TOURNAMENT MATRICES: SURVEY AND NEW RESULTS

By

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2007

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Tournaments are simple and complete direct graphs. In this thesis we survey and study particular cases of tournaments. Since the famous Seven Bridges Problem, which was analyzed by Leonard Euler in 1736 and stimulated the development of graph theory, graphs have been considered an important subject in mathematics and other applied sciences, such as physics, biology, chemistry, etc.

Over the last decades the study of graph spectra has been very interesting, because it characterizes the topological structure of a graph. But it turns out that this is not easy to attack. In this thesis we obtain new results about tournament matrices, in particular, about Brualdi-Li matrix and r-partite tournament matrices. The original inspiration of the thesis was to improve and extend the ideas introduced in **Algebraic Multiplicity of the eigenvalue of a bipartite tournament matrix**, by Yi-Zheng Fan and Jiong-Sheng Li published in *SIAM Journal on Matrix Analysis and Applications (SIMAX, 2002)*, and in **An upper bound on the Perron value of an almost regular tournament matrix**, by S. Kirkland, in *Linear Algebra and its Applications* (2003).

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MATRICES DE TORNEOS: ANÁLISIS Y NUEVOS RESULTADOS

Por

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Departamento: Departamento de Ciencias Matemáticas

Los **torneos** son grafos dirigidos simples y completos, pueden ser vistos como una combinación de teoría de grafos, análisis matricial y combinatoria. En esta tesis analizaremos casos particulares de torneos. Desde el famoso problema de los siete puentes, que fue analizado y solucionado por Leonard Euler en 1736 y que estimuló al desarrollo de su teoría, los grafos son tomados un tópico importante en matemáticas y en otras ciencias aplicadas tales como, física, biología, química, etc.

En las últimas décadas el estudio del espectro de un grafo es una aplicación interesante, porque caracteriza la estructura topológica de un grafo. En general, no es fácil atacar este tipo de problemas. En la tesis mostraremos nuevos resultados en matrices de torneos, particularmente en la matrix de Brualdi-Li y en matrices de torneos r-partitos. La inspiración original de la tesis fué mejorar y extender las ideas que aparecen en **Algebraic Multiplicity of the eigenvalue of a bipartite tournament matrix**, por Yi-Zheng Fan y Jiong-Sheng Li publicado en *SIAM J. on Matrix Analysis and Appl (SIMAX, 2002)* y en **An upper bound on the Perron value of an almost regular tournament matrix**, escrito por S. Kirkland en *Linear Algebra and its Appl.* (2003).

To my parents, the inspiration of my life.

To my wife and my son, my idols.

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TABLE OF CONTENTS

LIST OF SYMBOLS

$\mathbb{R}^n$	n-dimensional real vector space.
$\lambda_i(A)$	The ith eigenvalue of matrix A.
$\operatorname{Re}\lambda_i(A)$	The real part of $\lambda_i(A)$.
$\operatorname{Lm}\lambda_i(A)$	Imaginary part of $\lambda_i(A)$.
J_n	The $n \times n$ matrix with all entries equal to one.
$\mathbf{1}_n$	The vector columns with all entries equal to one.
A^*	Conjugate transpose of matrix A.
$\\| \cdot \\|_p$	The p-norm for matrices or vectors.

CHAPTER 1
INTRODUCTION

The thesis is organized as follows: In this chapter we describe the basic notions of graphs and tournaments. Chapter 2 overviews relevant details about two conjectures of Brualdi-Li matrix. Chapter 3 introduces the bipartite tournaments and some recent results. Chapter 4 provides a list of singular value properties for tournament matrices; Chapter 5 presents our results, concluding remarks, and future work in tournaments.

The tournaments are a class of directed graphs and are inspired from the *round robin competitions*. This topic has been of growing interest in the last decades. We will mainly focus on spectral properties and some related properties that were published recently in peer-reviewed journals. The original problem appears in the classical round robin tournament in reviewed building player ranking schemes. The research has motivated an extensive study of the combinatorial and spectral properties of tournament matrices, and therefore has motivated to write some good books (see [1, 5, 6, 11, 24]).

In [1] the authors state that: "the theory of graph spectra, is like an attempt to utilize linear algebra including, in particular, the well-developed theory of matrices for purposes of graph theory and its applications. However, that does not mean that the theory of graph spectra can be reduced to the theory of matrices; on the contrary, it has its own characteristic features and specific ways of reasoning fully justifying it to be treated as a theory in its own right."

1.1 Tournaments, Matrices and Graphs

Graph theory originated with the paper written by Leonhard Euler on the Seven Bridges of Königsberg and published in 1736. This is the first paper in the history of graph theory. The study of tournaments started around the first half of the last century, resulting in the publishing of **Topics on Tournaments** by John Moon, in which the author collected the most useful results. Tournaments have many applications in statistics, game theory and other related areas. For example in [18] it was proven that in round robin competition corresponding to T, a tournament matrix, the Kendall, Wei and Kamanujacharyula's ranking schemes agree with the ranking generated by the row sums of T.

If i, j are two vertices of a graph, we will use the notation $i \to j$ to represent the arc from i to j.

Definition 1. A **Tournament** of n vertices is a loop-free directed graph $\overrightarrow{G}$ with the property that for each pair of distinct vertices i and j, $\overrightarrow{G}$ contains exactly one of the arcs $i \to j$ or $j \to i$.

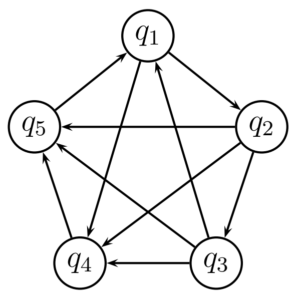
A **Tournament Matrix** is the $(0,1)$ adjacency matrix of a tournament, or equivalently, a $(0,1)$ matrix T such that $T + T^T = J - I$ where J denotes the all ones matrix.

Proposition 1. *The number of arcs in a tournament with n vertices is*

$$\frac{1}{2}n(n-1) = \binom{n}{2}.$$

For the proof note that each vertex has relations with the $n-1$ vertices, we can say that there are $n(n-1)$ arcs, but this count is double because we counted each arc two times, therefore the number of arcs for each tournament is $\frac{1}{2}n(n-1) = \binom{n}{2}$.

Example 1. *Figure 1–1 is an example of a tournament and its matrix.*



$$T = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1–1: Tournament and its matrix.

Clearly,

$$T + T^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} = J - I.$$

1.1.1 The Score Vector

Definition 2. If A is a tournament matrix, its *score vector* is defined as:

$$s = (s_1, s_2, \ldots, s_n)^T = A\mathbf{1}.$$

Note that if $(s_1, s_2, \ldots s_n)^T$ is a score vector of a tournament A, then

$$\sum_{i=1}^{n} s_i = \binom{n}{2}.$$

For $n = 2$ there are $2 = 2^{\binom{2}{2}}$ tournaments, for $n = 3$ we have $8 = 2^{\binom{3}{2}}$, in general we have:

Proposition 2. *Let V be a set of n vertices, then there exist $2^{\binom{n}{2}}$ different tournaments.*

There are n vertices, and $\binom{n}{2}$ arcs, and each arc has 2 possible directions, so we have 2 different tournaments, therefore the proposition is true.

The classical result about score vectors is the Landau's Theorem, see [21]. We give this result below.

Theorem 1. *A set of integers $S = (s_1, s_2, \dots, s_n)$, where $s_1 \leq s_2 \leq \cdots \leq s_n$ is a score vector of some tournament if and only if*

$$\sum_{i=1}^{n} s_i \geq \binom{k}{2},$$

for $k = 1, \dots, n$, with equality holding when $k = n$.

Proof. The proof is due to Ryser [24] in 1964. □

Recently Richard A. Brualdi and Jian Shen published a result about score vector, see [8].

1.1.2 Isomorphic Tournaments

Let T_1 and T_2 be two tournaments with vertices $\{1, 2, \dots, n\}$. We say that T_1 and T_2 are *isomorphic* if there exists a bijective function $\phi : \{1, \dots, n\} \to \{1, \dots, n\}$ such that

$$s \xrightarrow{T_1} t \quad \Longrightarrow \phi(s) \xrightarrow{T_2} \phi(t) \quad \text{or} \quad \phi(s) = \phi(t).$$

In [24] there is a classification for the number of non-isomorphic tournaments. Clearly, if two tournaments are isomorphic, then they have the same score vector, but the reciprocal proposition is not true. See for examples [24].

1.1.3 Permutation Matrix

Another equivalent way of characterizing isomorphic tournaments is using permutation matrices.

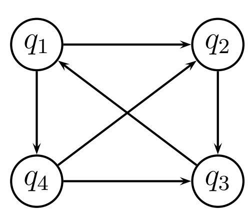
Definition 3. P is a *permutation matrix* if and only if P can be formed directly from I by reordering its rows or its columns.

Let T be a matrix and let P be a permutation matrix, then PT is the matrix formed by reordering the rows of T in same way that P reorders them. TP is same but the reordering is applied in its columns.

Proposition 3. *Let T_1 and T_2 be two tournaments with vertices $\{1, 2, \dots, n\}$, we say that T_1 and T_2 are* isomorphic *if there exists a permutation matrix P such that $T_2 = P^T T_1 P$.*

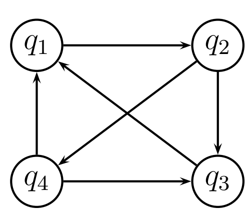
Example 2. *Let G_1 and G_2 be tournaments and T_1 and T_2 their tournament matrices respectively,*

G_1:



$$T_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

G_2:



$$T_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Then G_1 and G_2 are isomorphic. The permutation matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

note that isomorphs means relabeling the vertices of graph G_1 to obtain G_2.

Clearly, two isomorphic tournaments have same spectrum.

1.1.4 Paths and Cycles

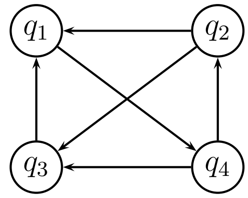
Definition 4. A *path* on a graph (also called a chain) is a sequence $\{x_1, x_2, \ldots, x_n\}$ such that $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ are graph edges and the x_i are distinct.

A closed path $\{x_1, x_2, \ldots, x_n, x_1\}$ on a graph is called a *cycle* or *circuit*.

Let $A^p = \left(a_{ij}^{(p)}\right)$. The number $a_{ij}^{(p)}$ is the (i, j) element of A^p. The next proposition is a classical result.

Proposition 4. *The $a_{ij}^{(p)}$, the element of A^p, is the number of paths of length p from vertex i to vertex j.*

Example 3. *The graph below has a path from q_1 to q_3 through q_4, this is a path of length 2,*



$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

clearly,

$$A^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

Note that the element $a_{13}^{(2)} = 1$ *means that there is one path of length 2 from* q_1 *to* q_3*, and* $a_{41}^{(2)} = 2$ *means that there exist two paths of length 2 from* q_4 *to* q_1*. And similarly,* $a_{44}^{(3)} = 2$ *means that there exist two paths of length 3 from* q_4 *to* q_4*, i.e., two cycles of length 3.*

1.1.5 Transitive Tournament

Definition 5. Let T be a tournament. T is *transitive* if for each vertices p, q and r, we have that if $p \to q$ and $q \to r$ then $p \to r$.

By reordering the vertices, the matrix for transitive tournament is upper triangular:

$$U = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

One characterization theorem is given in [24] for transitive tournament.

Theorem 2 ([24])**.** *The following statements are equivalent:*

1. T_n *is transitive.*
2. *Vertex* p_j *dominates node* p_i *if an only if* $j > i$*.*
3. T_n *has score vector* $(n-1, n-2, \ldots, 2, 1, 0)^T$*.*

4. *The score vector of* T_n *satisfies the equation*

$$\sum_{i=1}^{n} s_i^2 = \frac{n(n-1)(2n-1)}{6}.$$

5. T_n *contains no cycles.*
6. T_n *contains exactly* $\binom{n}{k+1}$ *paths of length* k*, if* $1 \leq k \leq n-1$.
7. T_n *contains exactly* $\binom{n}{k}$ *transitive subtournament* T_k*, if* $1 \leq k \leq n$.
8. *Each principal submatrix of* T_n *contains a row and column of zeros.*

1.1.6 Strongly Connected Tournament

Definition 6. A graph G is *strongly connected*, if there exists a path for each vertex i to each vertex j, $i \neq j$.

Example 4. *The tournament in figure 1–2 is strongly connected, it is easy to check the path from any vertex to any other vertex.*

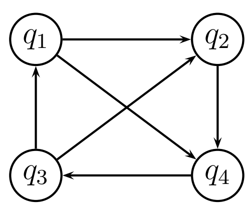


Figure 1–2: Strongly Connected Tournament

1.2 Bipartite Tournament

If we want to make a tournament where there are two disjoint teams I and II of players and each player on Team I plays against each player on Team II, we will have a new structure of tournament that we will call *bipartite tournament*. We note that they themselves do not have the structure of a tournament.

The following characterizes bipartite graphs: *if the greatest eigenvalue is equal to the negative of its smallest eigenvalue then the graph is bipartite [1].*

Definition 7. A tournament is bipartite if there is a partition of its set of vertices in two sets A, B, with $A \cap B = \emptyset$ such that there are no arcs between vertices that belong to the same set and for all $i \in A$ and $j \in B$, we have $i \to j$, or $j \to i$.

Little is known about bipartite tournaments. In [22], Li gave an upper bound for the spectral radius. Later, Sangwook Ree introduced Hypergraphs and, in the Conference on Hypergraphs in Hungary 2001, he spoke about the bipartite tournament matrices. He looked at the spectral bounds of bipartite tournament matrices, that is, tournament matrices of two teams, with arbitrary team size. He indicated that when bipartite matrices exist, players and teams of the matrices are evenly ranked.

Li showed that a bipartite tournament matrix can be both, regular and normal if and only if it has the same team size. Also, he found the condition that was necessary for the variance of the Perron vector (see Definition 11) of the bipartite tournament matrix to vanish.

We use the notation $T_{n_1 n_2}$ for bipartite tournament having sets $|A| = n_1$ and $|B| = n_2$. Clearly, the unique bipartite tournament, which is a tournament, is when $n_1 = n_2 = 1$.

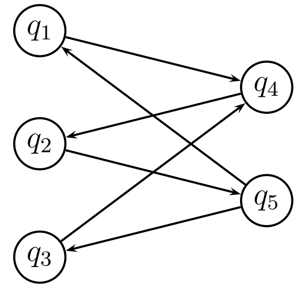
We may let

$$T_{n_1 n_2} = \left[\begin{array}{c|c} 0_{n_1} & B \\ \hline C & 0_{n_2} \end{array}\right], \tag{1.1}$$

where $B + C^T = J_{n_1 n_2}$, and J is a matrix having all entries equal to 1.

Example 5. *Let T_{32} be a bipartite tournament with $A = \{q_1, q_2, q_3\}$ and $B = \{q_4, q_5\}$. Its adjacency matrix and its graph are:*

$$T_{32} = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right]$$



1.3 r-partite Tournament

If we extend the same idea for bipartite tournament, into the case where we make a tournament with r teams where each player of team i plays with each player of all others teams j, $i \neq j$, we have a new structure of tournament which we will call an r-partite tournament, its matrix is

$$\begin{pmatrix} 0_{n_{11}} & B_{12} & \cdots & B_{1(r-1)} & B_{1r} \\ C_{21} & 0_{n_{22}} & \cdots & B_{2(r-1)} & B_{2r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{r-1,1} & C_{r-1,2} & \cdots & 0_{n_{r-1,r-1}} & B_{r-1,r} \\ C_{r1} & C_{r2} & \cdots & C_{r,r-1} & 0_{n_{rr}} \end{pmatrix},$$

where $B_{ij} + C_{ji}^T = J_{n_i,n_j}$ and all 0_i are square zero matrices.

Note that they themselves do not have the structure of tournament, unless the length of all subsets of vertices are one.

1.4 Eigenvalue Bounds for Tournament Matrices

Here we show two eigenvalue properties of tournament matrices. These facts are based on the equation $T + T^T = J - I$.

Proposition 5. *The real part of every eigenvalue of any tournament matrix T is at least $-1/2$.*

Proof. Let T be a tournament matrix, then

$$T + T^T = J - I, \tag{1.2}$$

and let λ be an eigenvalue of T and x its corresponding normalized eigenvector, i.e, $Tx = \lambda x$, and $x^*x = 1$. First we take the right of (1.2) then

$$\begin{aligned} x^*(J - I)x &= x^*Jx - x^*x = x^*Jx - 1 \\ &= y^*y - 1 \geq -1. \end{aligned}$$

On the other hand,

$$\begin{aligned} x^*(T + T^T)x &= x^*Tx + x^*T^*x \\ &= 2\mathrm{Re}\lambda, \end{aligned}$$

then we have

$$-1 \leq x^*(J - I)x = x^*(T + T^T)x = 2\mathrm{Re}\lambda,$$

and the result is obtained. □

Proposition 6. *The real part of every eigenvalue of any tournament matrix T is at most $(n-1)/2$, with equality holding if and only if T is a regular tournament matrix.*

For the proof you can see [4] or for the greatest eigenvalue $\rho(T)$ one can use the Levinger's inequality,

$$\rho(T) \leq \frac{\rho\left(T + T^T\right)}{2} = \frac{1}{2}\rho(J - I) = \frac{n-1}{2}.$$

The eigenvalues of T, $\lambda_i(T)$, $i = 1, 2, \ldots, n$, we will be ordered as $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots \geq |\lambda_n(T)|$. In Chapter 5 we show results about the eigenvalues of T.

1.5 Perron Frobenius Theory

In tournament theory we use only matrices with entries equal to 0 or 1.

Definition 8. An $n \times n$ matrix A with real entries is said to be **nonnegative** if $a_{ij} \geq 0$ for each i and j and **positive** if $a_{ij} > 0$. Similarly, a vector $\mathbf{x} = (x_1, \ldots, x_n)^t$ is said to be **nonnegative** if each $x_j \geq 0$ and **positive** if each $x_j > 0$.

Applications of these matrices are found in geometry and combinatorics see [2] and the Leontief input-output models in economics.

1.5.1 Irreducible Matrix

Definition 9. An $n \times n$ matrix A is said to be a *reducible matrix* if and only if for some permutation matrix P, the matrix $P^T AP$ is block upper triangular, i.e, it has this form

$$P^T AP = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right],$$

were A_{11} and A_{22} are of square order smaller than n.

If a square matrix is not reducible, it is said to be an *irreducible matrix.* The following conditions on an $n \times n$ matrix A are equivalent.

1. A is an irreducible matrix.
2. The digraph associated to A is strongly connected.
3. For each i and j, there exists some k such that $(A^k)_{ij} > 0$.
4. For any partition of the index set $\{1, 2, \ldots, n\}$ into nonempty disjoint sets I_1 and I_2 there exist $i \in I_1$ and $j \in I_2$ such that $a_{ij} \neq 0$.

Proposition 7. *Let A be an $n \times n$ non-negative matrix. Then A is irreducible if and only if $(I + A)^{n-1} > 0$.*

Proof. Let $y \in \mathbb{R}^n$ be such that $\mathbf{y} \geq 0$ and $\mathbf{y} \neq 0$ and write

$$\mathbf{z} = (I + A)\mathbf{y} = \mathbf{y} + A\mathbf{y}. \tag{1.3}$$

With a process that is shown in [20] we can say that $(I+A)^{n-1}\mathbf{y} \geq \mathbf{0}$, for any $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ and therefore the necessary condition is ready. The converse is easy because the graph associated with $(I + A)$ is strongly connected, and hence A is too. $\square$

Example 6. *Let A be a matrix of the form*

$$\begin{pmatrix} * & * & 0 & 0 & * \\ * & * & 0 & 0 & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & 0 & 0 & * \end{pmatrix},$$

it is reducible, because if we permute row 3 with 5, and then column 3 with 5, we obtain

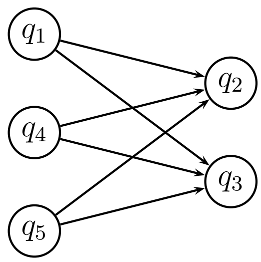
$$\begin{pmatrix} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix},$$

and therefore it is clearly block triangular.

Note that if a matrix has a row or column with all entries zero then the matrix is not strongly connected.

For certain applications, irreducible matrices are more useful than reducible matrices. In particular, the Perron-Frobenius Theorem (see next page) gives more information about the spectra of irreducible matrices than that of reducible matrices. It is known that the Perron Theory is for positive matrices and Frobenius extended similar properties for nonnegative matrices.

Example 7. *The digraph of figure 1–3 is a bipartite tournament,*



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Figure 1–3: Bipartite tournament and its reducible matrix.

Let

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

then

$$P^T A P = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

clearly A is reducible.

1.5.2 A Useful Theorem

Definition 10. Let A be a $(0,1)$-matrix, and $\rho(A)$ its spectral radius. Let h, *index of imprimitivity* or *index of cyclicity*, be the number of eigenvalues having modulus equal to the spectral radius. If $h = 1$ the matrix is called *primitive.*

Proposition 8. *If A if primitive then it is irreducible.*

Proof. See Berman's book [2]. $\square$

The next theorem is part of the famous Perron-Frobenius theory.

Theorem 3. *Let $A \geq 0$ be irreducible of order n. Then the following hold.*

1. *$\rho(A)$ is a simple eigenvalue, and any eigenvalue of A of the same modulus is also simple.*
2. *If A has h eigenvalues $\lambda_0 = re^{i\theta_0}, \lambda_1 = re^{i\theta_1}, \ldots, \lambda_{h-1} = re^{i\theta_{h-1}}$ of modulus $\rho(A) = r$, with $0 = \theta_0 < \theta_1 < \cdots < \theta_{h-1} < 2\pi$, then these numbers are the distinct roots of $\lambda^h - r^h = 0$.*
3. *More generally, the spectrum $S(A) = \lambda_0, \lambda_1, \cdots, \lambda_{n-1}$ goes over into itself under a rotation of the complex plane by $2\pi/h$.*
4. *If $h > 1$, there exists a permutation matrix P such that*

$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{h-1,h} \\ A_{h1} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where the zero blocks along the main diagonal are square.

Definition 11. The spectral radius $\rho(T)$ of a nonnegative irreducible matrix is called the **Perron value** and the corresponding eigenvector is a positive vector, which is called the **Perron vector** for T.

1.6 Regular and Almost Regular Tournament Matrices

Definition 12. A matrix T of a tournament is **regular** if the out-degree of all vertices of T is the same, i.e., if $T\mathbf{1} = ((n-1)/2)\mathbf{1}$, where $\mathbf{1}$ is the vector with all entries equal to one.

The definition is equivalent if each of the row sums of T is $\frac{n-1}{2}$. (Observe that necessarily n must be odd.) It is known in [3, 13] that for odd n, the matrix that maximize the Perron value over the class of $n \times n$ tournament most be a regular tournament matrix.

A matrix T of tournament where the first $n/2$ rows have sums equal to $(n-2)/2$ and the last $n/2$ row have sums equal to $n/2$ is called an **almost regular tournament matrix**.

An almost regular tournament matrix can be a principal submatrix of a regular tournament matrix, as we can see in the next example.

Example 8. *Let T be a regular tournament matrix of order 7*

$$T = \left(\begin{array}{cccccc|c} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right),$$

when $T\mathbf{1} = 3 \cdot \mathbf{1}$. Note that if we remove the last row and the last column we have

$$T' = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

almost regular tournament matrix such that $T'\mathbf{1} = \left(\begin{array}{c} 3 \cdot \mathbf{1} \\ \hline 2 \cdot \mathbf{1} \end{array}\right).$

1.7 Hypertournament and Generalization of Tournament Matrices

Hypertournament and generalized tournament matrices not only provide a means for inquiring into the properties of tournament matrices but also are the source for matrix analytic challenges of independent interest.

A matrix A is called an h-hypertournament if it has zero diagonal entries and $A + A^t = hh^t - I$ for some non-zero $h \in \mathbb{R}^n$. If $h = \mathbf{1}$, any ones vector, an h-hypertournament matrix A satisfies $A + A^t = J - I$, where J denotes the all ones matrix. If all the entries of a $\mathbf{1}$-hypertournament matrix A are in $\{0, 1\}$, then A is called a *tournament matrix*, and if all the entries of A are non-negative, then A is called a *generalized tournament matrix*.

Maybee and Pullman [23] show that every h-hypertournament matrix is (diagonally) similar to a $\mathbf{1}$-hypertournament matrix. Thus, the discussion of the spectral properties of an h-hypertournament matrix can be reduced to the case of $\mathbf{1}$-hypertournament matrices. It is further shown in [23] that

$$-\frac{1}{2} \le \operatorname{Re}\lambda \le \frac{n-1}{2}, \tag{1.4}$$

whenever λ is an eigenvalue of an h-hypertournament matrix. Moreover, the eigenvalues of a generalized tournament matrix satisfy (see [14])

$$|\mathrm{Im}\lambda| \le \frac{1}{2}\cot\left(\frac{\pi}{2n}\right). \tag{1.5}$$

CHAPTER 2
THE BRUALDI-LI MATRIX B_{2N}

2.1 Two Conjectures

Brualdi and Li conjectured that the matrix that minimizes the Perron value over the class of irreducible $n \times n$ tournament matrices is:

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0
\end{pmatrix} \tag{2.1}$$

Notice that the score vector of this matrix is $(1, 1, 2, 3, 4, \ldots, n-3, n-2, n-2)^T$. Let us denote this vector by σ_r. If, for a tournament matrix T, there is a permutation matrix P such that the score vector of PTP^T is σ_r, then we say that *the scores of T are equivalent to σ_r.*

Steve Kirkland et al. proved, the conjecture in 1996 [17], establishing two main results. First, they showed that if T is a tournament matrix which minimizes the Perron value over the class of irreducible tournament matrices of order n, then the score of T is equivalent to σ_r. Then they showed that among all the tournament matrices whose scores are equivalent to σ_n, the matrix given by (2.1) yields the smallest Perron value.

The second conjecture, made by Brualdi and Li in 1983 in [7], says that the matrix which maximizes the Perron value can be written as

$$B_{2n} = \left[\begin{array}{c|c} U_n & U_n^t \\ \hline U_n^t + I & U_n \end{array}\right],$$

where U_n denotes the matrix of order n with ones above the diagonal, and zeros on and below the diagonal. This type of matrix corresponds to a transitive tournament.

The first three B_{2n} matrices are

$$B_2 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right], \qquad B_4 = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array}\right], \qquad B_6 = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

The Brualdi and Li conjecture is still open, nevertheless there has been great progress made on it. This conjecture has been confirmed for small sizes, and there is supporting evidence for its validity asymptotically (as the order grows large) [9, 13, 19].

In [9] the authors prove that the Brualdi-Li matrix B_{2n}, has the largest Perron value among the matrices in

$$\mathcal{M}_n = \left\{ \begin{pmatrix} T & T^T \\ T^T + I & T \end{pmatrix} : T \text{ is an } n \times n \text{ tournament matrix} \right\}. \qquad (2.2)$$

Note that $\mathcal{M}_n$ contains the Brualdi-Li matrix B_{2n}.

2.2 Recent Results

The following are some results obtained recently:

- The first one is due to Brauner and Gentry [3], it was shown that if T is an $n \times n$ tournament matrix then

$$\rho(T) \leq \frac{n-1}{2},$$

with equality holding if and only if the tournament is *regular*.

- A result of Kirkland in [19] showed that for a sufficiently large even n, an $n \times n$ tournament matrix which maximizes the Perron value must be almost regular.
- Kirkland has also proved in [17] that

$$\rho(B_n) = \frac{n-1}{2} - \frac{e^2-1}{2(e^2+1)} + \mathcal{O}\left(\frac{1}{n^3}\right).$$

- Friedland obtained in [11] that for any matrix T of the almost regular tournament of order n

$$\rho(T) \leq \frac{n-1}{2} - \frac{3}{8n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

- The new bound obtained by S. Kirkland in [17] put the last two results together. For all sufficiently large even n, a tournament matrix T which maximizes the Perron value satisfies

$$\rho(T) = \frac{n-1}{2} - \frac{\gamma_n}{n} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

where

$$0.375 = \frac{3}{8} \leq \gamma_n \leq \frac{e^2-1}{2(e^2+1)} \approx 0.380797.$$

He found the best lower bound for γ_n, that is,

$$\frac{2\left(2^{2/3}\right) - 3^{4/3} + 13}{34} \approx 0.377453.$$

In [15], the authors give two forms of the characteristic polynomial of the Brualdi-Li tournament matrix. They use the first form to show that the roots of the characteristic polynomial are simple and that the Brualdi-Li tournament matrix is diagonalizable, and using the second form an expression is found for the coefficients of the powers of the variable λ in the characteristic polynomial. These coefficients give information about the cycle structure of the directed graph associated with the Brualdi-Li tournament matrix.

The most recent result about the spectral radius is given in [25], where it is proved that if T is an almost regular tournament matrix of order $n = 2m$, then

$$\rho(T) \geq \frac{m-1}{2} + \sqrt{\frac{m^2-1}{4}}. \tag{2.3}$$

2.3 The Determinant for B_{2n}

A beautiful and "simple" result for the Brualdi-Li matrix is the calculation of its determinant. We calculate this determinant in Chapter 5. If T is an $n \times n$ tournament matrix with $n > 1$, it is shown that for the particular subclass $\mathcal{M}_n$ of almost regular tournament matrices of order $2n$, like (2.2), the following is true [9]

$$\det(M_T) = (-1)(n-1)\det(T+I) + (-1)^{n-1} n \det(T),$$

when $M_T \in \mathcal{M}_n$. For Brualdi-Li matrix $T = U$ we have

$$\det(M_U) = (-1)(n-1)\det(U+I) + (-1)^{n-1} n \det(U) = 1 - n. \tag{2.4}$$

2.4 The Characteristic Polynomial for B_{2n}

For B_{2n}, you might think that, because of its simple structure, it is easy to find the characteristic polynomial, but in [15] this problem was solved ten years after the conjecture was formulated. In the proof they used the results in [16] and this paper uses results in [11] and [12].

In the first work they proved that the sequence $2n(n-\frac{1}{2}-\rho(B_{2n}))$ is convergent and found the limit. They also showed that asymptotically, the sequence is monotonically decreasing. This problem was established in [12] and was used to find the next theorem.

Theorem 4. *Suppose that $n \geq 2$, let B_{2n} be the Brualdi-Li matrix of order $2n$, and let $\rho(B_{2n})$ be its Perron value. Then*

$$2\rho^2(B_{2n}) - 2(n-1)\rho(B_{2n}) - (n-1) = \frac{1}{\left(\frac{\rho(B_{2n})+1}{\rho(B_{2n})}\right)^{2n} + 1} \tag{2.5}$$

In [15] it is shown that the equation

$$\left(2\lambda^2 - 2(n-1)\lambda - (n-1))\right)\left((1+\lambda)^{2n} + \lambda^{2n}\right) - \lambda^{2n} = 0. \tag{2.6}$$

is satisfied for the value $\rho(B_{2n})$.

It is easy to check that $\lambda = -\frac{1}{2}$ is a root of multiplicity 2 in (2.6). Observe that $-\frac{1}{2}$ is not in the spectrum of any tournament matrix because it is not an algebraic integer. This is true because the characteristic polynomial for any $(0,1)$-matrix the main coefficient is 1 and therefore it doesn't have a rational $1/2$ as a root.

Theorem 5. *Let $n \geq 2$ be an integer and B_{2n} the Brualdi and Li matrix. Then*

$$p(\lambda) = \frac{(2\lambda^2 - 2(n-1)\lambda - (n-1))\left((1+\lambda)^{2n} + \lambda^{2n}\right) - \lambda^{2n}}{(1+2\lambda)^2}, \tag{2.7}$$

is the characteristic polynomial of B_{2n}.

They used this polynomial to prove that its roots are simple and B_{2n} is diagonalizable. They also changed the last expression to find other expression, for the characteristic polynomial. This expression gives the information about the cycle structure of the direct graph associated with the Brualdi-Li tournament matrix.

Theorem 6. *The Brualdi and Li matrix B_{2n} has its characteristic polynomial $c(\lambda)$ equal to*

$$\lambda^{2n} - \sum_{j=0}^{n-1}(n-1-2j)(\lambda+1)^{2(n-j-1)}\lambda^{2j} \tag{2.8}$$

and for each k, such that $0 \leq k \leq 2n-2$, the coefficient of λ^k is

$$c_k = -\sum_{j=0}^{\lfloor k/2 \rfloor}(n-1-2j)\binom{2n-2j-2}{k-2j}.$$

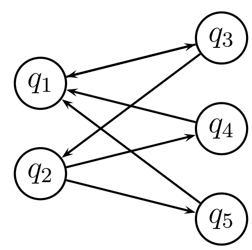
More recently, X. Yong has obtained further results about tournament matrices and the Brualdi-Li matrices [26].

CHAPTER 3
BIPARTITE TOURNAMENT

We should mention that all that will be presented in this chapter is referenced from [10] and the Richard A. Brualdi talk in the *Aveiro Graph Spectra Workshop* 2006. We used similar techniques of bipartite tournament matrices to consider the r-partite tournament matrices, which will be presented in chapter 5.

Example 9. *Let A be the matrix of a tournament show below.*

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Then calculate its spectrum

$$S(A) = \{i\sqrt[4]{2}, -i\sqrt[4]{2}, \sqrt[4]{2}, -\sqrt[4]{2}, 0\} \approx \{1.18921i, -1.18921i, 1.18921, -1.18921, 0\},$$

we see that the index of imprimitivity, $h(A) = 4$.

Example 10. *The matrix*

$$T_{3,3} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

has spectrum $\{i, -i, 1.4142136, -1.4142136, i, -i\}$ *and therefore has an index of imprimitivity equal to 2.*

$h(A)$ can be obtained from the associated directed graph $D(A)$ of A by Theorem 7. The relation between the index of imprimitivity and the associated graph is using the circuits of the associated graph. The following theorem is a classical result.

Theorem 7. *(See [2]) Let* $A \geq 0$ *be irreducible of order* n*. Let* S_i *be the set of all of the lengths* m_i *of circuits in* $D(A)$ *through the vertices* i*, and* $h_i = g.c.d_{m_i \in S_i}\{m_i\}$*. Then* $h_1 = h_2 = \cdots = h_n = h(A)$*.*

Lemma 1. *Let* T_{n_1,n_2} *be a bipartite tournament matrix. Then* $h(T_{n_1,n_2}) = 2$ *or* $h(T_{n_1,n_2}) = 4$*.*

See examples above, and for the proof see [10].

If the matrix T_{n_1,n_2}, $n_1 = n_2 = n$ (that is, the two teams have the same number of players n), then one can consider the spectral radius of $T_{n,n}$.

Note that the maximum spectral radius of bipartite tournament matrices of order $2n$ is less than n, and the minimum over irreducible bipartite tournaments matrix of order $2n$ is greater than 1.

Corollary 1. *Let* T_{n_1,n_2}*. Then the numbers of nonzero eigenvalues and distinct nonzero eigenvalues are both even.*

Theorem 8. *Let* T_{n_1,n_2} *be the corresponding bipartite tournament* G*. Then the following are equivalent.*

1. $h(A) = 4$.
2. *G has the structure of Figure 3–1.*

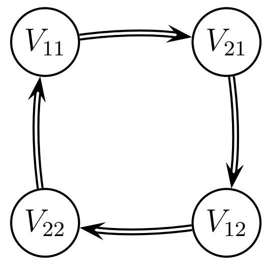


Figure 3–1: Structure of bipartite tournament with $h = 4$.

3. *The spectrum is* $S(A) = \{\rho(A), -\rho(A), i\rho(A), i\rho(A), 0^{n_1+n_2-4}\}$.

4. *The algebraic multiplicity of the eigenvalue 0 of* A *is* $n_1 + n_2 - 4$.

3.1 The Algebraic Multiplicity of the Eigenvalue 0

Lemma 2. *Let* T_{n_1,n_2} *be a bipartite tournament having the form of* (3.1), *where* $l_1 + l_2 + \cdots + l_k = n_1$, $m_1 + m_2 + \cdots + m_k = n_2$, $2 \le k \le n_2$. *Then* T_{n_1,n_2} *has exactly* $2k$ *nonzero eigenvalues and* $n_1 + n_2 - 2k$ *zero eigenvalues, and, for each of these eigenvalues, the algebraic multiplicity is the same as the geometric multiplicity.*

$$T_{n_1,n_2} = \begin{pmatrix} O_{l_1} & & & & J_{l_1,m_1} & & & \\ & O_{l_2} & & & & J_{l_2,m_2} & & \\ & & \ddots & & & & \ddots & \\ & & & O_{l_k} & & & & J_{l_k,m_k} \\ O_{m_1,l_1} & J_{m_1,l_2} & \cdots & J_{m_1,l_k} & O_{m_1} & & & \\ O_{m_2,l_1} & J_{m_2,l_2} & \ddots & \vdots & & O_{m_2} & & \\ \vdots & \ddots & \ddots & J_{m_{k-1},l_k} & & & \ddots & \\ O_{m_k,l_1} & \cdots & J_{m_k,l_{k-1}} & J_{m_k,l_k} & & & & O_{m_k} \end{pmatrix} \tag{3.1}$$

Theorem 9. *Let* $t = n_1 + n_2 - 2k$, $k = 2, 3, \ldots, n$. *For any* n_1, n_2 *there exists some matrix* T_{n_1,n_2} *whose eigenvalue 0 has the same algebraic and geometric multiplicity equal to* t.

More recently, we have obtained further results about r-partite tournament matrices in [27].

CHAPTER 4
SINGULAR VALUES OF TOURNAMENT MATRICES

In this chapter we survey some results about singular values of tournament matrices, including the the most recent results obtained by D. Gregory and S. Kirkland in [13].

The method for determining how close any matrix A_n is to a matrix of smaller rank involves factoring A into a product $U\Sigma V^*$, where U and V are orthogonal matrices of order n, and Σ is $n \times n$ matrix whose off-diagonal entries are all 0's and whose diagonal elements are $\sigma_1, \sigma_2, \ldots, \sigma_n$ and satisfy $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. The σ_i's determined by this factorization are unique and are called *singular values* of A, and the factorization $U\Sigma V^*$ is called the *singular value decomposition* of A.

We see that since V diagonalize A^*A, it follows that the $\mathbf{v}_j$'s are eigenvectors of A^*A, and similarly way for AA^*. Another way of calculating the singular values of A is to calculate the nonnegative square roots of the eigenvalues of A^*A or, equivalently, of AA^*. If the eigenvalues are also taken in nondecreasing order then $\sigma_i^2(A) = \lambda_i(A^*A) = \lambda_i(AA^*)$, $i = 1, \ldots, n$. In particular, $\sigma_1^2(A) = \rho(A^*A)$. The largest singular value, $\sigma_1(A)$, is also called *spectral norm* of A because $\sigma_1(A) = ||A||_2$, the operator norm induced by the usual Euclidean norm $||\cdot||_2$.

Definition 13. Let T be a tournament matrix of order n and $s = T\mathbf{1}$ its score vector. We will call $\alpha^2(T) = \frac{1}{n}\sum_i (s_i - \frac{n-1}{2})^2 = \frac{s^T s}{n} - (\frac{n-1}{2})^2$ the *score variance*.

For example for a regular tournament matrix T, $\alpha^2(T) = 0$, and for an almost regular tournament matrix T, $\alpha^2(T) = \frac{n}{8}$.

It is easily seen that if T is normal then the singular values of T are the module of its eigenvalues. A tournament matrix T is *nearly normal* in the sense that the rank one perturbation, i.e., $T - \frac{1}{2}J$ is a normal matrix. To see this, note that

$$T - \frac{1}{2}J = \frac{1}{2}D, \qquad \text{when } d_{ij} = \begin{cases} 1, & \text{if } t_{ij} = 1 \\ -1, & \text{if } t_{ij} = 0, \end{cases}$$

$$\left(T - \frac{1}{2}J\right)^* = \left(T^* - \frac{1}{2}J\right) = \frac{1}{2}E, \qquad \text{when } e_{ij} = \begin{cases} -1, & \text{if } t_{ij} = 1 \\ 1, & \text{if } t_{ij} = 0, \end{cases}$$

and

$$\left(T - \frac{1}{2}J\right)^* \left(T - \frac{1}{2}J\right) = \frac{1}{4}ED = \frac{1}{4}A \qquad \text{where } a_{ij} = \sum_j e_{ij}d_{ij}$$
$$\left(T - \frac{1}{2}J\right) \left(T - \frac{1}{2}J\right)^* = \frac{1}{4}DE = \frac{1}{4}B \qquad \text{where } b_{ij} = \sum_j d_{ij}e_{ji}$$

we see that $a_{ij} = b_{ij}$, this is because $e_{ij} = -e_{ji}$ and $d_{ij} = -d_{ij}$ for $i \neq j$, then $a_{ij} = b_{ij}$.

4.1 Majorization

Definition 14. Let $x, y \in \mathbb{R}^n$. We say that x is *weakly majorized* by y and write $x \prec_w y$ if for each $k = 1, \ldots, n$, the sum of the k largest entries of x is less than or equal to the sum of the k largest entries of y. We say that x is *majorized* by y and write $x \prec y$ if $x \prec_w y$ and $\sum x_i = \sum y_i$.

Example 11. *Let $x = (0, 2, 3, 4, 5, 6)^T$ and $y = (0, 2, 4, 4, 4, 7)^T$. Then $x \prec_w y$ and $x \not\prec y$ because*

k	$\sum_{i=1}^{k} x_i$	$\sum_{i=1}^{k} y_i$
1	0	0
2	2	2
3	5	6
4	9	10
5	14	14
6	20	21

Table 4–1: x weakly majorized by y.

From the necessary condition in Landau's theorem and the properties of majorization we have that $||T\mathbf{1}||_2 \leq ||U\mathbf{1}||_2$, then U is a transitive tournament matrix. This is equivalent to $\alpha^2(T) \leq \alpha^2(U) = \frac{n^2-1}{12}$, for all tournament matrices T of order n.

A similar and important result is shown below.

Theorem 10. *If U is the upper triangular tournament matrix of order $n \geq 2$, then, for all tournament matrices T of order n*

$$\sigma(T) \leq \sigma(U) = \frac{1}{2} \csc \frac{\pi}{4n-2}. \tag{4.1}$$

Equality holds if and only if M is the matrix of a transitive tournament.

Below we describe bounds on the minimum value of $\sigma_1(T)$ and the maximum value of $\sigma_n(T)$. These are easily verified.

Definition 15. A tournament matrix T of order $n \geq 2$ is called *doubly regular* if every pair of vertices in the associated tournament jointly dominates the same number of vertices (necessarily, $\frac{n-3}{4}$).

We see that T is doubly regular if and only if $T^T T = \frac{n+1}{4} I + \frac{n-3}{4} J$. Such matrices are also called *Hadamard tournament matrices* since they are coexistent with skew Hadamard matrices of order $n+1$.

Proposition 9. *Let T be a tournament matrix of order n and let $\sigma_i(T)$ be its singular values in nonincreasing order . Then*

1. $\sigma_1(T) \geq \frac{n-1}{2}$ *with equality holding if and only if M is regular.*
2. $\sigma_n(T) \leq \frac{\sqrt{n+1}}{2}$ *with equality holding if and only if M is doubly regular.*

Definition 16. Let T be a tournament matrix of order n. The *spread* of T, noted by $\mathrm{sp}(T)$, is $\max|\lambda - \mu|$, where the maximum is taken over all eigenvalue λ, μ of T.

Proposition 10. *Let T be a tournament matrix of order n. Then*

1. $\mathrm{sp}(T^T T) \geq \frac{n(n-3)}{4}$ *with equality holding if and only if T is doubly regular.*
2. $\mathrm{sp}(T^T T) \leq \frac{1}{4}\csc^2 \frac{\pi}{4n-2}$ *with equality holding if and only if T is transitive tournament matrix.*

Theorem 11. *Let T be a tournament matrix of order $n \geq 4$. Then T has precisely two distinct singular value if and only if T is doubly regular.*

The next proposition provides a lower bound on the spectral norm, $\sigma_1(M)$, of a tournament matrix of order n. When n is odd, it agrees with Proposition 9 and the regular tournament matrices are those that give equality. When n is even, it will yield the lower bound in Corollary 2 below. In that lower bound, equality holds only in the special case that $n = 2m$ where m is odd.

Proposition 11. *Let T be a tournament matrix of order $n \geq 2$ and let*

$$B = \begin{pmatrix} \left(\frac{n-1}{2}\right)^2 + \alpha^2 & \frac{n\alpha}{2} \\ \frac{n\alpha}{2} & \alpha^2 + \frac{1}{4} \end{pmatrix},$$

where α^2 is the score variance of T. Then

$$\sigma_1^2(T) \geq \rho(B) = \sigma^2 + \frac{1}{8}\left(n^2 - 2n + 2 + n\sqrt{(n-2)^2 + 16\alpha^2}\right).$$

Equality holds if and only if T has at least $n-2$ eigenvalues with real part $-\frac{1}{2}$.

Corollary 2. *If T is a tournament matrix of even order $n = 2m$, then*

$$\sigma_1^2(T) \geq \frac{1}{8}\left((n-2)^2 + n\sqrt{(n-2)^2+4}\right).$$

Equality is attained if and only if m is odd, and T is permutation similar to a matrix of the form

$$\begin{pmatrix} R & X \\ J - X^T & S \end{pmatrix},$$

where R and S are regular tournament matrices of order m and X is an $m \times m$ $\{0,1\}$-matrix with constant row and columns sums $(m-1)/2$.

If $n = 2m$ where m is odd, then the minimum spectral norm for tournament matrices of order n is given the lower bound in Corollary 2. Although we do not know minimum spectral norm for all cases where $n = 2m$, it is proven in Theorem 12 that any tournament matrix of even order that attains the minimum spectral norm must be almost regular. The following corollary to Proposition 9 will be needed in the proof.

Corollary 3. *If T is a tournament matrix of even order n and T is not almost regular, then*

$$\sigma_1^2(T) \geq \rho(B) = \frac{1}{8n}\left(n^3 - 4n^2 + 4n + 16 + n\sqrt{n^4 - 4n^3 + 8n^2 + 32n}\right),$$

where B is defined above with $\alpha^2 = \frac{1}{4} + \frac{2}{n}$.

Lemma 3. *Let R be a regular tournament matrix of odd order m and let*

$$M = \begin{pmatrix} R & 0 & R^T & 1 \\ 1^T & 0 & 0^T & 0 \\ R^T + I & 1 & R & 0 \\ 0^T & 1 & 1^T & 0 \end{pmatrix}$$

where 1 *and* 0 *are column m-vector. Then M is an almost regular tournament matrix of order $n = 2(m+1)$,* $\dim W_M = 4$, *and* $\sigma_1 = \frac{1}{2}$, $\sigma_2 = \sqrt{m}$, $\sigma_3 = \frac{1}{2}$.

Theorem 12. *If T is a tournament matrix of even order n with minimum spectral norm, then T is almost regular.*

The last result is same as the one given by Kirkland in [19].

CHAPTER 5
THE NEW PROPERTIES

In this chapter we give our results about the Brualdi-Li tournament matrix and r-partite tournament matrices.

5.1 A Simple Calculation of the Determinant for B_{2n}

Previously we proved that $B_{2n} - \frac{1}{2}J$ is normal, now we calculate its determinant and next the determinant for B_{2n}. To get this result we use the next theorem.

Theorem 13. *If A, B, C, D are square matrices of order n and $AC = CA$ the*

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|.$$

Proof. This result is a direct application of *Schur's complement.* □

Let B_{2n} be the Brualdi-Li matrix, and

$$B_{2n} - \frac{1}{2}J = \left(\begin{array}{c|c} U - \frac{1}{2}J & U^T - \frac{1}{2}J \\ \hline U^T + I - \frac{1}{2}J & U - \frac{1}{2}J \end{array} \right). \tag{5.1}$$

using that $U + U' + I = J$ then

$$\det\left(B_{2n} - \frac{1}{2}J\right) = \det\left(\left(U - \frac{1}{2}J\right)^2 - \left(U^T + I - \frac{1}{2}J\right)\left(U^T - \frac{1}{2}J\right)\right)$$
$$= \det\left(U^T + I - \frac{1}{2}J\right) = \det\left(\frac{1}{2}(U^T - U + I)\right)$$
$$= \frac{1}{2},$$

since the calculation of $\det(U^T - U + I)$ is easy: note that it is equal to 2^{n-1}.

For the calculation of the determinant of the Brualdi-Li matrix we use again Schur's complement with respect to $(U^T + I)$ in the exchange matrix from B_{2n}, i.e., $\det(PB_{2n})$, so

$$\left|\left(\begin{array}{c|c} I & 0 \\ \hline -U(U^T+I)^{-1}U & I \end{array}\right)\left(\begin{array}{c|c} U^T+I & U \\ \hline U & I \end{array}\right)\right| = \left|\left(\begin{array}{c|c} U^T+I & U \\ \hline 0 & U^T - U(U^T+I)^{-1}U \end{array}\right)\right|$$
$$= \det\left(U^T + I\right)\det\left(U^T - U(U^T+I)^{-1}U\right)$$
$$= 1 \cdot (-1)^{n-1}(n-1)$$
$$= (-1)^{n-1}(n-1).$$

To see this, observe that $U(U^T + I)^{-1}U = -U$ and $\det(U^T + U) = \det(J - I) = (-1)^{n-1}(n-1)$, when so exchange the block rows. The determinant for this matrix permutation

$$P = \left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right)$$

is $(-1)^n$ and therefore finally $\det(B_{2n}) = 1 - n$.

The above calculation is actually similar to the one presented in [9], this calculations is only for B_{2n} while the other calculation is for any tournament, when $U = T$.

5.2 The Perron Value of B_{2n}

Now we consider the Perron value of the Brualdi-Li matrix, we first present a proposition and next a theorem from Berman's book [2]. Next we give a proof for proposition 12.

Proposition 12. *Let $B_{2n}^{(0)}$ be the Brualdi-Li matrix. Let $k \geq 0$ and*

$$B_{2n}^{(k+1)} = \left(D^{(k)}\right)^{-1} B_{2n}^{(k)} D^{(k)},$$

where $D^{(k)} = \operatorname{diag}\left(B_{2n}^{(k)}\mathbf{1}\right)$. Then

$$B_{2n}^{(k)}\mathbf{1} \to \rho(B_{2n})\mathbf{1}, \text{ when } k \to \infty.$$

To prove the proposition, we use the following 2 facts:

1. The first one is to calculate $B_{2n}^{(k+1)} = \left(D^{(k)}\right)^{-1} B_{2n}^{(k)} D^{(k)}$.

$$D^{(k)} = \operatorname{diag}\left(d_1^{(k)}, d_2^{(k)}, \ldots, d_n^{(k)}\right),$$

then

$$B_{2n}^{(k+1)} = \left(D^{(k)}\right)^{-1} B_{2n}^{(k)} D^{(k)} = \left(b_{ij}^{(k+1)}\right), \tag{5.2}$$

where, $b_{ij}^{(k+1)} = b_{ij}^{(k)} \left(d_i^{(k)}\right)^{-1} d_j^{(k)}$ for $i, j = 1, \ldots, n$.

2. The second one is to calculate $B_{2n}^{(k+1)}\mathbf{1}$.

$$\begin{aligned} B_{2n}^{(k+1)}\mathbf{1} = \left(D^{(k)}\right)^{-1} B_{2n}^{(k)} D^{(k)} &= \left(\sum_{j=1}^{n} b_{ij}^{(k)} \left(d_i^{(k)}\right)^{-1} d_j^{(k)}\right) \\ &= \left(\left(d_i^{(k)}\right)^{-1} \sum_{j=1}^{n} b_{ij}^{(k)} d_j^{(k)}\right), \end{aligned} \tag{5.3}$$

for $i = 1, \ldots, n$.

Now, because

$$B_{2n} = \left(\begin{array}{c|c} U_n & U_n^T \\ \hline I + U_n^T & U_n \end{array} \right),$$

where U_n is a transitive tournament matrix of order n and therefore $B_{2n}\mathbf{1} = \left(\begin{array}{c} \frac{n-2}{2}\mathbf{1} \\ \hline \frac{n}{2}\mathbf{1} \end{array} \right)$ by definition, then

$$B_{2n}^{(1)} = \left(D^{(0)}\right)^{-1} B_{2n}^{(0)} D^{(0)} = \left(b_{ij}^{(0)} \left(d_i^{(0)}\right)^{-1} d_j^{(0)} \right) =$$

$$= \begin{cases} 1\frac{2}{n-2}\frac{n-2}{2} = 1, & \text{for } i = 1, \ldots, \frac{n}{2},\ j = i+1, \ldots, \frac{n}{2}, \\ 1\frac{2}{n-2}\frac{n}{2} = \frac{n}{n-2}, & \text{for } i = 1, \ldots, \frac{n}{2},\ j = \frac{n}{2}+1, \ldots, \frac{n}{2}+i-1, \\ 1\frac{2}{n}\frac{n-2}{2} = \frac{n-2}{n}, & \text{for } i = \frac{n}{2}+1, \ldots, n,\ j = 1, \ldots, i - \frac{n}{2}, \\ 1\frac{2}{n}\frac{n}{2} = 1, & \text{for } i = \frac{n}{2}+1, \ldots, n,\ j = i+1, \ldots, n, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$B_{2n}^{(1)}\mathbf{1} = \left(d_i^{(0)}\right)^{-1} \sum_{j=1}^{n} b_{ij}^{(0)} d_j^{(0)} =$$

$$= \begin{cases} \dfrac{2}{n-2} \displaystyle\sum_{j=1}^{n} b_{ij}^{(0)} d_j^{(0)} = \dfrac{2}{n-2} \left[\displaystyle\sum_{j=i+1}^{n/2} b_{ij}^{(0)} d_j^{(0)} + \sum_{j=\frac{n}{2}+1}^{n/2+i-1} b_{ij}^{(0)} d_j^{(0)} \right], \\ \dfrac{2}{n} \displaystyle\sum_{j=1}^{n} b_{ij} d_j = \dfrac{2}{n} \left[\displaystyle\sum_{j=\frac{n}{2}}^{i-1} b_{ij} d_j + \sum_{j=i+1}^{n/2+i-1} b_{ij} d_j \right], \end{cases}$$

$$= \begin{cases} \frac{2}{n-2} \left[\left(\frac{n}{2} - i\right) 1 \frac{n-2}{2} + (i-1) 1 \frac{n}{2} \right] = \frac{n}{2} - i + (i-1)\frac{n}{n-2}, \\ \frac{2}{n} \left[\left(i - \frac{n}{2}\right) 1 \frac{n-2}{2} + (n-i) 1 \frac{n}{2} \right] = \left(i - \frac{n}{2}\right) + n - i, \end{cases}$$

$$B_{2n}^{(1)}\mathbf{1} = \begin{cases} \frac{n^2-4n+4i}{2(n-2)}, & \text{for } i = 1, \ldots, \frac{n}{2}, \\ \frac{n^2+2n-4i}{2n}, & \text{for } i = \frac{n}{2}+1, \ldots, n. \end{cases}$$

Again,

$$B_{2n}^{(2)} = \left(D^{(1)}\right)^{-1} B_{2n}^{(1)} D^{(1)} = \left(b_{ij}^{(1)} \left(d_i^{(1)}\right)^{-1} d_j^{(1)}\right) =$$

$$= \begin{cases} 1\frac{2(n-2)}{n^2-4n+4i}\frac{n^2-4n+4i}{2(n-2)} = 1, & \text{for } i = 1,\ldots,\frac{n}{2},\ j = i+1,\ldots,\frac{n}{2}, \\ \frac{n}{n-2}\frac{2(n-2)}{n^2-4n+4i}\frac{n^2+2n-4i}{2n} = \frac{n^2+2n-4i}{n^2-4n+4i}, & \text{for } i = 1,\ldots,\frac{n}{2},\ j = \frac{n}{2}+1,\ldots,\frac{n}{2}+i-1, \\ \frac{n-2}{n}\frac{2n}{n^2+2n-4i}\frac{n^2-4n+4i}{2(n-2)} = \frac{n^2-4n+4i}{n^2+2n-4i}, & \text{for } i = \frac{n}{2}+1,\ldots,n,\ j = 1,\ldots,i-\frac{n}{2}, \\ 1\frac{2n}{n^2+2n-4i}\frac{n^2+2n-4i}{2n} = 1, & \text{for } i = \frac{n}{2}+1,\ldots,n,\ j = i+1,\ldots,n, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$B_n^{(2)}\mathbf{1} = \left(d_i^{(1)}\right)^{-1} \sum_{j=1}^{n} b_{ij}^{(1)} d_j^{(1)} =$$

$$= \begin{cases} \frac{2(n-2)}{n^2-4n+4i}\left[\left(\frac{n}{2}-i\right)1\frac{n^2-4n+4i}{2(n-2)} + (i-1)\frac{n}{n-2}\frac{n^2+2n-4i}{2n}\right] & \text{for } i = 1,\ldots,\frac{n}{2}, \\ \frac{2n}{n^2+2n-4i}\left[\left(i-\frac{n}{2}\right)\frac{n-2}{n}\frac{n^2-4n+4i}{2(n-2)} + (n-i)1\frac{n^2+2n-4i}{2n}\right] & \text{for } i = \frac{n}{2}+1,\ldots,n, \end{cases}$$

$$= \begin{cases} \frac{n^3-6n^2+16ni-16i^2+8i-4n}{2(n^2-4n+4i)}, & \text{for } i = 1,\ldots,\frac{n}{2}, \\ \frac{n^3+8n^2-24ni+16i^2}{2(n^2+2n-4i)}, & \text{for } i = \frac{n}{2}+1,\ldots,n. \end{cases}$$

Finally,

$$B_n^{(3)} = \left(D^{(2)}\right)^{-1} B_n^{(2)} D^{(2)} = \left(b_{ij}^{(2)} \left(d_i^{(2)}\right)^{-1} d_j^{(2)}\right) =$$

$$= \begin{cases} 1, & \text{for } i = 1,\ldots,\frac{n}{2},\ j = i,\ldots,\frac{n}{2}, \\ \frac{n^3+8n^2-24ni+16i^2}{n^3-6n^2+16ni-16i^2+8i-4n}, & \text{for } i = 1,\ldots,\frac{n}{2},\ j = \frac{n}{2}+i,\ldots,n, \\ \frac{n^3-6n^2+16ni-16i^2+8i-4n}{n^3+8n^2-24ni+16i^2}, & \text{for } i = \frac{n}{2}+1,\ldots,n,\ j = 1,\ldots,i-\frac{n}{2}, \\ 1, & \text{for } i = \frac{n}{2}+1,\ldots,n,\ j = i,\ldots,n, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$B_n^{(3)}\mathbf{1} = \left(d_i^{(2)}\right)^{-1}\sum_{j=1}^{n} b_{ij}^{(2)} d_j^{(2)} =$$

$$= \begin{cases} \left(\frac{n}{2}-i\right)+(i-1)\frac{n^3+8n^2-24ni-16i^2}{n^3-6n^2+16ni-16i^2+8i-4n}, \\ \left(i-\frac{n}{2}\right)\frac{n^3-6n^2+16ni-16i^2+8i-4n}{n^3+8n^2-24ni-16i^2}+(n-i), \end{cases}$$

$$= \begin{cases} \frac{n^4-8n^3+44in^2-96ni^2+64in-20n^2+64i^3-48i^2}{2(n^3-6n^2+16in-16i^2+8i-4n)}, & \text{for } i=1,\dots,\frac{n}{2}, \\ \frac{n^4+22n^3+4n^2-64i^3-92in^2+128ni^2+16i^2-16in}{2(n^3+8n^2+16i^2-24in)}, & \text{for } i=\frac{n}{2}+1,\dots,n, \end{cases}$$

Theorem 14. *Let B_{2n} be the Brualdi-Li matrix, and*

$$B_{2n}^{(k)}\mathbf{1} = (r_1, r_2, \dots, r_n)^T, \quad \mathbf{1} = (1,\dots,1)^T.$$

Then

$$\min_{1\le i\le n} r_i \le \rho(B_{2n}) \le \max_{1\le i\le n} r_i.$$

Proof. This result is from a theorem in Berman's book (page 37. [2]) □

Here we presente a better result below.

Theorem 15. *Let $A=(a_{ij})\ge 0$ of order n and $p_i=\sum_{j\ne i} a_{ij}$, for $i=1,\dots,n$. Then*

$$\min_{i\ne j}\left\{a_{ii}+a_{jj}+\sqrt{(a_{ii}-a_{jj})^2+4p_ip_j}\right\} \le \rho(A)$$
$$\le \max_{i\ne j}\left\{a_{ii}+a_{jj}+\sqrt{(a_{ii}-a_{jj})^2+4p_ip_j}\right\}.$$

Proof. If $Ax=\rho(A)x$ then

$$(a_{ii}-\rho(A))x_i = -\sum_{i=1}^{n} a_{ij}x_j.$$

Let $x_p = \max_{1\leq j\leq n} x_j$, $x_q = \max_{\substack{j\neq p\\ 1\leq j\leq n}} x_j$. Then

$$(a_{pp} - \rho(A))x_p \leq P_p x_q \tag{5.4}$$

$$(a_{qq} - \rho(A))x_q \leq P_q x_q \tag{5.5}$$

multiplication of (5.4) and (5.5) yields

$$(a_{pp} - \rho(A))(a_{qq} - \rho(A)) \leq P_p P_q,$$

solving for $\rho(A)$ we obtain the bounds. □

If $A = (a_{ij})$, $a_{ij} \geq 0$, $A\mathbf{1} = (s_1, s_2, \ldots, s_n)^T$ we have

$$\min_{i\neq j} s_i s_j \leq \rho(A) \leq \max_{i\neq j} s_j s_j$$

5.3 The Inverse of Brualdi-Li Matrix

In this section we calculate the inverse of the Brualdi-Li matrix. For this we design a little script function in **Scilab** by which we can obtain B_{2n} for any n.

```
function [B, U] = matrixBL(n)

 S = ones(n/2,n/2);
 U = S -  tril(S);
 B = [U U’; (U’ + eye()) U];

endfunction
```

Through calculations, we see that if B_{2n} is the Brualdi and Li matrix, then its inverse has the following form:

$$B_{2n}^{-1} = \left(\begin{array}{c|c} u & 1 \\ \hline (C_{2n-1} - W)^{-1} & v \end{array}\right) \tag{5.6}$$

where $u = (-1, 0, \ldots, 0, 0)$, $v = (0, 0, \ldots, 0, -1)^T$ of order $2n-1$,

$$C_{2n-1} = \operatorname{circ}(\underbrace{1, \ldots, 1}_{n-1}, \underbrace{0, \ldots, 0}_{n}),$$

is a circulant matrix of order $2n-1$ and

$$W = \left(\begin{array}{c|c} 0_{\frac{n}{2}, \frac{n}{2}-1} & 0_{\frac{n}{2}, \frac{n}{2}} \\ \hline J_{\frac{n}{2}-1, \frac{n}{2}-1} & 0_{\frac{n}{2}-1, \frac{n}{2}} \end{array} \right). \tag{5.7}$$

For example for $n = 4$ we have

$$B_8 = \left(\begin{array}{c|ccccccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array} \right)$$

and

$$B_8^{-1} = \left(\begin{array}{ccccccc|c} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 \end{array} \right)$$

The general case is

$$B_{2n}^{-1} = \frac{1}{n-1}\left(\begin{array}{cccccccccc|c}
1-n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & n-1 \\
\hline
1 & 2-n & 1 & \cdots & 1 & 2-n & 1 & \cdots & 1 & 1 & 0 \\
1 & 1 & 2-n & 1 & \cdots & 1 & 2-n & 1 & \cdots & 1 & 0 \\
\vdots & \vdots & & \ddots & \ddots & & \ddots & \ddots & & \vdots & \vdots \\
1 & 1 & \cdots & & 2-n & 1 & \cdots & & 1 & 2-n & 0 \\
2-n & 1 & 1 & \cdots & 1 & 2-n & 1 & \cdots & & 1 & 0 \\
1 & 2-n & 1 & \cdots & & 1 & 2-n & 1 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots & & & \ddots & \ddots & & \vdots & \vdots \\
1 & 1 & \cdots & 2-n & 1 & 1 & \cdots & & 1 & 2-n & 0 \\
1 & 1 & \cdots & 1 & 2-n & 1 & & \cdots & & 1 & 1-n
\end{array}\right), \tag{5.8}$$

where the block has three diagonals, the first one start in entry $(1,2)$, second one start in entry $(1,n)$, and last in entry $(n+1,1)$.

Theorem 16. *The Brualdi-Li matrix is ill-conditioned.*

Proof. We need to show that $\|B_{2n}\|\|B_{2n}^{-1}\|$ is very large, for any norm. Note that $\|B_{2n}\|_\infty = \|B_{2n}\|_1 = n$. On the other hand, direct calculations shows that $\|B_{2n}^{-1}\|_\infty = \|B_{2n}^{-1}\|_1 = \frac{4n-5}{n-1}$. Then,

$$\mathrm{cond}_\infty(B_{2n}) = \mathrm{cond}_1(B_{2n}) = \|B_{2n}\|_1\|B_{2n}^{-1}\|_1 = n\frac{(4n-5)}{n-1} > 4n-5,$$

for any n. Therefore the matrix is ill-conditioned for large n. $\square$

For the Euclidean, we know that

$$\|B_{2n}\|_2 = \lambda_1^{1/2}\left((B_{2n})^T B_{2n}\right)$$

and

$$\lambda_1\left((B_{2n})^T B_{2n}\right) \geq \max_{i \leq i \leq n} c_{ii}$$

where $c_{ii} = \sum_{j=1}^{2n} b_{ij}^2 = \sum_{j=1}^{2n} b_{ij}$. Therefore,

$$\mathrm{cond}_2(B_{2n}) = \|B_{2n}\|_2\|B_{2n}^{-1}\|_2 \geq \sqrt{n}\cdot 1 = \sqrt{n}. \tag{5.9}$$

For example if $n = 100$, then $\mathrm{cond}_\infty(B_{2n}) > 4n - 5 = 395$, and $\mathrm{cond}_2(B_{2n}) > \sqrt{n} = 10$.

5.4 r-partite Tournament

The results shown here are similar to the presentation [10]. We first make an extension of bipartite tournaments to 3-partite tournaments. We consider if $r > 3$.

One of the questions to prove a result that we will present later is: *in all reducible matrices, can we find a vertex with off-degree or in-degree equal to zero?* The answer is negative. Observe the following example.

Example 12. *We see that*

$$T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

is a reducible matrix. This is true because if

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is a permutation matrix, then we have

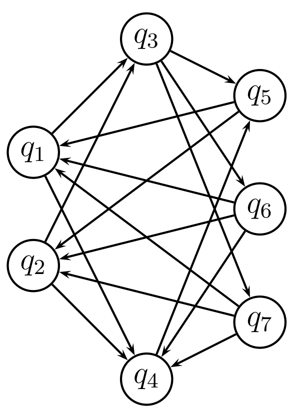
$$PTP^T = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

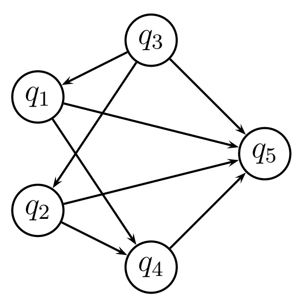
is a triangular matrix by blocks and T does not have any row or column of zeros.

Another question is, *does any irreducible matrix have at least one nonzero element in each column and each row?* In this case the answer is positive. Let us see the following examples:

Example 13. *T_{223} represents the matrix of a 3-partite tournament (left). This matrix is irreducible because is easy to check the graph is strongly connected. On the other hand it is clear that the 3-partite tournament matrix T_{221} (right) is reducible because the graph is not strongly connected, you can see this in vertex number 5, which does not connect with any vertices*

$$T_{223} = \left(\begin{array}{cc|cc|ccc} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right) \qquad T_{221} = \left(\begin{array}{cc|cc|c} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right)$$





Let $\mathcal{J}$ be the 3-partite tournament matrix having the structure

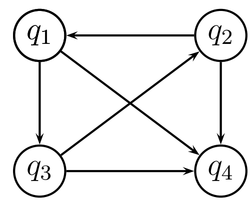
$$\mathcal{J} = \begin{pmatrix} 0 & J_{n_1} & 0 \\ 0 & 0 & J_{n_2} \\ J_{n_3} & 0 & 0 \end{pmatrix}, \tag{5.10}$$

where J_{n_i} for $i = 1, 2, 3$ is a square matrix of order n_i. It's easy to check that, this type of matrix is irreducible.

The next example is for an irreducible matrix, i.e when the graph is strongly connected.

Example 14. *Let A be a tournament of order* 4*,*

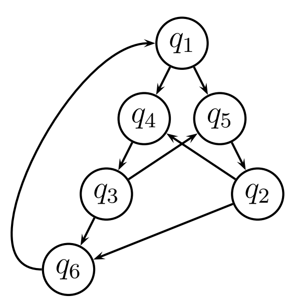
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



Note that the matrix has, in all rows and all columns, at least one nonzero element and note that its graph is strongly connected.

For any n ,we thought that if T_n, tournament matrix, has in-degree and out-degree in all its vertices the $T_{n-1}^{(i)}$, the tournament obtained to remove the vertex i, has same property. We try to use this approach, but that is not true, let us observe the following example:

Example 15. *In this example we see that if we remove any vertex, then the resulting graph will not have the previous property.*



Definition 17. A circuit of length m in a graph $G(V, E)$ is a sequence of arcs $(i_1, i_2), (i_2, i_3), \ldots, (i_{m-1}, i_m), (i_m, i_1)$ of V.

A circuit of length n, will be called an n-cycle.

Lemma 4. *Let $T_{n_1 n_2}$ be a bipartite tournament whose vertices have in-degree and out-degree. Then $T_{n_1 n_2}$ has a 4-cycle.*

Proof. Let T be a bipartite tournament matrix. By Proposition 4 each element in the main diagonal of T^4 is a number a 4-cycle, we suppose that $n_1 = n_2$ and let A_{nn} be of the form $A_{nn} = \left[\begin{smallmatrix} 0_n & B \\ C & 0_n \end{smallmatrix}\right]$, and $A^4 = \left[\begin{smallmatrix} BCBC & 0 \\ 0 & CBCB \end{smallmatrix}\right]$, note that $BCBC = (BJ - BB^T)^2$, and $CBCB = (BB^T - JB)^2$, therefore we need to prove that any element in the same position of $BCBC$ in the row a column is different from zero (similarly for $BCBC$). First we use $BJ - BB'$, it has of form $(BJ - BB')_{ij} = r_{ij}$ such that $r_{ij} = \sum_{k=1}^{n} b_{ik}(1 - b_{jk}) \geq 0$, if $i = j$ it is clearly, that $b_{ik}(1 - b_{jk}) = 0$. Now if $i \neq j$ there exist k_0, j_0 such that $b_{ik_0} = 1$ and $b_{j_0 k_0} = 0$. Existence of k_0, j_0 is ascertained by the hypothesis and for $r_{j_0 i}$ there exists k' such that $b_{j_{0k'}} = 1$. The proof is finished if we can prove that $b_i k' = 0$. We will continue by contradiction. We suppose that k' exists for $b_{j_0 k'} = 1$ and not for $b_{ik'} = 0$, so we need to change for other j_0, but if we continue this way, the ith row has all entries equal to one, this is not possible because this contradicts the hypothesis that all vertices have in-degree and out-degree, therefore $b_{ik'} = 0$ and all vertices have 4-cycle. $\square$

Example 16. *Consider*

$$\underset{B}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}} \quad \underset{BJ - BB'}{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 & 1 \\ 2 & 2 & 2 & 0 & 1 \\ 2 & 3 & 2 & 1 & 0 \end{pmatrix}} \quad \underset{BCBC = (BJ - BB')^2.}{\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 3 & 6 & 2 & 2 & 1 \\ 6 & 6 & 6 & 1 & 3 \\ 6 & 9 & 4 & 3 & 4 \\ 7 & 8 & 5 & 2 & 6 \end{pmatrix}},$$

Note that the element 6 of the diagonal of $(BJ - BB')^2$ in position 2 comes from the inner product of row 2, $(1, 0, 1, 0, 1)$ with column $(1, 0, 2, 2, 3)^T$, which has at least *one nonzero element in the same position because for row 2 of B, we can find other different row itself such that $b_{ik}(1-b_{jk})$ there exist j_0, k_0.*

In [10] the authors prove that if A is an irreducible bipartite tournament matrix then $h(A)=2$ or $h(A)=4$. For r-partite tournament matrices with $r=3$, if A is irreducible we have the only cases $h=1$ or $h=r=3$. For example:

$$T_{112} = \left(\begin{array}{c|c|cc} 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right),$$

is irreducible and $h=1$, but

$$T_{112} = \left(\begin{array}{c|c|cc} 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} 0 & J_{11} & 0 \\ 0 & 0 & J_{12} \\ J_{21} & 0 & 0 \end{pmatrix},$$

has a property like equation (5.10) and $h=3$. For $r>3$ we have the next theorem.

Theorem 17. *Let $r>3$. If A is a r-partite tournament matrix then A is primitive, i.e, $h=1$.*

Proof. Without loss of generality we only have to consider the case $r=4$. Let A be a 4-partite tournament matrix. Labeling the vertices of the associated graph we have

$$A = \begin{pmatrix} 0_{n_1} & A_{12} & A_{13} & A_{14} \\ A_{21} & 0_{n_2} & A_{23} & A_{24} \\ A_{31} & A_{32} & 0_{n_3} & A_{34} \\ A_{41} & A_{42} & A_{43} & 0_{n_4} \end{pmatrix}, \tag{5.11}$$

when all 0_{n_i} are square zero matrices an $A_{ij} + A'_{ji} = J_{n_i n_j}$. Suppose that A is irreducible, and we will proceed by contradiction. Suppose that $h=4$, then by

Theorem 3 part 4 there exists a permutation P such that

$$PAP^T = \begin{pmatrix} 0 & \mathcal{A}_{12} & 0 & 0 \\ 0 & 0 & \mathcal{A}_{23} & 0 \\ 0 & 0 & 0 & \mathcal{A}_{34} \\ \mathcal{A}_{41} & 0 & 0 & 0 \end{pmatrix}, \tag{5.12}$$

where the zero blocks along the main diagonal are square. Clearly this matrix does not satisfy the definition of 4-partite tournament. If $h = 3$, then the new matrix has the form

$$PAP^T = \begin{pmatrix} 0 & \mathcal{A}_{12} & 0 \\ 0 & 0 & \mathcal{A}_{23} \\ \mathcal{A}_{31} & 0 & 0 \end{pmatrix}, \tag{5.13}$$

this matrix can be only a 3-partite tournament matrix. Similar case for $h = 2$.

We consider other values $h > 4$ and complete the proof for $r = 4$, we consider one more case, for example $h = 5$. In this case the matrix has the form

$$PAP^T = \begin{pmatrix} 0 & \mathcal{A}_{12} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{A}_{23} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{A}_{34} & 0 \\ 0 & 0 & 0 & 0 & \mathcal{A}_{45} \\ \mathcal{A}_{41} & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.14}$$

We divide its vertices into subsets V_1, V_2, V_3, V_4, V_5 such that each arc is from V_i to V_{i+1} for some $1 \leq i \leq 4$, or V_5 to V_1. We use this partition to obtain the 4-partite tournament matrix again. We can join V_1 and V_3, since there exist no arcs between, them so we obtain the matrix below

$$\left(\begin{array}{cc|c|c|c} 0 & 0 & J & 0 & 0 \\ 0 & 0 & 0 & J & 0 \\ \hline 0 & J & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & J \\ \hline J & 0 & 0 & 0 & 0 \end{array}\right),$$

and clearly it is not the 4-partite tournament matrix.

The same argument can be applied to consider for $h > 4$. So we can claim that $h = 1$. $\square$

In the next section we state the conclusion and possible future work in this area.

5.5 Conclusion and Future Work

We observed that the tournament matrices are special $(0, 1)$-matrices.

Since Perron-Frobenius theory about nonnegative matrices, many people have paid much attention to the topics because of their many applications in the real world. We can use their eigenvalue properties to attack the problems. However, it is not easy to obtain the spectrum of graphs, and there are still many open problems. For example, which graphs have distinct eigenvalues? This is important in graph spectra because the spectrum characterizes the topological structure of a graph. According to our understanding, the combination of graph theory, matrix analysis and combinatorics makes this topic really interesting.

In all the considerations above, finding a better bound for Perron value of Brualdi-Li matrix is not easy and it seems that we need different techniques. We hope that the similarity techniques in the paper of Savchenko in [25] can help attack the problem. The maximization problem for spectral radius seems hard and ....

Anyway, in this attempt we learned other techniques or properties of Brualdi-Li matrix.

The study of r-partite tournament matrix is currently in infancy, but very active, we continue our research on the topic and we hope that we can publish our papers soon.

We are working on the following problems:

1. Calculate the algebraic multiplicity of eigenvalue 0 and calculate the number of distinct eigenvalues in the r-partite tournament matrices.

2. Explore the spectrum of bipartite tournaments: Let $\mathcal{T}(R,S)$ denote the set of all bipartite tournaments with score vectors R and S, for given R and S, determine $\max \rho(T)$ and $\min \rho(T)$ over all matrix in $\mathcal{T}(R,S)$. Currently there are many people working on the nearly regular bipartite tournaments.

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