 we have proven $1 \Rightarrow 3 \Rightarrow 4$, we have that $s_f$ is also an idempotent radical. Furthermore, Corollary 6.3c tells us that $s_q = s_f$.

## 2.7 Limitations

Now we will consider what happens when a pre-radical is either not idempotent or not a radical. But first, we introduce a new pre-radical.

For any $G \in \underline{\mathbf{Grp}}$, let $s(G)$ be the join of all subgroups of $G$ that are in $\underline{\mathbf{Sim}}$, where $\underline{\mathbf{Sim}}$ denotes the subclass of all simple groups. It is easy to verify that $s$ is an idempotent pre-radical. However, it is not a radical. For example, consider the group $Z_{12}$ (additive group of integers modulo 12). As we stated earlier, the only simple abelian groups are of the form $Z_p$ where $p$ is prime. Hence, $s(Z_{12})$ is isomorphic to $Z_6$, which makes $Z_{12}/s(Z_{12})$ isomorphic to $Z_2$. Thus, $s(Z_{12}/s(Z_{12})) \neq e$. We will refer to $s$ as the simple pre-radical.

For the next two proofs, let $r$ be a pre-radical, and let $\underline{\mathbf{B}} = \{M \mid r(M) = M\}$ and $\underline{\mathbf{C}} = \{M \mid r(M) = e\}$.

Claim: $r$ is non-idempotent radical $\Rightarrow \underline{\mathbf{B}} = \underline{\mathbf{C}}^l$ & $\underline{\mathbf{C}} \neq \underline{\mathbf{B}}^r$

*Proof.*

Assume $r$ is a non-idempotent radical.

By Corollary 6.1, $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$. Suppose $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$. Then by Corollary 6.4, $r$ is an idempotent radical. $\Rightarrow\Leftarrow$

Therefore, $\underline{\mathbf{C}} \neq \underline{\mathbf{B}}^r$. ■

Example:

Consider $r = k$ (commutator radical). Here, $\underline{\mathbf{B}} = \underline{\mathbf{Perfect}}$ (sub class of all perfect groups), and $\underline{\mathbf{C}} = \underline{\mathbf{Ab}}$. We know that $\underline{\mathbf{Ab}} \neq \underline{\mathbf{Perfect}}^r$, since $\underline{\mathbf{Perfect}}^r = \underline{\mathbf{HypoAb}}$ (sub class of all hypoabelian groups). Since $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$, we have that $\underline{\mathbf{Perfect}} = \underline{\mathbf{Ab}}^l$.

Claim: $r$ is idempotent pre-radical, but not a radical $\Rightarrow \underline{\mathbf{B}} \neq \underline{\mathbf{C}}^l$ & $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$

*Proof.*

Assume $r$ is an idempotent pre-radical, but not a radical.

By Corollary 6.2, $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r$. Suppose $\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$. They by Corollary 6.4, $r$ is an idempotent radical. $\Rightarrow\Leftarrow$

Therefore, $\underline{\mathbf{B}} \neq \underline{\mathbf{C}}^l$. ■

Example:

Consider $r = s$ (simple pre-radical). Here, $\underline{\mathbf{C}} = \underline{\mathbf{Simfree}}$. Although $s$ is the join of simple subgroups, $\underline{\mathbf{B}}$ does not consist only of simple groups, since it also contains non-simple groups that are generated by its simple subgroups (such as $Z_6$). This is due to the fact that for any group $G$, $s(G)$ is not necessarily a simple subgroup (i.e. simple subgroups are not closed under joins). For example, $s(S_3) = S_3$, but $S_3$ is not a simple group [S3]. If we compare with the perfect radical $p$, we notice that $p(G)$ *is* a perfect subgroup and that $\{M \mid p(M) = M\}$ is precisely all the perfect groups.

$\underline{\mathbf{B}} \neq \underline{\mathbf{Simfree}}^l = \underline{\mathbf{Simquo}}$, since $Z_{12} \notin \underline{\mathbf{B}}$ yet $Z_{12} \in \underline{\mathbf{Simquo}}$. It's easy to show that $\underline{\mathbf{Simfree}} = \underline{\mathbf{Sim}}^r$. Since $\underline{\mathbf{Sim}} \subseteq \underline{\mathbf{B}} \subseteq \underline{\mathbf{Simquo}}$ and $\underline{\mathbf{Sim}}^r = \underline{\mathbf{Simfree}} = \underline{\mathbf{Simquo}}^r$, it follows that $\underline{\mathbf{Simfree}} = \underline{\mathbf{B}}^r$.

Now we list a couple of examples when one of the two conditions of a torsion theory isn't met.

$\underline{\mathbf{B}} \neq \underline{\mathbf{C}}^l$ & $\underline{\mathbf{C}} = \underline{\mathbf{B}}^r \Rightarrow T_B$ may not be a radical

Let $\underline{\mathbf{B}} = \underline{\mathbf{Sim}}$ and $\underline{\mathbf{C}} = \underline{\mathbf{Simfree}}$. We know that $\underline{\mathbf{Sim}} \neq \underline{\mathbf{Simfree}}^l = \underline{\mathbf{Simquo}}$ and $\underline{\mathbf{Simfree}} = \underline{\mathbf{Sim}}^r$. In this case, $T_B = s$ (simple pre-radical). Hence, $T_B$ is not a radical. Note that since $T_C = s_f$, $T_C$ is indeed an idempotent radical; this agrees with Corollary 6.2. By Corollary 6.3b, $s(G) \subseteq s_f(G)$ for any group $G$.

$\underline{\mathbf{B}} = \underline{\mathbf{C}}^l$ & $\underline{\mathbf{C}} \neq \underline{\mathbf{B}}^r \Rightarrow T_C$ may not be idempotent

Let $\underline{\mathbf{B}} = \underline{\mathbf{Perfect}}$ and $\underline{\mathbf{C}} = \underline{\mathbf{Ab}}$. As we know, $\underline{\mathbf{Perfect}} = \underline{\mathbf{Ab}}^l$ and $\underline{\mathbf{Ab}} \neq \underline{\mathbf{Perfect}}^r$. By polarity (and as discussed in the Examples section), we have $\underline{\mathbf{Ab}} \subset \underline{\mathbf{Perfect}}^r = \underline{\mathbf{HypoAb}}$.

Observe that for any $G \in \underline{\mathbf{Grp}}$, the intersection of all subgroups $H$ of $G$ such that $G/H$ is abelian is precisely $G'$ (See Theorem 7). Hence, we have that $T_C = k$ (commutator radical). Notice that $k$ is not idempotent. Take, for example, $S_3$. The commutator subgroup of $S_3$ is its 3-element subgroup $A_3$, so $k(S_3) = A_3$ [S3]. But every 3-element group is abelian, so $k(k(S_3)) = k(A_3) = e$. Thus, $k(S_3) \neq k(k(S_3))$. Note that since $T_B = p$, $T_B$ is indeed an idempotent radical; this agrees with Corollary 6.1. By Corollary 6.3a, $p(G) \subseteq k(G)$ for any group $G$.

# 3 Closure Operators in $\underline{\mathbf{Grp}}$

## 3.1 Definition and Examples

**Definition 15** *A* closure operator C *on* $\underline{\mathbf{Grp}}$ *is a family* $\{(\,)_X^C\}_{X\in\underline{\mathbf{Grp}}}$ *of functions on the subgroup lattices with the following properties that hold for each* $X \in \underline{\mathbf{Grp}}$*:*

**Expansiveness** $M \subseteq (M)_X^C$ *for all* $M < X$*;*

**Order Preservation** $M \subseteq N \Rightarrow (M)_X^C \subseteq (N)_X^C$ *for subgroups* $M,\ N$ *of* $X$*;*

**Continuity** $\forall f \in Hom(X,Y)\ \forall M < X,\ f((M)_X^C) \subseteq (f(M))_Y^C$.

(See [C, DT] for a general definition of a closure operator on an abstract category.)

Under the *Order Preservation* condition, the *Continuity* condition is equivalent to the following: $\forall f \in \mathrm{Hom}(X,Y)\ \forall N < Y,\ (f^{-1}(N))_X^C \subseteq f^{-1}((N)_Y^C)$ [C, DT]. When no confusion is likely, we will simply write $M^C$ instead of $(M)_X^C$.

**Examples**

For $M < X$ where $X$ is an arbitrary group:

1. $M^{C_i} = M$ is a trivial closure operator.

2. $M^{C_\triangleleft} = <M>_\triangleleft$ is called the *normal closure* of $M$.

3. The intersection of all normal subgroups $K$ of $X$ containing $M$ such that $X/K$ is abelian.

4. The intersection of all normal subgroups $K$ of $X$ containing $M$ such that $X/K$ is torsion-free (also named the *torsion-free normal closure*).

5. The subgroup generated by $M$ and by all perfect subgroups of $X$.

6. The subgroup generated by $M$ and by all simple subgroups of $X$.

The last five examples are claimed to be closure operators in [C]. In order to become more familiar with this concept, we now proceed to prove that $C_\triangleleft$ is a closure operator.

*Proof.*
$M \subseteq M^{C_\triangleleft}$ (Expansiveness)

Clearly, $M \subseteq <M>_\triangleleft = M^{C_\triangleleft}$.

$M \subseteq N \Rightarrow M^{C_\triangleleft} \subseteq N^{C_\triangleleft}$ (Order Preservation)

Let $N < X$ be such that $M \subseteq N$. Then $M \subseteq N \subseteq <N>_\triangleleft$. So $<N>_\triangleleft$ is a normal group that contains $M$, but by definition $<M>_\triangleleft$ is the smallest normal group that contains $M$, so it must be that $<M>_\triangleleft \subseteq <N>_\triangleleft \Rightarrow M^{C_\triangleleft} \subseteq N^{C_\triangleleft}$.

$f(M^{C_\triangleleft}) \subseteq f(M)^{C_\triangleleft}$ (Continuity)

Let $Y \in \underline{\mathbf{Grp}}$ and $f \in \mathrm{Hom}(X, Y)$ be arbitrary.

Let $y \in f(M^{C_\triangleleft}) = f(<M>_\triangleleft)$. Then $\exists x \in <M>_\triangleleft$ such that $f(x) = y$. Then by Definition 9, $x$ is in the intersection of all normal subgroups of $X$ that contain $M$. Let $K \triangleleft Y$ be arbitrary such that $f(M) \subseteq K$. Then $M \subseteq f^{-1}(f(M)) \subseteq f^{-1}(K)$. Hence, $f^{-1}(K)$ is a normal subgroup of $X$ that contains $M$. But $x$ is in the intersection of all such subgroups, so it must be that $x \in f^{-1}(K) \Rightarrow y = f(x) \in K$. Since $K$ was arbitrary, $y$ is in all normal subgroups of $Y$ that contain $f(M)$, and thus in their intersection. This means $y \in <f(M)>_\triangleleft = f(M)^{C_\triangleleft}$. Therefore, $f(M^{C_\triangleleft}) \subseteq f(M)^{C_\triangleleft}$. ■

Immediately, we see the similarities between pre-radicals and closure operators. By Lemma 2, both share the Order Preservation (Monotonicity) and Continuity requirements. However, a radical gives you something smaller (a subgroup, with the trivial group as a lower bound) whereas a closure operator gives you something bigger (a containing group, with the whole group as an upper bound). This accounts for the slight difference in the Continuity condition; we apply the radical to the whole group, yet apply the closure operator to a subgroup. The similarities allow us to use pre-radicals as a means of creating closure operators.

### 3.2 Pre-radical yields Closure Operators

Claim: Let $r : \underline{\mathbf{Grp}} \to \underline{\mathbf{Grp}}$ be a pre-radical, and let $M < X \in \underline{\mathbf{Grp}}$. Then

1. $M^{C_r} = M \cdot r(X)$
2. $M^{C^r} = \pi^{-1}(r(X/<M>_\triangleleft))$

yield closure operators, where $\pi : X \to X/{<}M{>}_{\triangleleft}$ denotes the quotient morphism. These closure operators have been derived from their counterparts in **Ab** [C, DT], and are the least and last (respectively) closure operators such that $M = \{e\} \Rightarrow M^{C_r} = r(X) = M^{C^r}$ [DT]. Note that since $r(X) \triangleleft X$, $M \vee r(X) = M \cdot r(X)$ [H].

*Proof.*

First, we prove $C_r$ is a closure operator.

Extension

Clearly, $M \subseteq M \cdot r(X) = M^{C_r}$.

Order Preservation & Continuity

Let $Y \in$ **Grp** and $f \in \mathrm{Hom}(X, Y)$ be arbitrary. Let $N < Y$ be such that $f(M) \subseteq N$. Since $r$ is a pre-radical, we have $f(r(X)) \subseteq r(Y)$. Hence, $f(M) \cdot f(r(X)) \subseteq N \cdot r(Y)$.

Let $y \in f(M^{C_r})$. Then $\exists x \in M^{C_r} = M \cdot r(X)$ such that $f(x) = y$. Hence, $x = m_1 x_1$, where $m_1 \in M$ and $x_1 \in r(X)$. Since $f$ is a homomorphism, we have $y = f(x) = f(m_1 x_1) = f(m_1) f(x_1) \in f(M) \cdot f(r(X)) \subseteq N \cdot r(Y)$. Thus, $y \in N \cdot r(Y) = N^{C_r}$. Hence, $f(M^{C_r}) \subseteq N^{C_r}$.

*Order Preservation* is now the particular case where $Y = X$ and $f$ is the identity homomorphism. *Continuity* is the particular case where $N = f(M)$.

Therefore, $C_r$ is a closure operator. Now, on to $C^r$.

Extension

$\mathrm{Ker}\,\pi = {<}M{>}_{\triangleleft} \Rightarrow M \subseteq {<}M{>}_{\triangleleft} \subseteq \pi^{-1}(r(X/{<}M{>}_{\triangleleft})) = M^{C^r}$. So $M \subseteq M^{C^r}$.

Order Preservation & Continuity

Let $Y \in$ **Grp** and $f \in \mathrm{Hom}(X, Y)$ be arbitrary. Let $N < Y$ be such that $f(M) \subseteq N$. Since the normal closure holds under the Continuity and Order Preservation conditions, we have $f({<}M{>}_{\triangleleft}) \subseteq {<}f(M){>}_{\triangleleft} \subseteq {<}N{>}_{\triangleleft}$.

Let $\bar{\pi} : Y \to Y/{<}N{>}_{\triangleleft}$ denote the quotient morphism, and let $\psi : X/{<}M{>}_{\triangleleft} \to Y/{<}N{>}_{\triangleleft}$ be defined by $\psi(x{<}M{>}_{\triangleleft}) = f(x){<}N{>}_{\triangleleft}$. We observe that $\psi$ is a homomorphism (Theorem 3) and that $\psi \circ \pi = \bar{\pi} \circ f$. Then, $\pi^{-1} \circ \psi^{-1} = (\psi \circ \pi)^{-1} = (\bar{\pi} \circ f)^{-1} = f^{-1} \circ \bar{\pi}^{-1}$.

Since $r$ is a pre-radical, we have $\psi(r(X/<M>_{\triangleleft})) \subseteq r(Y/<N>_{\triangleleft})$. Hence, $r(X/<M>_{\triangleleft}) \subseteq \psi^{-1}(\psi(r(X/<M>_{\triangleleft}))) \subseteq \psi^{-1}(r(Y/<N>_{\triangleleft}))$, so $M^{C^r} = \pi^{-1}(r(X/<M>_{\triangleleft})) \subseteq \pi^{-1}(\psi^{-1}(r(Y/<N>_{\triangleleft}))) = (\pi^{-1} \circ \psi^{-1})(r(Y/<N>_{\triangleleft})) = (f^{-1} \circ \bar{\pi}^{-1})(r(Y/<N>_{\triangleleft})) = f^{-1}(\bar{\pi}^{-1}(r(Y/<N>_{\triangleleft}))) = f^{-1}(N^{C^r})$. Then, $M^{C^r} \subseteq f^{-1}(N^{C^r}) \Rightarrow f(M^{C^r}) \subseteq f(f^{-1}(N^{C^r}))$. But $f(f^{-1}(N^{C^r})) \subseteq N^{C^r}$. Thus, $f(M^{C^r}) \subseteq N^{C^r}$.

*Order Preservation* is now the particular case where $Y = X$ and $f$ is the identity homomorphism. *Continuity* is the particular case where $N = f(M)$.

Therefore, $C^r$ is a closure operator. ■

## 3.3 Weak Heredity and Heredity

**Definition 16** *Given a closure operator $C$ and $M < X \in \underline{\mathbf{Grp}}$, we say that $M$ is* closed *if $M^C = M$.*

**Definition 17** *Given a closure operator $C$ and $M < X \in \underline{\mathbf{Grp}}$, we say that $M$ is* dense *if $M^C = X$.*

**Definition 18** *A closure operator $C$ is said to be* weakly hereditary *if for any $M < X \in \underline{\mathbf{Grp}}$, we have $(M)^C_Y = Y$, where $Y = (M)^C_X$. In other words, a closure operator is weakly hereditary if for any $M < X \in \underline{\mathbf{Grp}}$, $M$ is dense in its closure. ($(M)^C_X$ may be denoted as $M^C_X$)*

**Definition 19** *A pre-radical $r$ is said to be* hereditary *if for every subgroup $M$ of an arbitrary group $X$, we have $r(M) = r(X) \cap M$.*

**Definition 20** *A closure operator $C$ is said to be* hereditary *if for all subgroups $H, M$ of an arbitrary group $X$ such that $H \subseteq M$, we have $H^C_M = H^C_X \cap M$.*

**Theorem 8** *Let $r$ be a pre-radical. Then:* [DT]

1. *$C_r$ weakly hereditary $\iff$ $r$ idempotent, $C^r$ weakly hereditary $\Rightarrow$ $r$ idempotent.*

2. *$C_r$ is hereditary $\iff$ $r$ is hereditary, $C^r$ is hereditary $\Rightarrow$ $r$ is hereditary.*

Before we prove this theorem, we introduce a useful lemma:

**Lemma 7** *Let $r$ be an idempotent pre-radical and let $M$, $N$ be sugroups of $X \in \underline{\mathbf{Grp}}$. Then*

*a. If $M \subseteq N$, then $r(M \vee r(N)) = r(M) \vee r(N)$.*

*b. $r(M \cdot r(X)) = r(M) \cdot r(X)$.*

*c. If $M \subseteq N$ and $L < X$, then $(M \cdot L) \cap N = M \cdot (L \cap N)$.*

*Proof.*
(a) Assume $M \subseteq N$. Since $r(N) \subseteq N$ and $r$ is monotone (Lemma 2), it follows that $M \vee r(N) \subseteq N \Rightarrow r(M \vee r(N)) \subseteq r(N) \subseteq r(M) \vee r(N)$. Hence, $r(M \vee r(N)) \subseteq r(M) \vee r(N)$.

Conversely, $M \subseteq M \vee r(N) \Rightarrow r(M) \subseteq r(M \vee r(N))$ and $r(N) \subseteq M \vee r(N) \Rightarrow r(r(N)) \subseteq r(M \vee r(N)) \Rightarrow r(N) \subseteq r(M \vee r(N))$ (because $r$ is idempotent). Since both $r(M)$ and $r(N)$ are contained in $r(M \vee r(N))$, it follows that $r(M) \vee r(N) \subseteq r(M \vee r(N))$.

Therefore, $r(M \vee r(N)) = r(M) \vee r(N)$.

(b) Since $M \subseteq X$, then by (a) we have $r(M \vee r(X)) = r(M) \vee r(X)$. But $r(X) \triangleleft X$, so $M \vee r(X) = M \cdot r(X)$ and $r(M) \vee r(X) = r(M) \cdot r(X)$. Therefore, $r(M \cdot r(X)) = r(M) \cdot r(X)$.

(c) Assume $M \subseteq N$ and $L < X$.

Let $x \in (M \cdot L) \cap N$. Then $x \in M \cdot L \Rightarrow x = ml$ where $m \in M, l \in L$. Since $m \in M \subseteq N$ and $x = ml \in N$, then $l \in N$. Hence, $l \in L \cap N$. Thus, $x = ml \in M \cdot (L \cap N)$.

Let $y \in M \cdot (L \cap N)$. Then $y = mn$ where $m \in M, n \in L \cap N$. Since $m \in M \subseteq N$ and $n \in N$, then $y = mn \in N$. However, $n \in L$, so $y = mn \in M \cdot L$. Thus, $y \in (M \cdot L) \cap N$.

Therefore, $(M \cdot L) \cap N = M \cdot (L \cap N)$. ■

*Proof.* (Theorem 8)

Let $X \in \underline{\mathbf{Grp}}$ be arbitrary. First, we'd like to recall that for $M < X$, $M = \{e\} \Rightarrow M_X^{C_r} = r(X) = M_X^{C^r}$.

(1) Assume $r$ is idempotent and let $M$ be any subgroup of $X$.

Let $Y = M_X^{C_r}$. Then $M_Y^{C_r} = M \cdot r(Y) = M \cdot r(M_X^{C_r}) = M \cdot r(M \cdot r(X)) = M \cdot r(M) \cdot r(X)$ (Lemma 7b) $= M \cdot r(X)$ (since $r(M) \subseteq M) = M_X^{C_r} = Y$.

Thus, $C_r$ is weakly hereditary.

Assume $C_r$ and $C^r$ are both weakly hereditary. Then, for any $M < X$, we have $M \cdot r(M^{C_r}) = M^{C_r}$ and $\pi^{-1}(r(M^{C^r}/ <M>_{\triangleleft})) = M^{C^r}$. In particular, let $M = \{e\}$ (the trivial subgroup).

Then $M^{C_r} = r(X)$. Hence, $M \cdot r(M^{C_r}) = M^{C_r} \Rightarrow r(r(X)) = r(X)$.

Since $M^{C^r} = r(X), \pi^{-1}(r(M^{C^r}/ <M>_{\triangleleft})) = M^{C^r} \Rightarrow \pi^{-1}(r(r(X)/\{e\})) = r(X) \Rightarrow \pi^{-1}(r(r(X))) = r(X) \Rightarrow \pi^{-1}(r(r(X))/\{e\}) = r(X) \Rightarrow r(r(X)) = r(X)$.

Thus, in either case, $r$ is idempotent.

(2) Assume $r$ is hereditary, then $r(G) = r(X) \cap G$ for all $G < X$. Let $M, N$ be subgroups of $X$ with $M \subseteq N$.

$M_X^{C_r} \cap N = (M \cdot r(X)) \cap N = M \cdot (r(X) \cap N)$ (Lemma 7c) $= M \cdot r(N) = M_N^{C_r}$.

Thus, $C_r$ is hereditary.

Assume $C_r$ and $C^r$ are both hereditary. Then, for any $M < N < X$, we have $M_N^{C_r} = M_X^{C_r} \cap N$ and $M_N^{C^r} = M_X^{C^r} \cap N$. In particular, let $M = \{e\}$.

Then $M_N^{C_r} = M_X^{C_r} \cap N \Rightarrow r(N) = r(X) \cap N$.

Also, $M_N^{C^r} = M_X^{C^r} \cap N \Rightarrow r(N) = r(X) \cap N$.

Thus, in either case, $r$ is hereditary.

Note that, unlike in the category of abelian groups, we were not able to prove $r$ idempotent $\Rightarrow C^r$ weakly hereditary. Let $<M>_{\triangleleft Y}$ be the intersection of all normal subgroups of $Y$ that contain $M$ (where $Y = M_X^{C^r}$). To prove weak heredity of $C^r$ we would need a different quotient morphism $\pi_Y : Y \to Y/ <M>_{\triangleleft Y}$ so that $M_Y^{C^r} = \pi_Y^{-1}(r(Y/ <M>_{\triangleleft Y})$. However, although $<M>_{\triangleleft Y}$ is a normal subgroup of $Y$, it may not be a normal subgroup of $X$. This, along with the difference in quotient morphisms, were obstacles in establishing an equivalence.

Similarly, we were not able to prove $r$ hereditary $\Rightarrow C^r$ hereditary. To prove heredity of $C^r$ we would need to use $r$'s heredity in one of the following two ways:

$$r(N/ <M>_{\triangleleft X}) = r(X/ <M>_{\triangleleft X}) \cap N/ <M>_{\triangleleft X}$$
$$r(N/ <M>_{\triangleleft N}) = r(X/ <M>_{\triangleleft N}) \cap N/ <M>_{\triangleleft N}$$

In the first case, we cannot be sure that $<M>_{\triangleleft X}$ is contained in $N$. In the second case, we cannot be sure that $<M>_{\triangleleft N}$ is a normal subgroup of $X$, since normality is not transitive (see [D8] for a counterexample).

### 3.4 Closure Operators expressed by means of a Pre-radical

For $M < X \in \underline{\mathbf{Grp}}$, one of our examples of a closure operator (Example 5) was the subgroup generated by $M$ and all perfect subgroups of $X$. This is a particular example of the closure operator $C_r$, where we let $r$ be the perfect radical $p$. We recall that $p(X)$, the largest perfect subgroup of $X$, is equivalent to the subgroup generated by all perfect subgroups of $X$. Hence, we obtain $M^{C_r} = M^{C_p} = M \cdot p(X)$. Since $p$ is idempotent, we have that $C_p$ is weakly hereditary (Theorem 8).

Similar to Example 5 in its definition, Example 6 can be similarly defined by making use of the simple pre-radical $s$. Thus, $C_s$ provides us with the closure operator in question, where $M^{C_s} = M \cdot s(X)$.

Another example of a closure operator (Example 3) was the intersection of all normal subgroups $K$ of $X$ containing $M$ such that $X/K$ is abelian. This can also be expressed as $M \cdot X'$, where $X'$ is the commutator subgroup of $X$ [C]. Since $k(X) = X'$, we let $r$ be the commutator radical to obtain $M^{C_r} = M^{C_k} = M \cdot k(X) = M \cdot X'$.

What about Example 4? Since it's similar to Example 3, can it also be expressed with a pre-radical? We can declare $t(X)$ to be the subgroup generated by all the torsion elements of $X$, and easily verify that $t$ is a pre-radical, thus providing us with a closure operator $C_t$, where $M^{C_t} = M \cdot t(X)$. However, this closure operator is not the same as the one listed in Example 4. This is due to the fact that $t$ is not a radical (i.e. $M/t(M)$ is not necessarily torsion-free). In contrast, $k$ is a radical (since $M/k(M)$ is abelian), and that's why it can be used to express the closure operator of Example 3. Note that in the category of abelian groups, $t$ is a radical, so $C_t$ does coincide with Example 4 in $\underline{\mathbf{Ab}}$.

We define the following lemma to try to generalize some of the cases when a radical can be used to express a closure operator with a definition similar to Example 3:

**Lemma 8** *Let $r$ be a radical and let $\underline{\mathbf{C}} = \{M \mid r(M) = e\}$. If $\forall X \in \underline{\mathbf{Grp}}$ $r(X) = \cap\{M \mid M \triangleleft X, X/M \in \underline{\mathbf{C}}\}$, then for any subgroup $G$ of an arbitrary group $X$, $G \cdot r(X) = \cap\{N \mid G < N, N \triangleleft X, X/N \in \underline{\mathbf{C}}\}$.*

*Proof.*

Assume $\forall X \in \underline{\mathbf{Grp}}$ $r(X) = \cap\{M \mid M \triangleleft X, X/M \in \underline{\mathbf{C}}\}$. Let $G$ be any subgroup of an arbitrary group $X$.

($\subseteq$) Let $h \in G \cdot r(X)$.

Then $h = gx$ for some $g \in G$, $x \in r(X)$. Let $K$ be an arbitrary normal subgroup of $X$ such that $G < K$ and $X/K \in \underline{\mathbf{C}}$. Then, since $g \in G < K$, it follows that $g \in K$. We have that $x \in r(X) = \cap\{M \mid M \triangleleft X, X/M \in \underline{\mathbf{C}}\}$. But $K$ is one of those subgroups, so $x \in K$. Since $g$ and $x$ are both in $K$, then $gx = h \in K$. However, $K$ was arbitrary, so $h \in \cap\{N \mid G < N, N \triangleleft X, X/N \in \underline{\mathbf{C}}\}$.

($\supseteq$) Let $k \in \{N \mid G < N, N \triangleleft X, X/N \in \underline{\mathbf{C}}\}$.

$k \in \cap\{N \mid G < N, N \triangleleft X, X/N \in \underline{\mathbf{C}}\} \subseteq \cap\{M \mid M \triangleleft X, X/M \in \underline{\mathbf{C}}\} = r(X)$. Hence, $k \in r(X) \Rightarrow k \in G \cdot r(X)$. ■

Example:

We know that the perfect radical, $p$, is idempotent. By Corollary 6.3c, $p(G)$ is also equal to the intersection of all the hypoabelian subgroups of $G$ for any group $G$.

Hence, for any subgroup $M$ of an arbitrary group $X$, the intersection of all normal subgroups $K$ of $X$ that contain $M$ such that $X/K$ is hypoabelian is equal to $M \cdot p(X)$ (See Example 5).

### 3.5 Closure Operator yields Pre-radical

We have previously proven that given a pre-radical $r$, we can use it to construct a closure operator (actually, we constructed two, namely $C_r$ and $C^r$). Now, we claim the following:

Claim: Given a closure operator $C$, let $X \in \underline{\mathbf{Grp}}$ be arbitrary. Then $w(X) = \{e_X\}_X^C$ yields a pre-radical, where $e_X$ is the identity element in $X$.

*Proof.*

$w(X) < X$

Clearly, $w(X) = \{e_X\}_X^C < X$

$f : X \to Y \Rightarrow f(w(X)) \subseteq w(Y)$

Let $Y \in$ **Grp** and $f \in \text{Hom}(X, Y)$ be arbitrary. By the continuity condition of the closure operator $C$, we have:

$$f(w(X)) = f(\{e_X\}_X^C) \subseteq (f(e_X))_Y^C = \{e_Y\}_Y^C = w(Y)$$

Therefore, $w$ is a pre-radical. ■

We now have proven there is a correspondence between closure operators and pre-radicals. However, this correspondence is not bijective, since both $C_r$ and $C^r$ will yield the same pre-radical using the construction method mentioned above (which happens to be the same pre-radical used in the construction of $C_r$ and $C^r$; i.e. $w = r$).

Note that if $v$ is any existing pre-radical, then we can define $w$ as follows: $w(X) = (v(X))_X^C$. Using $v$'s continuity, we can modify the above proof to show that this new version of $w$ also yields a pre-radical. Our proof above would then represent the particular case where $v$ is the trivial pre-radical ($v(X) = e$).

# 4 Conclusions and Future Work

By proving the Equivalence Theorem, we have successfully established a relationship between torsion theories and idempotent radicals in the category of groups. Applications of the theorem illustrate how we can use an idempotent radical to find a torsion theory, or use a torsion theory to find an idempotent radial. If we do not have an idempotent radical, but instead have either a radical or an idempotent pre-radical, the Corollaries help us obtain an idempotent radical from which we can then obtain a torsion theory. In addition, the Corollaries help us gain insight into some of the properties of the subclasses generated by radicals or idempotent pre-radicals.

We have also established a relationship between pre-radicals and closure operators in the category of groups, listing different examples. Furthermore, we proved that certain properties of one imply certain (possibly different) properties of the other. Clearly, this indirectly establishes a relationship between torsion theories and closure operators.

Of the two methods of constructing a closure operator from a pre-radical, one was slightly neglected in favor of the other due to the latter's simplicity. A more balanced approach and set of examples using both methods would be ideal. There are other properties of pre-radicals and closure operators that were not explored in this work; doing so would result in a better and deeper understanding of the relationship we have established here. Also, there were implications which were not proven to be equivalences as in the abelian case. Further investigation could either fully establish the equivalences or provide counterexamples.

# 5 Miscellaneous Findings

We conclude with a couple of results that we encountered while working on our main research topic. Even though they are not strictly related to it, we found them interesting enough to be included.

Claim: Let $\underline{\mathbf{B}}$ and $\underline{\mathbf{C}}$ be subclasses of $\underline{\mathbf{Grp}}$. Then

1. $\underline{\mathbf{B}} \subseteq \underline{\mathbf{B}}^{rl}$
2. $\underline{\mathbf{B}} \subseteq \underline{\mathbf{C}} \Rightarrow \underline{\mathbf{C}}^r \subseteq \underline{\mathbf{B}}^r$
3. $\underline{\mathbf{B}}^{rlr} \subseteq \underline{\mathbf{B}}^r$

as well as the dual statements with $r$ and $l$ interchanged [L].

*Proof.*
(1) Let $B_1 \in \underline{\mathbf{B}}$.

Let $B_2 \in \underline{\mathbf{B}}^r$ be arbitrary. Then, $\forall B \in \underline{\mathbf{B}}$ $\mathrm{Hom}(B, B_2) = e$. In particular, since $B_1 \in \underline{\mathbf{B}}$, $\mathrm{Hom}(B_1, B_2) = e$. But $B_2$ was arbitrary, so $\forall B_r \in \underline{\mathbf{B}}^r$ $\mathrm{Hom}(B_1, B_r) = e$. Thus, $B_1 \in \underline{\mathbf{B}}^{rl}$.

Therefore, $\underline{\mathbf{B}} \subseteq \underline{\mathbf{B}}^{rl}$.

(2) Assume $\underline{\mathbf{B}} \subseteq \underline{\mathbf{C}}$.

Let $C_1 \in \underline{\mathbf{C}}^r$. Let $B_1 \in \underline{\mathbf{B}}$ be arbitrary. Since $\underline{\mathbf{B}} \subseteq \underline{\mathbf{C}}$ (by hypothesis), we have $B_1 \in \underline{\mathbf{C}}$. But $C_1 \in \underline{\mathbf{C}}^r$, so $\forall C \in \underline{\mathbf{C}}$ $\mathrm{Hom}(C, C_1) = e$. In particular, since $B_1 \in \underline{\mathbf{C}}$, $\mathrm{Hom}(B_1, C_1) = e$. However, $B_1$ was arbitrary, so $\forall B \in \underline{\mathbf{B}}$ $\mathrm{Hom}(B, C_1) = e$. Thus, $C_1 \in \underline{\mathbf{B}}^r$.

Therefore, $\underline{\mathbf{C}}^r \subseteq \underline{\mathbf{B}}^r$.

(3) This follows from (1) and (2), by letting $\underline{\mathbf{C}} = \underline{\mathbf{B}}^{rl}$.

The dual statements can be similarly proven.

Let $X \subseteq G \in$ **Grp** and let $\bar{X} = \{\, gxg^{-1} \mid g \in G,\ x \in X \,\}$. In the particular case where $X$ is a normal subgroup of $G$, we have $\bar{X} = X = <X>_{\triangleleft}$. However, for the more general case, we state the following:

Claim: $<\bar{X}> = <X>_{\triangleleft}$.

*Proof.*

First, we show that $\bar{X} \subseteq <X>_{\triangleleft}$. Let $x \in \bar{X}$. Then $x = g_1 x_1 g_1^{-1}$ for some $g_1 \in G,\ x_1 \in X$. Since $X \subseteq <X>_{\triangleleft},\ x_1 \in <X>_{\triangleleft}$. Now, since $g_1 \in G,\ x_1 \in <X>_{\triangleleft}$ and $<X>_{\triangleleft}$ is normal in $G$, $x = g_1 x_1 g_1^{-1} \in <X>_{\triangleleft}$. Thus, $\bar{X} \subseteq <X>_{\triangleleft}$.

Hence, $<X>_{\triangleleft}$ is a subgroup of $G$ that contains $\bar{X}$. But, by definition, $<\bar{X}>$ is the intersection of all the subgroups of $G$ that contain $\bar{X}$. So it must be that $<\bar{X}> \subseteq <X>_{\triangleleft}$.

We now proceed to show that $<\bar{X}>$ is a normal subgroup of $G$ that contains $X$. Clearly, $X \subseteq \bar{X} \subseteq <\bar{X}>$, and by definition $<\bar{X}>$ is a subgroup of $G$, so all that is left to prove is that $<\bar{X}>$ is normal.

Let $g \in G$ and $x \in <\bar{X}>$ be arbitrary. Then $x = a_1^{n_1} a_2^{n_2} \ldots a_t^{n_t}$ where $a_i \in \bar{X},\ n_i$ is an integer. By definition of $\bar{X}$, we have that $a_i = g_i x_i g_i^{-1}$, where $g_i \in G,\ x_i \in X$. Hence, $g a_i g^{-1} = g(g_i x_i g_i^{-1})g^{-1} = (gg_i)x_i(g_i^{-1}g^{-1}) = (gg_i)x_i(gg_i)^{-1} \in \bar{X}$. Since $\bar{X} \subseteq <\bar{X}>$, we have $g a_i g^{-1} \in <\bar{X}>$. But $<\bar{X}>$ is a group, so $(g a_i g^{-1})^{n_i} = g a_i^{n_i} g^{-1} \in <\bar{X}>$. Thus, letting $e$ denote the identity element in $G$, $gxg^{-1} = g a_1^{n_1} a_2^{n_2} \ldots a_t^{n_t} g^{-1} = g a_1^{n_1} e a_2^{n_2} e \ldots e a_t^{n_t} g^{-1} = g a_1^{n_1}(g^{-1}g) a_2^{n_2}(g^{-1}g) \ldots (g^{-1}g) a_t^{n_t} g^{-1} = (g a_1^{n_1} g^{-1})(g a_2^{n_2} g^{-1}) \ldots (g a_t^{n_t} g^{-1}) \in <\bar{X}>$. Since $g$ and $x$ were arbitrary, we have $gxg^{-1} \in <\bar{X}>$ for all $g \in G,\ x \in <\bar{X}>$. Thus, $<\bar{X}>$ is a normal subgroup of $G$ that contains $X$. But $<X>_{\triangleleft}$ is the intersection of all the normal subgroups of $G$ that contain $X$. So it must be that $<X>_{\triangleleft} \subseteq <\bar{X}>$.

Therefore, $<\bar{X}> = <X>_{\triangleleft}$. ■

Let $X, Y \in$ **Grp** and $f \in \mathrm{Hom}(X, Y)$. In addition, let $M < X$. Due to the Extension and Continuity conditions of the normal closure, we have $f(M) \subseteq f(<M>_{\triangleleft}) \subseteq <f(M)>_{\triangleleft}$. If $M$ was a normal subgroup and $f$ was an epimorphism, then we would instead have an equality. However, for the more general case, we state the following:

Claim: $<f(<M>_{\triangleleft})>_{\triangleleft} = <f(M)>_{\triangleleft}$.

*Proof.*

Since $f(<M>_{\lhd}) \subseteq <f(M)>_{\lhd}$, we have that $<f(M)>_{\lhd}$ is a normal subgroup that contains $f(<M>_{\lhd})$. But by definition, $<f(<M>_{\lhd})>_{\lhd}$ is the smallest normal subgroup that contains $f(<M>_{\lhd})$, so $<f(<M>_{\lhd})>_{\lhd} \subseteq <f(M)>_{\lhd}$.

$M \subseteq <M>_{\lhd} \Rightarrow f(M) \subseteq f(<M>_{\lhd}) \Rightarrow <f(M)>_{\lhd} \subseteq <f(<M>_{\lhd})>_{\lhd}$ (by Order Preservation of the normal closure).

Therefore, $<f(<M>_{\lhd})>_{\lhd} = <f(M)>_{\lhd}$. ■

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