KILLING VECTOR FIELDS FOR A SPECIAL CLASS OF METRICS

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Abstract

A one parameter local group of isometries of Riemannian or more generally pseudo-Riemannian manifolds is generated by a Killing vector field which is subjected to the commonly named Killing equations. The latter constitute an over determined system of first order partial differential equations which are even linear and homogeneous. However, in general such a system is not completely integrable.

A brief presentation of the well-known results on the existence of nontrivial Killing vector fields (i.e. nontrivial solutions of Killing equations) is provided. These results also suggest a method of constructing Killing vector fields, which consists basically of studying consequences of the so called integrability conditions. That last part requires usually quite involved symbolic computations and therefore can be aided by the appropriate computer programs.

The method is applied to a class of pseudo Riemannian structures that depends on two arbitrary holomorphic functions of one complex variable. Some constraints on these functions arise as a consequence of the existence of nontrivial Killing vector fields. The nature of these constraints and an explicit form of a Killing field are presented as the final result.

Resumen

Un grupo de isometrías de un parámetro de una variedad Riemanniana o más generalmente de una variedad seudo- Riemanniana es generado por un campo vectorial de Killing, el cual esta sujeto a las ecuaciones comúnmente llamadas de Killing. Estas últimas se convierten en un sistema sobredeterminado de ecuaciones diferenciales parciales de primer orden, las cuales, también son lineales y homogéneas. En general este sistema no es completamente integrable.

Se provee una presentación de los bien conocidos resultados de la existencia de campos vectoriales de Killing (esto es, las soluciones de las ecuaciones de Killing). Estos resultados sugieren un método de construcción de campos vectoriales de Killing, el cual básicamente consiste de estudiar las consecuencias de las así llamadas condiciones de integrabilidad. Esta última parte requiere computaciones simbólicas y por lo tanto se usa un programa de computación simbólica apropiado.

El método es aplicado a una clase de estructura seudo-Riemanniana que depende de dos funciones holomórficas arbitrarias de una variable compleja. Surgen algunas restricciones sobre estas funciones como consecuencia de la existencia de campos vectoriales de Killing. La naturaleza de las restricciones y una forma explicita de los campos de Killing son presentados como el resultado final.

Dedicate

A Faride y Rafael mis amados padres, quienes me han enseñado el valor del trabajo duro, la importancia de la persistencia y a quienes les debo lo que soy y seré, y a mis hermanos, ustedes han sido mi inspiración.

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To Cynthia Díaz Quiles "Who finds a friend finds a treasure"

Contents

Chapter 1

Introduction

"Differential Geometry has a long history as a field of Mathematics and yet its rigorous foundation in the realm of contemporary mathematics is relatively new. In fact, some "old" results still require to be reformulated in terms of Modern Mathematics"[1]. Given any mathematical structure a question arises about a group of its automorphisms. And so, for a Riemannian or a more general pseudo-Riemannian manifold one asks about a group of transformations[2] which preserve inner products of tangent vectors. Such a group is called a group of isometries of the underlying manifold. Riemannian manifolds are natural generalizations of Euclidean spaces and originally they emerge in studies of a geometry of two dimensional surfaces in a three dimensional Euclidean space. Pseudo-Riemannian manifolds are their further abstraction.

Since a discovery of The Special Relativity Theory and later on The General Relativity Theory, pseudo-Riemannian manifolds have become a basic ingredient of physical theories of space-times, in which the presence of a nontrivial gravitational field is attributed to a nonzero curvature of the underlying pseudo-Riemannian metric structure. The metric structure is in turn subject to the so

[1] [Kobayashi, 1963] p. v

[2] Let M be a $C^\infty-$manifold. A transformation of M is a diffeomorphism of M onto itself.

called Einstein equations. One of many activities in that area is devoted to the studies of special solutions of Einstein equations and in particular their geometric properties. One of the standard questions asked then is that about a group of isometries and their natural generalizations: groups of homothetic transformations and groups of conformal transformations.

In [Rózga, 2001] a class of metric structures is presented, whose properties are the subject of this work. They are a particular case of nonexplicit solutions of Einstein equations discussed in details in [Plebański, 1978]. They generalize a class of solutions presented in [Plebański et al., 1998]. A further discussion of that class is provided in [Plebański and Rózga, 2002]. A problem of solutions of Killing equations for such metric structures is treated in [Rózga, 2002]. The method that is outlined there can be employed also to study an analogous problem for the metric structures of [Rózga, 2001].

With these as bases, the purpose of this work is to find an explicit form of the so called "infinitesimal isometries" or "Killing vector fields" for the class of pseudo-Riemannian structures presented in [Rózga, 2001]. By studying and using existing methods associated with one-parameter (local) groups of isometries of pseudo-Riemannian manifolds we determine the conditions under which the generalized pseudo-Riemannian structures presented in [Rózga, 2001] admit nontrivial one-parameter (local) groups of isometries.

In Chapter 2 we give a brief presentation of basic concepts such as vector fields, tensor algebra, tensor fields, affine connection, parallelism, torsion and curvature fields, and also, pseudo-Riemannian structures and some important identities.

In Chapter 3 we present some relevant tensors such as Riemann, Ricci, Einstein and Weyl tensors. In Chapter 4 we present Killing structures, which includes one parameter groups of transformations, definitions of Killing vector fields and Killing Equations. We close this chapter with some illustrative examples.

Finally in Chapter 5 we present the metric which is the principal subject of this work, some geometric properties and its consequences. Also, we present the method that we use to find explicit expressions for Killing vector fields and for the functions which the metric and the Killing vector fields depend on. At the end of this chapter we can find the results and a brief discussion.

Chapter 2

Review of Literature

2.1 Introduction

We follow the basic definitions and notation used in [Kobayashi, 1963] and [Helgason, 1962]. Many of our definitions have two approaches. The local approach which helps us to view what happens in an open neighborhood, and the general one, used in the modern Differential Geometry.

During all our work we are using the *Einstein summation convention* of summing over repeated indices.

2.2 Basic Definitions

Let M be a topological space[1]. We assume that M satisfies the Hausdorff separation axiom[2], in this case, M is called a *Hausdorff space*.

An *open chart* on M is a pair (U, φ) where U is an open subset of M and φ

[1] [Munkres, 2000] p. 76

[2] The Hausdorff separation axiom states that any two different points in M can be separated by disjoint open sets

is an homeomorphism[3] of U onto an open subset of $\mathbf{R}^m$.

Let $\mathcal{O} \subset R^n$ and $\mathcal{O}' \subset R^m$ be open subsets of two Euclidean spaces R^n and R^m where n and m are the dimensions of the spaces. The mapping

$$\varphi : \mathcal{O} \to \mathcal{O}',$$

is called differentiable if the coordinates $y_j\left(\varphi\left(p\right)\right)$ of $\varphi\left(p\right)$ are (indefinitely) differentiable functions of coordinates $u^i\left(p\right)$, for $p \in \mathcal{O}$.

Definition 1 *Let M be a Hausdorff space. A* differentiable structure *on M of dimension m is a collection of open charts $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ on M where $\varphi_\alpha(U_\alpha)$ is an open subset of R^m such that the following conditions are satisfied:*

$M1.$ $M = \underset{\alpha\in A}{\cup} U_\alpha.$

$M2.$ For each pair $\alpha,\ \beta \in A$ the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a differentiable mapping of $\varphi_\alpha(U_\alpha \cap U_\beta)$ onto $\varphi_\beta(U_\alpha \cap U_\beta)$.

$M3.$ The collection $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ is a maximal family of open charts for which $M1$ and $M2$ hold.

A *differentiable manifold* (or C^∞ *manifold* or simply *manifold*) of dimension m is a Hausdorff space with a differentiable structure of dimension m. *If* M is a manifold, a *local chart* on M or *local coordinate system* on M is by definition a pair $(U_\alpha, \varphi_\alpha)$ where $\alpha \in A$. If $p \in U_\alpha$ and $\varphi_\alpha\left(p\right) = \left(u^1\left(p\right), \ldots, u^m\left(p\right)\right)$, the set U_α is called a *coordinate neighborhood* of p and the numbers $u^i(p)$ are called *local coordinates* of p. The mapping

$$\varphi_\alpha : q \mapsto \left(u^1\left(q\right), \ldots, u^m\left(q\right)\right), \qquad q \in U_\alpha,$$

[3]Let A and B be topological spaces; $f : A \to B$ is called a homeomorphism between A and B if and only if f is a continuous bijection with a continuous inverse. Also called a continuous transformation.

is often denoted by $\{u^1, \ldots, u^m\}$. Let f be a real-valued function on C^∞manifold M. The function f is called *differentiable at a point* $p \in M$ if there exists a local chart $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$ such that the composite function $f \circ \varphi_\alpha^{-1}$ is a differentiable function on $\varphi_\alpha(U)$. The function f is called *differentiable* if it is differentiable at each point $p \in M$. By a *differentiable curve* of class C^k in M, we shall mean a differentiable mapping of class C^k of a closed interval $[a, b]$ into M.

2.3 Vector Fields and 1-Forms

This section starts with the concept of derivation which is useful to define a vector field, and further a tangent vector. We also provide some important notation and local expressions.

Definition 2 *Let* $\mathbf{A}$ *be an algebra over a field* $\mathbf{K}$. *A* derivation *of* $\mathbf{A}$ *is a mapping* $B : \mathbf{A} \to \mathbf{A}$ *such that:*

- B is a linear mapping; that is, $B(\alpha f + \beta g) = \alpha Bf + \beta Bg$ for $\alpha, \beta \in \mathbf{K}$, $f, g \in \mathbf{A}$;
- $B(fg) = f(Bg) + (Bf)g$ for $f, g \in \mathbf{A}$

Definition 3 *A* vector field $\mathbf{X}$ *on a* C^∞ *manifold* M *is a derivation of the algebra* $\mathfrak{F}(M)$[4].

Let us denote the set of all vector fields on M by $\mathfrak{D}^1$(or $\mathfrak{D}^1(M)$).

Now, we shall define a tangent vector (or simply a *vector*) at a point p of M.

[4] $\mathfrak{F}(M) \equiv C^\infty(M)$ denotes the set of all differentiable functions on M. See [Helgason, 1962] p.5

Definition 4 *Let $\gamma(t)$ be a curve of class C^1, $a \leq t \leq b$, such that $\gamma(t_o) = p$. The vector tangent to the curve $\gamma(t)$ at p is a mapping*

$$X : \mathfrak{F}(p) \rightarrow \mathbb{R}$$

defined by

$$Xf = \left(\frac{df(\gamma(t))}{dt}\right)_{t=0}.$$

Given any curve $\gamma(t)$ with $p = \gamma(t_o)$, let $u^j = \gamma^j(t)$, $j = 1, ..., m$, be its equations in terms of the local coordinate system $\{u^1, ..., u^m\}$ then

$$\left(\frac{df(\gamma(t))}{dt}\right)_{t=0} = \sum_j \left(\frac{\partial f}{\partial u^j}\right)_p \left(\frac{d\gamma^j(t)}{dt}\right)_{t_0}.$$

In other words, Xf is the derivative of f in the direction of the curve $\gamma(t)$ at $t = t_{o.}$The vector X satisfies the following conditions:

1. X is a linear mapping of $\mathfrak{F}(p)$ into $\mathbb{R}$;

2. $X(fg) = f(p)X(g) + g(p)X(f)$, for all $f, g \in \mathfrak{F}(p)$.

Definition 5 *Now, let X_p denote the linear mapping*

$$\begin{aligned} X_p &: \mathfrak{F}(p) \rightarrow \mathbf{R}, \\ X_p &: f \mapsto (Xf)(p), \end{aligned}$$

for $p \in M$ and $X \in \mathfrak{D}^1$. Then the set

$$M_p = \left\{X_p : X \in \mathfrak{D}^1(M)\right\},$$

is called the tangent space to M *at* p, *and its elements are called* tangent vectors *to* M *at* p.

Definition 6 *A vector field X on a manifold M is an assignment of a vector X_p to each point p of M.*

In terms of a local coordinate system $\{u^1, \ldots, u^m\}$, a vector field $\mathbf{X}$ may be expressed by $\mathbf{X} = \xi^j \left(\frac{\partial}{\partial u^j}\right)$, where ξ^j are functions defined in the coordinate neighborhood, called the *components* of $\mathbf{X}$ with respect to $\{u^1, \ldots, u^m\}$.

Now, for vector fields we can define the *Lie bracket* $[\mathbf{X}, \mathbf{Y}]$ as the following derivation of $\mathfrak{F}(M)$; for $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{D}^1(M)$:

$$[\mathbf{X}, \mathbf{Y}]\, f = \mathbf{X}(\mathbf{Y}f) - \mathbf{Y}(\mathbf{X}f).$$

In local coordinate system $\{u^1, \ldots, u^m\}$, we write

$$\mathbf{X} = \xi^j \left(\frac{\partial}{\partial u^j}\right), \qquad \mathbf{Y} = \eta^j \left(\frac{\partial}{\partial u^j}\right).$$

Then,

$$[\mathbf{X}, \mathbf{Y}]\, f = \left[\xi^k \left(\frac{\partial \eta^j}{\partial u^k}\right) - \eta^k \left(\frac{\partial \xi^j}{\partial u^k}\right)\right] \left(\frac{\partial f}{\partial u^j}\right). \tag{2.1}$$

Therefore, $[\mathbf{X}, \mathbf{Y}]$ is a vector field whose components with respect to $\{u^1, \ldots, u^m\}$ are given by $\left[\xi^k \left(\frac{\partial \eta^j}{\partial u^k}\right) - \eta^k \left(\frac{\partial \xi^j}{\partial u^k}\right)\right]$, $j = 1, ..., m$.

As is customary we shall often write $\theta(\mathbf{X})\mathbf{Y} = [\mathbf{X}, \mathbf{Y}]$. The operator $\theta(\mathbf{X})$ is called the *Lie Derivate* with respect to $\mathbf{X}$.

In particular, we have the *Jacobi Identity*:

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0, \tag{2.2}$$

for $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{D}^1(M)$ or, otherwise written

$$\theta(\mathbf{X})\left([\mathbf{Y}, \mathbf{Z}]\right) = [\theta(\mathbf{X})\mathbf{Y}, \mathbf{Z}] + [\mathbf{Y}, \theta(\mathbf{X})\mathbf{Z}].$$

The tangent space M_p has its dual[5] vector space M_p^*, which is called the vector space of *covectors* at p.

[5] Let A be a commutative ring with identity element, E a module over A. Let E^*, called the *dual* of E, denote the set of all $A-$*linear* mappings of E into A.

Definition 7 *An assignment of a covector to each point $p \in M$ is called* 1-form *or* differential form of degree 1. *For each function $f \in \mathfrak{F}(M)$, the* total differential $(df)_p$ *of f at p is defined by*

$$\langle (df)_p, \mathbf{X} \rangle = \mathbf{X}\mathrm{f} \qquad\qquad for\ \mathbf{X} \in M_p,$$

where $\langle , \rangle$ denotes the value of the first entry on the second entry as a linear functional on M_p or the value of the second entry as a linear functional on M_p^*, (since $M_p^{**} = M_p$).

If $\{u^1, \ldots, u^m\}$ is a local coordinate system in a neighborhood of p, then the total differentials $(du^1)_p, \ldots, (du^m)_p$ form a basis for M_p^*. In fact, they form a dual basis of the basis $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^m)_p$ for M_p.

In a coordinate neighborhood of p, every 1-form ω can be uniquely written as

$$\omega = f_j du^j,$$

where f_j are certain functions defined in that neighborhood and called the components of ω with respect to $\{u^1, \ldots, u^m\}$. The 1-form ω is said to be differentiable if f_j are differentiable (this conditions is independent on the choice of a local coordinate system).

A 1-form can be defined also as an $\mathfrak{F}(M)-$linear mapping of the $\mathfrak{F}(M)-$module of the vector fields $\mathfrak{D}^1(M)$ into $\mathfrak{F}(M)$. The two definitions are related by

$$(\omega(\mathbf{X}))_p = \langle \omega_p, \mathbf{X}_p \rangle, \qquad \mathbf{X} \in \mathfrak{D}^1(M), \quad p \in M.$$

2.4 Tensor Fields

Now we shall extend the notions of vector fields and 1-forms to tensor fields with the help of the following notation. Let $\mathfrak{D}_s$ denote the $\mathfrak{F}$-module of all $\mathfrak{F}$-multilinear mappings of $\underbrace{\mathfrak{D}^1 \times \mathfrak{D}^1 \times \ldots \times \mathfrak{D}^1}_{s\ times}$ into $\mathfrak{F}$, where $s \geq 1$ is an integer. Similarly $\mathfrak{D}^r$ denotes the $\mathfrak{F}$-module of all $\mathfrak{F}$-multilinear mappings $\underbrace{\mathfrak{D}_1 \times \mathfrak{D}_1 \times \ldots \times \mathfrak{D}_1}_{r\ times}$ into $\mathfrak{F}$. More generally, let $\mathfrak{D}^r_s$ denote the $\mathfrak{F}$-module of all $\mathfrak{F}$-multilinear mappings of $\underbrace{\mathfrak{D}_1 \times \ldots \times \mathfrak{D}_1}_{r\ times} \times \underbrace{\mathfrak{D}^1 \times \ldots \times \mathfrak{D}^1}_{s\ times}$ into $\mathfrak{F}$. We often write $\mathfrak{D}^r_s(M)$ instead of $\mathfrak{D}^r_s$. We have $\mathfrak{D}^r_0 = \mathfrak{D}^r$, $\mathfrak{D}^0_s = \mathfrak{D}_s$, and we put $\mathfrak{D}^0_0 = \mathfrak{F}$.

A *tensor field* $\mathbf{T}$ on M of type (or degree) (r, s) is by definition an element of $\mathfrak{D}^r_s(M)$. It is said to be *contravariant* of degree r, and *covariant* of degree s. In particular, the tensor fields of type $(0, 0)$ are differentiable functions,a tensor field of type $(1, 0)$ is a vector field, and of type $(0, 1)$ is a 1-form.

Now for $p \in M$, we define $\mathfrak{D}^r_s(p)$, a set of all $\mathbb{R}$-multilinear mappings of

$$\underbrace{M_p^* \times M_p^* \times \ldots \times M_p^*}_{r\ times} \times \underbrace{M_p \times M_p \times \ldots \times M_p}_{s\ times},$$

into $\mathbb{R}$. $\mathfrak{D}^r_s(p)$ is a vector space over $\mathbb{R}$ and it is isomorphic with the tensor product (defined in Section (2.4.1))

$$\underbrace{M_p \otimes M_p \otimes \ldots \otimes M_p}_{r\ times} \otimes \underbrace{M_p^* \otimes M_p^* \otimes \ldots \otimes M_p^*}_{s\ times},$$

denoted shortly by $\mathfrak{D}^r_s(p) = \otimes^r M_p \otimes^s M_p^*$. We also put $\mathfrak{D}^0_0(p) = \mathbb{R}$.

Now given $T \in \mathfrak{D}^r_s(M)$ we can define for each $p \in M$ an element $\mathbf{T}_p \in \mathfrak{D}^r_s(p)$ according to

$$\mathbf{T}_p\left((\theta_1)_p, \ldots, (\theta_r)_p, (Z_1)_p, \ldots, (Z_s)_p\right) = \mathbf{T}(\theta_1, \ldots, \theta_r, Z_1, \ldots, Z_s)(p),$$

where $\theta_i \in \mathfrak{D}_1(M)$, $Z_j \in \mathfrak{D}^1(M)$, $i = 1, \ldots, r$, $j = 1, \ldots, s$.

The same equations permit us to identify $\mathfrak{D}^r_s(M)$ with the set of corresponding differentiable mappings, from M into $\cup_{p \in M} \mathfrak{D}^r_s(p)$, providing us therefore with alternative definition of a tensor field; as it was in case of vector fields and $r-$ forms.

2.4.1 Algebraic Operations on Tensor Fields

There are two important algebraic operations on tensor fields, which arise from the corresponding operations on tensors (at a point). They are the *tensor product* and *contractions*.

Tensor product

We define the *tensor product* of two vector spaces U and V, denoted by $U \otimes V$ and also called *the tensor direct product* as follows. Let $M(U,V)$ be the vector space which has the set $U \times V$ as a basis, i.e., the vector space generated by the pairs (u,v) where $u \in U$ and $v \in V$. Let N be the vector subspace of $M(U,V)$ generated by elements of the form

$$(u+u',v)-(u,v)-(u',v)\,, \quad (u,v+v')-(u,v)-(u,v')\,,$$
$$(ru,v)-r\,(u,v)\,, \quad (u,rv)-r\,(u,v)\,,$$

where $u,\ u' \in U,\ v,\ v' \in V$ and r any scalar . The definition is the same no matter which scalar field is used. We set $U \otimes V = M(U,V)/N$. For every pair (u,v) considered as an element of $M(U,V)$, its image by the natural projection $M(U,V) \rightarrow U \otimes V$ will be denoted by $u \otimes v$. Define the canonical bilinear mapping ϕ of $U \times V$ into $U \otimes V$ by:

$$\phi\,(u,v) = u \otimes v, \;\text{ for }\; (u,v) \in U \times V.$$

The following rules are satisfied,

$$\begin{aligned}
(u_1+u_2)\otimes v &= u_1 \otimes v + u_2 \otimes v,\\
u \otimes (v_1+v_2) &= u \otimes v_1 + u \otimes v_2,\\
r\,(u \otimes v) &= (ru)\otimes v = u \otimes (rv)\,.
\end{aligned}$$

One basic consequence is that

$$0 \otimes v = u \otimes 0 = 0.$$

A vector basis$\{u_i\}$ of U and $\{v_j\}$ of V gives a basis for $U \otimes V$, namely $\{u_i \otimes v_j\}$, for all pairs (i, j). An arbitrary element of $U \otimes V$ can be written uniquely as $\sum a_{i,j} u_i \otimes v_j$, where $a_{i,j}$ are scalars. If U is a n dimensional and V is a k dimensional vector space, then $U \otimes V$ has dimension nk.

The tensor product $\otimes$ of two tensor fields $S \in \mathfrak{D}^r_s$ and $T \in \mathfrak{D}^k_l$ is defined as $\mathfrak{F}-$bilinear mapping $(S, T) \to S \otimes T$ of $\mathfrak{D}^r_s \times \mathfrak{D}^k_l$ into $\mathfrak{D}^{r+k}_{s+l}$, which sends

$$(X_1 \otimes \cdots \otimes X_r \otimes X_1^* \otimes \cdots \otimes X_s^*, Z_1 \otimes \cdots \otimes Z_k \otimes Z_1^* \otimes \cdots \otimes Z_l^*) \in \mathfrak{D}^r_s \times \mathfrak{D}^k_l,$$

into

$$(X_1 \otimes \cdots \otimes X_r \otimes Z_1 \otimes \cdots \otimes Z_k \otimes X_1^* \otimes \cdots \otimes X_s^* \otimes Z_1^* \otimes \cdots \otimes Z_l^*) \in \mathfrak{D}^{r+k}_{s+l},$$

for X_i, $Z_j \in \mathfrak{D}^1$ and X_q^*, $Z_h^* \in \mathfrak{D}_1$. In terms of components, if S is given by $S^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}$ and T is given by $T^{m_1 \cdots m_k}{}_{n_1 \cdots n_l}$, then

$$(S \otimes T)^{i_1 \cdots i_{r+k}}{}_{j_1 \cdots j_{s+l}} = S^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} T^{i_{r+1} \cdots i_{r+k}}{}_{j_{s+1} \cdots j_{s+l}}.$$

We define the tensor algebra $\mathfrak{D} = \mathfrak{D}(M)$ over M to be the direct sum of the $\mathfrak{F}-$modules $\mathfrak{D}^r_s(M)$. $\mathfrak{D}(M) = \overset{\infty}{\underset{r,s=o}{\oplus}} \mathfrak{D}^r_s(M)$, so that an element of $\mathfrak{D}$ is of the form $\sum_{r,s=0}^{\infty} K^r_s$, where $K^r_s \in \mathfrak{D}^r_s(M)$ are zero except for a finite number of them. Then $\mathfrak{D}(M)$ is an algebra over $\mathfrak{F}$, the product $\otimes$ being defined pointwise, i.e., if K, $L \in \mathfrak{D}(M)$ then

$$(K \otimes L)_p = K_p \otimes L_p, \text{ for all } p \in M.$$

Similarly, if $p \in M$ we consider the direct sum $\mathfrak{D}(p) = \overset{\infty}{\underset{r,s=o}{\oplus}} \mathfrak{D}^r_s(p)$ which is an associative algebra over the ring $\mathbb{R}$. Therefore, it is a tensor algebra over $M_{p.}$ The submodules

$$\mathfrak{D}^* = \sum_{r=0}^{\infty} \mathfrak{D}^r, \text{ and } \mathfrak{D}_* = \sum_{s=0}^{\infty} \mathfrak{D}_s,$$

are subalgebras of $\mathfrak{D}$ and the subspaces

$$\mathfrak{D}^* (p) = \sum_{r=0}^{\infty} \mathfrak{D}^r (p)\,, \text{ and } \mathfrak{D}_* (p) = \sum_{s=0}^{\infty} \mathfrak{D}_s (p)\,,$$

are subalgebras of $\mathfrak{D} (p)$.

Contraction

Let $p \in M$, and $r,\ s,\ i,\ j$ be integers such that $r, s,\ i,\ j\ \geq 1,\ 1 \leq i \leq r$, and $1 \leq j \leq s$. Consider the $\mathbb{R}$-linear mapping $\mathbf{C}^i_j : \mathfrak{D}^r_s(p) \to \mathfrak{D}^{r-1}_{s-1}(p)$ defined by

$$\mathbf{C}^i_j(e_1 \otimes \ldots \otimes e_r \otimes f_1 \otimes \ldots \otimes f_s) = \langle e_i, f_j \rangle \,(e_1 \otimes \ldots \widehat{e_i} \ldots \otimes e_r \otimes f_1 \otimes \ldots \otimes \widehat{f_j} \otimes \ldots \otimes f_s),$$

where $e_1, \ldots, e_r \in M_p$, $\ f_1, \ldots f_s \in M^*_p$. (The symbol$\,\widehat{}\,$ over a letter means that the letter is missing). Now, there exists a unique $\mathfrak{F}$-linear mapping $\mathbf{C}^i_j : \mathfrak{D}^r_s(M) \to \mathfrak{D}^{r-1}_{s-1}(M)$ such that

$$(\mathbf{C}^i_j(T))_p = \mathbf{C}^i_j(T_p),$$

for all $T \in \mathfrak{D}^r_s(M)$ and all $p \in M$. This mapping satisfies the relation

$$\mathbf{C}^i_j(X_1 \otimes \ldots \otimes X_r \otimes \omega_1 \otimes \ldots \otimes \omega_s) = \langle X_i, \omega_j \rangle \,(X_1 \otimes \ldots \widehat{X_i} \ldots \otimes X_r \otimes \omega_1 \otimes \ldots \widehat{\omega}_j \otimes \ldots \otimes \omega_s),$$

for all $X_1, \ldots, X_r \in \mathfrak{D}^1,\ \omega_1, \ldots, \omega_s \in \mathfrak{D}_1$. The mapping $\mathbf{C}^i_j$ is called the *contraction* of *i-th* contravariant index and the *j-th* covariant index.

2.5 The Grassmann Algebra

Let $\wedge^s \mathfrak{D}_1$ (or $\wedge^s \mathfrak{D}_1 (M)$) be the set of alternate $\mathfrak{F}-$multilinear mappings of $\underbrace{\mathfrak{D}^1 \times \mathfrak{D}^1 \times \ldots \times \mathfrak{D}^1}_{s\ times}$ into $\mathfrak{F}$, where s is an integer ≥ 1 and M denotes a C^∞ manifold. We put $\wedge^0 \mathfrak{D}_1 = \mathfrak{F}$, $\wedge^1 \mathfrak{D}_1 = \mathfrak{D}_1$ and let $\wedge \mathfrak{D}_1$ denote the direct sum $\wedge \mathfrak{D}_1 = \sum_{s=0}^{\infty} \wedge^s \mathfrak{D}_1$ of the $\mathfrak{F}$-modules $\wedge^s \mathfrak{D}_1$. The elements of $\wedge \mathfrak{D}_1$are called *exterior differential forms* on M. The elements of $\wedge^s \mathfrak{D}_1$ are called *differential* $s - forms$, or just $s - forms$.

Now we will define a mapping called *Alternation,* denoted by A. Let $\mathfrak{G}_s$ denote the group of permutations of the set $\{1, \ldots, s\}$. Each $\sigma \in \mathfrak{G}_s$ induces an $\mathfrak{F}-linear$ mapping of $\mathfrak{D}^1 \times \mathfrak{D}^1 \times \ldots \times \mathfrak{D}^1$ onto itself, given by

$$\sigma : (X_1, \ldots, X_s) \mapsto \left(X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(s)}\right), \qquad \left(X_i \in \mathfrak{D}^1\right).$$

Thus, for each $\Omega \in \mathfrak{D}_s$, the mapping $\Omega \circ \sigma^{-1}$ is well defined and the mapping

$$\sigma \cdot \Omega : \Omega \mapsto \Omega \circ \sigma^{-1}$$

is a one-to-one $\mathfrak{F}-linear$ mapping of $\mathfrak{D}_s$ onto itself. Let $\epsilon(\sigma) = 1$ or -1 depending if σ is an even or an odd permutation. Consider the linear transformation

$$\begin{aligned} A_s &: \mathfrak{D}_s \to \mathfrak{D}_s \\ A_s(\Omega_s) &= \frac{1}{s!} \sum_{\sigma \in \mathfrak{G}_s} \epsilon(\sigma)\, \sigma \cdot \Omega_s, \qquad \Omega_s \in \mathfrak{D}_s. \end{aligned}$$

If $s = 0$, we put $A_s(\Omega_s) = \Omega_s$. We extend A_s to an $\mathfrak{F}-linear$ mapping

$$A : \mathfrak{D}_* \to \mathfrak{D}_*$$

by

$$A(\Omega) = \sum_{s=0}^{\infty} A_s(\Omega_s)$$

where $\Omega = \sum_{s=0}^{\infty} \Omega_s$, $\Omega_s \in \mathfrak{D}_s$. For any θ, $\omega \in \wedge \mathfrak{D}_1$ we can now define the *exterior product*

$$\theta \wedge \omega = A(\theta \otimes \omega).$$

this turn $\wedge \mathfrak{D}^1$ into an associative algebra. The module $\wedge \mathfrak{D}^1$ of alternate $\mathfrak{F} -$ *multilineal* functions with the exterior product is called the *Grassmann algebra of the manifold M*.

For each $p \in M$ we can also define the Grassmann algebra $\wedge\mathfrak{D}_1(p)$ of the tangent space M_p. The elements of $\wedge\mathfrak{D}_1(p)$ are the alternate, $\mathbb{R}-$*multilinear*, real-valued functions on M_p and the product (also denoted by $\wedge$) satisfies

$$\theta_p \wedge \omega_p = (\theta \wedge \omega)_p\,, \qquad \theta,\ \omega\ \in \wedge\mathfrak{D}^1.$$

This turns $\wedge\mathfrak{D}_1(p)$ in an associative algebra containing the dual M_p^*. If $\theta, \omega \in M_p^*$, then

$$\theta \wedge \omega = -\omega \wedge \theta. \tag{2.3}$$

If $\theta^1, \ldots, \theta^l \in M_p^*$ and $\omega^i = \sum_{j=1}^{l} a_{ij}\theta^j$, where $a_{ij} \in \mathbb{R}$, then, as a consequence of (2.3)

$$\omega^1 \wedge \ldots \wedge \omega^l = \det(a_{ij})\, \theta^1 \wedge \ldots \wedge \theta^l.$$

Now we write some important relationships. Let $f,\ g \in \mathfrak{F}(M)\,,\ \theta \in \wedge^r\mathfrak{D}_1$, $\omega \in \wedge^s\mathfrak{D}_1$, $X_i \in \mathfrak{D}^1$. Then,

$$f \wedge g = fg,$$

$$(f \wedge \theta)(X_1, ..., X_r) = f\theta(X_1, ..., X_r)\,,$$

$$(\omega \wedge g)(X_1, ..., X_s) = g\omega(X_1, ..., X_s)\,,$$

$$(\theta \wedge \omega)(X_1, ..., X_{r+s}) =$$

$$\frac{1}{(r+s)!}\sum_{\sigma \in \mathfrak{G}_{r+s}} \epsilon(\sigma)\, \theta\left(X_{\sigma(1)}, ..., X_{\sigma(r)}\right) \omega\left(X_{\sigma(r+1)}, ..., X_{\sigma(r+s)}\right),$$

$$\theta \wedge \omega = (-1)^{rs}\, \omega \wedge \theta.$$

2.5.1 Exterior Differentiation

Let $\wedge\mathfrak{D}(M)$ be the Grassmann algebra over M. The *Exterior differentiation* d can be characterized[6] as follows:

1. d is a unique $\mathbb{R}-$linear mapping of $\wedge\mathfrak{D}(M)$ into itself such that $d(\wedge^r\mathfrak{D}) \subset \wedge^{r+1}\mathfrak{D}$;

[6] [Helgason, 1962] p.20

2. For any function $f \in \wedge^0 \mathfrak{D}^1$, df is the total differential, i.e., the 1-form given by $df\,(X) = Xf$

3. If $\omega \in \wedge^r \mathfrak{D}^1$ and $\lambda \in \wedge^s \mathfrak{D}^1$, then

$$d\,(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^r\, \omega \wedge d\lambda;$$

4. $d \circ d = 0$.

In terms of a local coordinate system, if

$$\omega = \sum_{i_1 < \cdots < i_r} f_{i_1 \cdots i_r} du^{i_1} \wedge \cdots \wedge du^{i_{1r}},$$

then

$$d\omega = \sum_{i_1 < \cdots < i_r} df_{i_1 \cdots i_r} \wedge du^{i_1} \wedge \cdots \wedge du^{i_{1r}}.$$

Now, we introduce an important concept in our work, the affine connection.

2.6 Affine Connection

Definition 8 *An* affine connection *on a manifold M is a rule $\boldsymbol{\nabla}$ which assigns to each $X \in \mathfrak{D}^1(M)$ a linear mapping $\boldsymbol{\nabla}_X$ of vector space $\mathfrak{D}^1(M)$ into itself satisfying the following conditions:*

(∇_1) $\boldsymbol{\nabla}_X(Y+Z) = \boldsymbol{\nabla}_X Y + \boldsymbol{\nabla}_X Z;$

(∇_2) $\boldsymbol{\nabla}_{fX+gY} Z = f \cdot \boldsymbol{\nabla}_X Z + g \cdot \boldsymbol{\nabla}_Y Z;$

(∇_3) $\boldsymbol{\nabla}_X(fY) = f \cdot \boldsymbol{\nabla}_X Y + (Xf)Y,$
all $f,\ g \in \mathfrak{F}(M),\ X,\ Y,\ Z \in \mathfrak{D}^1(M).$

The operator $\boldsymbol{\nabla}_X$ is called *Covariant Differentiation* with respect to X. $\boldsymbol{\nabla}_X$ can be extended to tensor fields.

Theorem 1 *There exists a unique extension of* $\boldsymbol{\nabla}_X : \mathfrak{D}^1 \to \mathfrak{D}^1$*, to the tensor algebra* $\mathfrak{D}(M)$*, denoted by the same symbol* $\boldsymbol{\nabla}_X$*, which possesses the following properties:*

(i) $\boldsymbol{\nabla}_X$ is a derivation of the tensor algebra $\mathfrak{D}(M)$.

(ii) $\boldsymbol{\nabla}_X$ preserves types of tensors.

(iii) $\boldsymbol{\nabla}_X$ commutes with contractions.

Example 1 *Let* $g \in \mathfrak{D}_2,\ X,\ Y,\ Z \in \mathfrak{D}^1$*, then by(iii) and (i)*

$$\begin{aligned}
\boldsymbol{\nabla}_X g(Y,Z) &= \boldsymbol{\nabla}_X C_1^1 C_2^2 \left[(g \otimes Y \otimes Z)\right] \\
&= C_1^1 C_2^2 \left[\boldsymbol{\nabla}_X \left(g \otimes Y \otimes Z\right)\right] \\
&= C_1^1 C_2^2 \left[(\boldsymbol{\nabla}_X g) \otimes Y \otimes Z + g \otimes \boldsymbol{\nabla}_X Y \otimes Z + g \otimes Y \otimes \boldsymbol{\nabla}_X Z\right],
\end{aligned}$$

Since,$\boldsymbol{\nabla}_X g(Y,Z) = Xg\,(Y,Z)$. *Thus,*

$$Xg\,(Y,Z) = (\boldsymbol{\nabla}_X g)(Y,Z) + g(\boldsymbol{\nabla}_X Y, Z) + g(Y, \boldsymbol{\nabla}_X Z), \tag{2.4}$$

where C_1^1, C_2^2 *are the respective contractions.*

As a simple application of Theorem 1 we obtain a relation between$\boldsymbol{\nabla}_X\ \omega$ and $\boldsymbol{\nabla}_X\ Y$,where X, $Y \in \mathfrak{D}^1$ and $\omega \in \mathfrak{D}_1$. Indeed, by (i)

$$\boldsymbol{\nabla}_X \left(Y \otimes \omega\right) = (\boldsymbol{\nabla}_X Y) \otimes \omega + Y \otimes \boldsymbol{\nabla}_X \omega,$$

so (iii) implies

$$X\omega\left(Y\right) = \omega\left(\boldsymbol{\nabla}_X Y\right) + \left(\boldsymbol{\nabla}_X \omega\right)\left(Y\right). \tag{2.5}$$

Next, we discuss local representations of covariant derivatives of tensor fields.

Let $(U, \{u^1, \ldots, u^m\}))$ be a local coordinate system on M, and let $\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^m}$ be the *natural basis* of M_p, and $du^1, \ldots, du^m$ its dual basis with respect to these coordinates, where p is a point of U. Then the connection coefficients Γ^k_{ij} with respect to $\{u^1, \ldots, u^m\}$ are defined by

$$\boldsymbol{\nabla}_{\partial/\partial u^i}\left(\frac{\partial}{\partial u^j}\right) = \Gamma^k_{ij}\frac{\partial}{\partial u^k}. \tag{2.6}$$

Consequently, by (2.5)

$$\boldsymbol{\nabla}_{\partial/\partial u^i}\left(du^j\right) = -\Gamma^k_{ij}du^k.$$

Let $X \in \mathfrak{D}^1$ and $\omega \in \mathfrak{D}_1$. We can write

$$X = \xi^i\frac{\partial}{\partial u^i} \qquad \text{and} \quad \omega = \alpha_i du^i \qquad\qquad \text{for } 1 \leq i \leq m.$$

Denoting $\boldsymbol{\nabla}_{\partial/\partial u^i}$ by $\boldsymbol{\nabla}_i$, we obtain,

$$\boldsymbol{\nabla}_j X = \boldsymbol{\nabla}_j\left(\xi^i\frac{\partial}{\partial u^i}\right) = \xi^k\Gamma^i_{jk}\frac{\partial}{\partial u^i} + \frac{\partial \xi^i}{\partial u^j}\frac{\partial}{\partial u^i}.$$

A semicolon is used to denote covariant differentiation with respect to a natural vector basis. Thus, the components of $\boldsymbol{\nabla}_j X$, denoted by $\xi^i_{;j}$, are of the form

$$\xi^i_{;j} = \frac{\partial \xi^i}{\partial u^j} + \xi^k\Gamma^i_{jk}, \tag{2.7}$$

and the covariant derivate of a vector field $X = \xi^i\frac{\partial}{\partial u^i}$ in direction of a vector field $Y = \eta^j\frac{\partial}{\partial u^j}$ at p is given by

$$\boldsymbol{\nabla}_Y X = \left(\eta^j\xi^i_{;j}\right)\left(\frac{\partial}{\partial u^i}\right).$$

Similarly,

$$\boldsymbol{\nabla}_j\omega = \boldsymbol{\nabla}_j\left(\alpha_i du^i\right) = \frac{\partial \alpha_i}{\partial u^j}du^i - \alpha_k\Gamma^k_{ji}du^i,$$

thus, the components of $\boldsymbol{\nabla}_j\omega$, denoted by $\alpha_{i;j}$, are

$$\alpha_{i;j} = \frac{\partial \alpha_i}{\partial u^j} - \alpha_k\Gamma^k_{ji}. \tag{2.8}$$

The extension of the above argument to tensor field T of type (r, s) is straightforward. The covariant derivate of the tensor field whose local components are

$T^{i_1 \dots i_r}_{j_1 \dots j_s}$ has the components denoted by $T^{i_1 \dots i_r}{}_{j_1 \dots j_s ; k}$ and they are

$$T^{i_1 \dots i_r}{}_{j_1 \dots j_s ; k} = \frac{\partial T^{i_1 \dots i_r}{}_{j_1 \dots j_s}}{\partial u^k} + \sum_{h=1}^{r} \Gamma^{i_h}_{kl} T^{i_1 \dots i_{h-1} l i_{h+1} \dots i_r}_{j_1 \dots j_s} - \sum_{g=1}^{s} \Gamma^{l}_{k j_g} T^{i_1 \dots i_r}{}_{j_1 \dots j_{g-1} l j_{g+1} \dots j_s} . \tag{2.9}$$

Consequently for all $T \in \mathfrak{D}^r_s$,

$$(\boldsymbol{\nabla}_Y T)_p = \left(\eta^k T^{i_1 \dots i_r}{}_{j_1 \dots j_s ; k}\right)_p \cdot \left(\frac{\partial}{\partial u^{i_1}}\right)_p \otimes \dots \otimes \left(\frac{\partial}{\partial u^{i_r}}\right)_p \otimes \left(du^{j_1}\right)_p \otimes \dots \otimes \left(du^{j_s}\right)_p .$$

For any vector field X, the tensor field $\boldsymbol{\nabla}_X T$ is also of degree (r, s). We have thus a rule which associates a tensor field of degree (r, s) to each vector field −i.e., a tensor field of degree $(r, s+1)$. We denote this tensor field by ∇T.

Indeed, let us assume that $T \in \mathfrak{D}^r_s(M)$. Then we define $\boldsymbol{\nabla} T \in \mathfrak{D}^r_{s+1}(M)$ according to,

$$(\boldsymbol{\nabla} T)(\omega_1, \cdots, \omega_r, , X, Y_1, \dots, Y_s) = (\boldsymbol{\nabla}_X T)(\omega_1, \cdots, \omega_r, Y_1, \dots, Y_s), \tag{2.10}$$

for all $X, Y_1, \dots, Y_s \in \mathfrak{D}^1(M)$ and $\omega_1, \cdots, \omega_r \in \mathfrak{D}_1(M)$, where

$$\begin{aligned} (\boldsymbol{\nabla}_X T)(\omega_1, \cdots, \omega_r, Y_1, \dots, Y_s) &= X T(\omega_1, \cdots, \omega_r, Y_1, \dots, Y_s) \\ - \sum_{i=1}^{r} T(\omega_1, \cdots, \boldsymbol{\nabla}_X \omega_i, \cdots, Y_s) &+ \sum_{j=1}^{s} T(\omega_1, \cdots \boldsymbol{\nabla}_X Y_j, \cdots, Y_s) \end{aligned} \tag{2.11}$$

2.7 Parallelism

Soon we will see that the concept of affine connection is intimately tied to the geometric concept of parallelism. Let M be a $\mathbf{C}^\infty$ manifold. A *curve* in M is a regular[7] mapping of an open interval $I \subset \mathbf{R}$ into M. The restriction of a curve to a closed subinterval is called a *curve segment.* The curve segment is called *finite*

[7] Let M, N be differentiable manifolds. A mapping $\phi : M \to N$ is called regular at $p \in M$ if ϕ is differentiable at p and ϕ^*_p is one-to-one mapping of M_p into $N_{\phi(p)}$. ϕ^* is defined in Section 4.1

if the interval is finite. Let $\gamma : t \to \gamma(t)$ $(t \in I)$ be a curve in M. Differentiation with respect to the parameter will often be denoted by a dot $(\dot{})$.

If the vector fields X and Y have local representations

$$X = X^i \frac{\partial}{\partial u^i} \text{ and } Y = Y^j \frac{\partial}{\partial u^j} \qquad \text{on } U,$$

then $\boldsymbol{\nabla}_X Y$ has a local representation

$$\boldsymbol{\nabla}_X Y = \sum_k \left(\sum_i X^i \frac{\partial Y^k}{\partial u^i} + \sum_{i,j} \Gamma^k_{ij} X^i Y^j \right) \frac{\partial}{\partial u^k} \qquad \text{on } U. \tag{2.12}$$

The vector field $\boldsymbol{\nabla}_X Y$ is said to be the *covariant derivative* of Y with respect to X. If $X = \frac{\partial}{\partial u^i}$, then the components of $\boldsymbol{\nabla}_X(Y) = \boldsymbol{\nabla}_i(Y)$ are denoted by $Y^k_{;i}$ and

$$Y^k_{;i} = \frac{\partial Y^k}{\partial u^i} + \Gamma^k_{ij} Y^j.$$

Suppose now that to each $t \in I$ it is associated a vector $Y(t) \in M_{\gamma(t)}$. Assume $Y(t)$ vary differentiably with t. Let $X(t) = \dot{\gamma}(t)$ $(t \in I)$. Then there exist vector fields X, $Y \in \mathfrak{D}^1$ such that

$$X_{\gamma(t)} = X(t), \text{ and } \quad Y_{\gamma(t)} = Y(t) \quad (t \in J).$$ [8]

The family of tangent vectors $Y(t) = Y^i(t) \frac{\partial}{\partial u^i}$, $(t \in J)$ is said to be *parallel* with respect to γ_J (or parallel along γ_J) if

$$(\boldsymbol{\nabla}_X Y)_{\gamma(t)} = 0, \tag{2.13}$$

for all $t \in J$. To show that this definition is independent of the choice of X and $Y \in \mathfrak{D}^1$, we express (2.13) in coordinates $\{u^1, \ldots, u^m\}$. For simplicity we put $u^i(t) = u^i(\gamma(t))$, $X^i(t) = X^i(\gamma(t))$, $Y^i(t) = Y^i(\gamma(t))$, $(t \in J)$, $(1 \le i \le m)$.

[8] J is a compact subinterval of I such that the finite curve segment $\gamma_J : t \to \gamma(t)$ $(t \in J)$ has no double points and such that $\gamma(J)$ is contained in a coordinate neigborhood U.

Then $X^i(t) = \dot{u}^i(t)$; X and Y being as before and using the chain rule in (2.12) we obtain

$$\frac{dY^k}{dt} + \Gamma^k_{ij}\frac{du^i}{dt}Y^j = 0 \qquad (t \in J)\ . \tag{2.14}$$

This equation involves X and Y only through their values *on* the curve. Then the condition for parallelism is independent of the choice of X and Y.

Definition 9 *Let $\gamma : t \to \gamma(t)$ $(t \in I)$ be a curve in M. The curve γ is called* geodesic *if the family of tangent vectors $\dot{\gamma}(t)$ is parallel with respect to γ. In other words, γ is geodesic if*

$$\boldsymbol{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0. \qquad \text{(geodesic equation)}$$

A geodesic γ is called maximal if it is not a proper restriction of any geodesic.

The geodesic equation may be expressed as a system of m second order ordinary differential equations. Suppose γ_J is a finite geodesic segment contained in a coordinate neighborhood U where the coordinates $\{u^1, \ldots, u^m\}$ are valid. Then the equation 2.14 implies

$$\frac{d^2u^k}{dt^2} + \Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt} = 0 \qquad (t \in J)\,. \qquad \text{(geodesic equation in coordinates)}$$

Finally, a vector field X is called *geodesic* if

$$\boldsymbol{\nabla}_X X = 0.$$

2.8 Torsion and Curvature Fields

Let M be a manifold with an affine connection $\boldsymbol{\nabla}$. The *torsion tensor* T of $\boldsymbol{\nabla}$ is the $\mathfrak{F}$-bilinear function T: $\mathfrak{D}^1 \times \mathfrak{D}^1 \to \mathfrak{D}^1$ given by

$$T(X, Y) = \boldsymbol{\nabla}_X Y - \boldsymbol{\nabla}_Y X - [X, Y], \qquad \text{(torsion tensor)}$$

for all $X, Y \in \mathfrak{D}^1(M)$. In particular,

$$T(fX, Y) = T(X, fY) = fT(X, Y),$$

for $f \in \mathfrak{F}(M)$. Because of that the value $T(X,Y)|_p$ depends only on the values of X_p and Y_p. Consequently, T determines a bilinear mapping $M_p \times M_p \to M_p$ at each point $p \in M$. Using the skew symmetry ($[X,Y] = -[Y,X]$) of the Lie bracket for vector fields it is easy to see that $T(X,Y) = -T(Y,X)$. Hence T is skew symmetric. Let $\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^m}$ be the natural basis with respect the local coordinates $\{u^1, \dots, u^m\}$. Since $\left[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right] = 0$ for all $1 \le i, j \le m$, it follows that

$$T\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \left(\Gamma^k_{ij} - \Gamma^k_{ji}\right)\frac{\partial}{\partial u^k}.$$

Clearly, the torsion tensor provides a measure of the nonsymmetry of the connection coefficients. Hence, $T = 0$ if and only if these coefficients are symmetric in their subscripts. An affine connection $\boldsymbol{\nabla}$ with $T = 0$ is said to be *torsion free* or *symmetric.*

The curvature R of $\boldsymbol{\nabla}$ is a function which assigns to each pair $X,\ Y \in \mathfrak{D}^1$ the $\mathfrak{F}$-linear mapping $R(X,Y) : \mathfrak{D}^1 \to \mathfrak{D}^1$ given by

$$R(X,Y)Z = \boldsymbol{\nabla}_X\boldsymbol{\nabla}_Y Z - \boldsymbol{\nabla}_Y\boldsymbol{\nabla}_X Z - \boldsymbol{\nabla}_{[X,Y]}Z, \qquad \text{(curvature)}$$

for all $Z \in \mathfrak{D}^1(M)$. The curvature R provides a measure of the noncommutativity of $\boldsymbol{\nabla}_X$ and $\boldsymbol{\nabla}_Y$. The torsion and the curvature represent tensor fields. Note that R have the following properties: $R(X,Y)Z = -R(Y,X)Z$ and $R(fX,\ gY)hZ = fghR(X,\ Y)Z$ for all $f,\ g,\ h \in \mathfrak{F}(M),\ X,\ Y,\ Z \in \mathfrak{D}^1(M)$.

Indeed, the *torsion tensor field* is defined as an $\mathfrak{F}$-multilinear mapping

$$T : \mathfrak{D}_1 \times \mathfrak{D}^1 \times \mathfrak{D}^1 \to \mathfrak{F}(M),$$

by,

$$T(\omega, X, Y) = \omega(T(X,\ Y)),$$

for all X, $Y \in \mathfrak{D}^1$ and $\omega \in \mathfrak{D}_1$,which is an element of $\mathfrak{D}^1_2$.

T is given in local coordinates by

$$T = T^k_{\ ij} \frac{\partial}{\partial u^k} \otimes du^i \otimes du^j,$$

where

$$T^k_{\ ij} = \Gamma^k_{ij} - \Gamma^k_{ji}. \qquad \text{(torsion components)}$$

Similarly, the *curvature tensor field* is defined as an $\mathfrak{F}$-multilinear mapping

$$\begin{aligned} R &: \mathfrak{D}_1 \times \mathfrak{D}^1 \times \mathfrak{D}^1 \times \mathfrak{D}^1 \ \rightarrow \mathfrak{F}(M) \\ R &: (\omega,\ Z,\ X,\ Y) \mapsto \omega(R(X,\ Y)Z), \end{aligned}$$

according to

$$R(\omega, Z, X, Y) = \omega\left(R(X, Y)Z\right), \tag{2.15}$$

for all X, Y, $Z \in \mathfrak{D}^1$ and $\omega \in \mathfrak{D}_1$. Therefore R is an element of $\mathfrak{D}^1_3$.

In local coordinates R is given by

$$R = R^i_{jkl} \frac{\partial}{\partial u^i} \otimes du^j \otimes du^k \otimes du^l,$$

where the *curvature components* $R^i_{\ jkl}$ are of the form,

$$R^i_{\ jkl} = \frac{\partial \Gamma^i_{lj}}{\partial u^k} - \frac{\partial \Gamma^i_{kj}}{\partial u^l} + \left(\Gamma^a_{lj}\Gamma^i_{\ ka} - \Gamma^a_{kj}\Gamma^i_{\ ma}\right). \tag{2.16}$$

Thus, the tensor fields T and R are of type $(1, 2)$ and $(1, 3)$, respectively.

Notice that $R^i_{\ jkl} = -R^i_{\ jlk}$. Furthermore, if $X = X^i \frac{\partial}{\partial u^i}$, $Y = Y^i \frac{\partial}{\partial u^i}$, $Z = Z^i \frac{\partial}{\partial u^i}$ and $\omega = \omega_i du^i$, then

$$R(X, Y)Z = R^i_{\ jkl} Z^j X^k Y^l \frac{\partial}{\partial u^i},$$

and,

$$R(\omega, Z, X, Y) = \omega\left(R(X, Y)Z\right) = R^i_{\ jkl} \omega_i Z^j X^k Y^l.$$

Consequently, one has $R\left(du^i, \frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^l}\right) = R^i_{jkl}$.

Let $p \in M$ and suppose $X_1, \ldots, X_m$ is a basis for the vector fields in some neighborhood N_p of p, that is, each vector field X on N_p can be written as $X = \sum_i f_i X_i$ where $f_i \in \mathfrak{F}(N_p)$. We define the functions Γ^k_{ij}, $T^k{}_{ij}$, $R^k{}_{lij}$ on N_p by the formulas

$$\begin{aligned}
\boldsymbol{\nabla}_{X_i} X_j &= \Gamma^k_{ij} X_k. \\
T(X_i, X_j) &= T^k{}_{ij} X_k. \\
R(X_i, X_j) X_l &= R^k{}_{lij} X_k.
\end{aligned}$$

2.9 The Pseudo-Metric Structure and the Riemannian Connection

A *pseudo–Riemannian metric* g for a manifold M is a smooth symmetric tensor field of type $(0,2)$ on M which assigns to each point $p \in M$ a nondegenerate[9] inner product $g_p : M_p \times M_p \to \mathbb{R}$ of signature $(-, \ldots, -, +, \ldots, +)$. If the components of g in local coordinates are g_{ij}, then the nondegeneracy assumption is equivalent to the condition that the determinant of the matrix (g_{ij}) is nonzero.

Definition 10 *Let M be a C^∞-manifold. A pseudo-Riemannian structure on M is a tensor field g (or pseudo–Riemannian metric g) of type (0,2) which satisfies:*

(a) $g(X, Y) = g(Y, X)$ (symmetric)

(b) For each $p \in M$, g_p is a nondegenerate bilinear form on $M_p \times M_p$.

[9] Here *nondegenerate* means that for each nontrivial vector $v \in M_p$ there is some $w \in M_p$ such that $g_p(v, w) \neq 0$.

A *pseudo Riemannian manifold* is a connected C^∞-manifold with a pseudo-Riemannian structure. If g_p is positive definite for each $p \in M$, we speak of a Riemannian structure and Riemannian manifold.

If the matrix (g_{ij}) has s negative eigenvalues and $r = n - s$ positive eigenvalues, then the signature of g will be denoted (r, s). For each fixed $p \in M$, there exist local coordinates $(U, \{u^1, \ldots, u^m\})$ such that $g_p = g|_{M_p}$can be represented as the diagonal matrix $diag\{-1, \ldots, -1, 1, \ldots, 1\}$. For each pseudo-Riemannian manifold (M, g) there is an associated pseudo-Riemannian manifold $(M, -g)$ obtained by replacing g with $-g$. Aside from some minus change of sign, there is no essential difference between (M, g) and $(M, -g)$. Thus, results for spaces of signature (s, r) may always be translated into corresponding results for spaces of signature (r, s) by appropriate sign changes and inequality reversals.

Theorem 2 *On a pseudo-Riemannian manifold there exists one and only one affine connection satisfying the following two conditions:*

(i) The torsion tensor T is 0, i.e., $\boldsymbol{\nabla}_X Y - \boldsymbol{\nabla}_Y X = [X, Y]$ for all $X,\ Y \in \mathfrak{D}^1$;

(ii) The parallel displacement preserves the inner product on the tangent spaces, i.e.,$\boldsymbol{\nabla}_X g = 0,\ X \in \mathfrak{D}^1$.

A proof of this theorem can be found in [Helgason, 1962] p.48

From (2.4), the local representation form of (i) and (ii) is

$$\boldsymbol{\nabla}_i g_{jk} = 0 = \frac{\partial g_{jk}}{\partial u^i} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl},$$

so that,

$$\frac{\partial g_{jk}}{\partial u^i} = \Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl.}$$

Now due to (i), the symmetry of the connection coefficients, we have:

$$\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} = 2\Gamma^l_{ij} g_{lk}.$$

So *the connection coefficients* are,

$$\Gamma^{l}_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{kj}}{\partial u^{i}} + \frac{\partial g_{ik}}{\partial u^{j}} - \frac{\partial g_{ij}}{\partial u^{k}} \right), \tag{2.17}$$

where g^{ij} are defined by

$$g^{ia} g_{aj} = \delta^{i}_{j} \qquad \text{For } 1 \leq \text{i, j} \leq \text{m},$$

and represent local components of the $(2,0)$ tensor called the inverse pseudo-metric tensor and denoted further by g^{-1}.

2.10 Some Important Identities

2.10.1 Bianchi Identities

The curvature R and the torsion T satisfy the so called Bianchi identities. The general form of the *first Bianchi identity* is

$$\begin{aligned} R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = \\ T(T(X,Y),Z) + T(T(Y,Z),X) + T(T(Z,X),Y) \\ + (\nabla_X T)(Y,Z) + (\nabla_Y T)(Z,X) + (\nabla_Z T)(X,Y), \end{aligned} \tag{2.18}$$

Similarly, the *second Bianchi identity*

$$\begin{aligned} (\nabla_Z R)(X,Y) + (\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + R(T(X,Y),Z) \\ + R(T(Y,Z),X) + R(T(Z,X),Y) = 0 \end{aligned} \tag{2.19}$$

A particular case occurs when the affine connection is torsion free. Then the first and second Bianchi identities become simpler,

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{2.20}$$

and

$$(\nabla_Z R)(X,Y) + (\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) = 0 \tag{2.21}$$

respectively.

In local coordinates,

$$R^l{}_{ikj} + R^l{}_{jik} + R^l{}_{kji} = 0 \qquad \text{(first Bianchi identity)}$$

$$R^l{}_{kji;h} + R^l{}_{khj;i} + R^l{}_{kih;j} = 0, \qquad \text{(second Bianchi identity)}$$

since $\Gamma^l_{ji} = \Gamma^l_{ij}$. A demonstration of those identities can be found in [Okubo, 1987] p.130

2.10.2 Ricci Identity

The second mixed covariant derivatives of tensor fields do not commute. In fact, let $\omega = \omega_i du^i$ be an arbitrary 1-form, a computation shows that

$$\omega_{k;ij} - \omega_{k;ji} = R^l{}_{kij}\omega_l - \omega_{k;l}T^l{}_{ij}. \tag{2.22}$$

Equation (2.22) is referred to as *Ricci Identity.*

In the case where the affine connection is torsion free the Ricci Identity become as follows,

$$\omega_{k;ij} - \omega_{k;ji} = R^l{}_{kij}\omega_l. \tag{2.23}$$

We arrive at equation (2.23) from the commutativity of partial derivatives, the symmetry of connection coefficients, the Leibnitz rule of differentiation, and finally, from the definition of curvature R, more specifically from the curvature components equation (2.16). Similarly, from the definition of curvature R,we can achieve the Ricci identity for vector fields.

More generally, we can get the Ricci identity for a tensor S of type (r,s).

$$S^{i_1\ldots i_r}{}_{j_1\ldots j_s;kl} - S^{i_1\ldots i_r}{}_{j_1\ldots j_s;lk} = \sum_{a=1}^{s} S^{i_1\ldots i_r}{}_{j_1\ldots j_{a-1}mj_{a+1}\cdots j_s} R^m{}_{j_a kl}$$
$$-\sum_{b=1}^{r} S^{i_1\ldots i_{b-1}mi_{b+1}\ldots i_r}{}_{j_1\ldots j_s} R^{i_b}{}_{mkl} - S^{i_1\ldots i_r}{}_{j_1\ldots j_s;m} T^m{}_{kl}. \qquad (2.24)$$

When covariant differentiations are used in place of ordinary differentiations, the Ricci's identity must be used in place of the ordinary condition of commutativity for ordinary differentiations. If covariant differentiation is applied to (2.9) we obtain *the second covariant derivative* of T in components $T^{i_1\ldots i_r}{}_{j_1\ldots j_s;lk}$. These quantities are not symmetric in the indices k and l as in the case of ordinary second derivatives. In torsion-free case (2.24) is reduced to:

$$S^{i_1\ldots i_r}{}_{j_1\ldots j_s;kl} - S^{i_1\ldots i_r}{}_{j_1\ldots j_s;lk} = \sum_{a=1}^{s} S^{i_1\ldots i_r}{}_{j_1\ldots j_{a-1}mj_{a+1}\cdots j_s} R^m{}_{j_a kl}$$
$$-\sum_{b=1}^{r} S^{i_1\ldots i_{b-1}mi_{b+1}\ldots i_r}{}_{j_1\ldots j_s} R^{i_b}{}_{mkl}. \qquad (2.25)$$

We can get a detailed explanation of (2.23) in [Detweiler, 2003] and for (2.25) in [Goldberg, 1970]

2.11 Natural Isomorphism of Tangent and Cotangent Spaces and Induced Isomorphisms

Let (M,g) be a pseudo-Riemannian m dimensional manifold. Let p be a fixed point of M and M_p, M_p^* the corresponding tangent and cotangent spaces.

We define the linear mapping

$$\flat : M_p \to M_p^*,$$

according to

$$\flat(X) = X^\flat = g(X,\cdot),$$

for $X \in M_p$.

The following proposition is just a well known fact from Linear Algebra, called by some authors, Riesz representation theorem.

Proposition 3 $\flat : M_p \to M_p^*$ *is an isomorphism of vector spaces.*

Proof. *One-to-one.* Indeed, let $X_1,\ X_2 \in M_p$ and $X_1^\flat = X_2^\flat$ then

$$\begin{aligned} g(X_1, Y) &= g(X_2, Y) \qquad \text{for all} \quad Y \in M_p \\ \text{so} \quad g(X_1, Y) - g(X_2, Y) &= 0 \\ g(X_1 - X_2, Y) &= 0. \end{aligned}$$

Since g is nondegenerate we have that $X_1 = X_2$. Then $\flat$ is one-to-one.
Let $\tilde{\omega} \in M_p^*$. We look for a $\widetilde{X} \in M_p$ such that $\widetilde{X}^\flat = \tilde{\omega}$, in other words $g(\widetilde{X}, \cdot) = \tilde{\omega}$ or $g(\widetilde{X}, Y) = \tilde{\omega}(Y)$ for all $Y \in M_p$. In fact it suffices to prove that for a basis of M_p. So, let $X_1, \ldots, X_m$ be a basis of M_p and $\omega^1, \ldots, \omega^m$ its dual basis. Now, $\tilde{\omega} = \tilde{\omega}_i \omega^i$ and $\widetilde{X} = \widetilde{X}^j X_j$, where $\widetilde{X}^j$ and $\tilde{\omega}_i$ are the components of $\widetilde{X}$ and $\tilde{\omega}$ respectively.

$$\begin{aligned} g\left(\widetilde{X}, X_k\right) &= \tilde{\omega}(X_k) \\ g\left(\widetilde{X}^j X_j, X_k\right) &= \tilde{\omega}_j \omega^j(X_k) \\ \widetilde{X}^j g(X_j, X_k) &= \tilde{\omega}_i \delta_k^j \\ \widetilde{X}^j g_{jk} &= \tilde{\omega}_k. \end{aligned}$$

If (g^{ij}) denotes the inverse matrix of (g_{ij}), then $\widetilde{X}^j = \tilde{\omega}_k g^{kj}$. Therefore $\flat$ is an isomorphism of M_p onto M_p^* . ■

Now, let $\# = \flat^{-1}$, be the inverse function of $\flat$. If $\tilde{\omega} \in M_p^*$ then we denote $\#(\tilde{\omega})$ by $\tilde{\omega}^\#$. Another way to define $\tilde{\omega}^\#$ is the following.

Definition 11 *Let* $\tilde{\omega} \in M_p^*$ *then* $\tilde{\omega}^\#$ *is a unique element of* M_p, *such that* $g(\tilde{\omega}^\#, Y) = \langle \tilde{\omega}, Y \rangle$ *for all* $Y \in M_p$

We can check that the components of $\tilde{\omega}^{\#}$ are the same as the components of $\tilde{X}$, which is defined by $\tilde{X}^{\flat} = \tilde{\omega}$, (see Proposition 3).

$$\tilde{\omega}^{\#} = (\tilde{\omega}^{\#})^i X_{i.}$$

Since $\tilde{\omega} = \tilde{\omega}_i \omega^i$, and using a basis $\{X_k\}$ instead of Y, we obtain,

$$(\tilde{\omega}^{\#})^i g(X_i, X_k) = \tilde{\omega}_k.$$

In as much g is nondegenerative, so

$$\tilde{\omega}_k = (\tilde{\omega}^{\#})^i g_{ik},$$

and then

$$(\tilde{\omega}^{\#})^j = \tilde{\omega}_k g^{kj}.$$

Now the mappings $\flat$, $\#$ and the identities mappings of M_p, and M_p^* can be used to determine the induced isomorphism of the tensor spaces of types (r, s) and (l, m), where $r + s = l + m$.

We provide examples of such induced isomorphisms.

Example 2 *We define:*

$$\# \otimes id \otimes id : M_p^* \otimes M_p^* \otimes M_p^* \to M_p \otimes M_p^* \otimes M_p^*.$$

Let $T \in M_p^* \otimes M_p^* \otimes M_p^*$, *then*

$$[(\# \otimes id \otimes id)\, T]\, (\omega, X, Y) = T\left(\omega^{\#}, X, Y\right),$$

for all X, $Y \in M_p$ *and* $\omega \in M_p^*$. *In local components,*

$$T_{j_1 j_2 j_3} \to g^{j_1 i_1} T_{j_1 j_2 j_3} = T^{i_1}{}_{j_2 j_3},$$

Example 3 *We define:*

$$\# \otimes \flat \otimes id : M_p^* \otimes M_p \otimes M_p^* \to M_p \otimes M_p^* \otimes M_p^*,$$

by,

$$\left[(\# \otimes \flat \otimes id) \, T \right] (\omega, X, Y) = T \left(\omega^\#, X^\flat, Y \right),$$

for $T \in M_p^* \otimes M_p \otimes M_p^*$*, all* X*,* $Y \in M_p$ *and* $\omega \in M_p^*$*. In local components,*

$$T_{j_1}{}^{i_2}{}_{j_3} \to g_{i_2 j_2} g^{i_1 j_1} T_{j_1}{}^{i_2}{}_{j_3} = T^{i_1}{}_{j_2 j_3}.$$

Remark 1 *The isomorphisms of the tensor spaces at* p *determine the corresponding isomorphisms of the modules of tensor fields. Those new isomorphisms are denoted by the same symbols as the ones of respective tensor spaces.*

Finally we point out a property of the inverse pseudo-metric tensor g^{-1} which can be used to define g^{-1} as well.

Proposition 4 $g^{-1} = (\# \otimes \#) \, g$

Proof. Let $h : M_p^* \times M_p^* \to \mathbb{R}$ be a tensor of type $(2, 0)$, determined by $h(\omega, \tilde{\omega}) = g(\omega^\#, \tilde{\omega}^\#)$. Then the components of h are,

$$\begin{aligned} h^{ij} &= h(\omega^i, \omega^j) \\ &= g\left((\omega^i)^\#, (\omega^j)^\# \right). \end{aligned}$$

However, it can be shown that $(\omega^i)^\# = g^{ik} X_k$. Indeed,

$$g\left(\left(\omega^i \right)^\#, X_k \right) = \omega^i \left(X_k \right) = \delta^i_k$$

and

$$\left(\omega^i \right)^\# = g^{ik} X_k$$

is a unique solution of that equations. So,

$$\begin{aligned} g\left(g^{ik}X_k, g^{jr}X_r\right) &= g^{ik}g^{jr}g(X_k, X_r) \\ &= g^{ik}g^{jr}g_{kr} \\ &= g^{ik}\delta_k^j = g^{ij}, \end{aligned}$$

and therefore $h^{ij} = g^{ij}$. $\blacksquare$

Chapter 3

Some Important Tensors

3.1 Riemannian Tensor and Sectional Curvature.

Now we shall define the *Riemann Tensor* which is also known as the *covariant curvature tensor.* It is obtained if the upper index of the curvature tensor is lowered. In local components,

$$R_{ijkl} = g_{ai} R^{a}{}_{jkl}. \qquad \text{(covariant curvature components)}$$

Alternatively, one may define the Riemann tensor R as the $(0,4)$ tensor such that

$$R(W, Z, X, Y) = g(W, R(X, Y)Z),$$

which possesses the following properties,

$$\text{Antisymmetry} \;:\; R(W, Z, X, Y) = -R(Z, W, X, Y,) = -R(W, Z, Y, X) \tag{3.1}$$

$$\text{Cyclicity} \;:\; R(W, Z, X, Y) + R(W, Y, Z, X) + R(W, X, Y, Z) = 0 \tag{3.2}$$

$$\text{Symmetry} \;:\; R(W, Z, X, Y) = R(X, Y, W, Z) \tag{3.3}$$

Therefore components of so understood Riemann tensor satisfy the following identities:

$$R_{ijkl} = R_{klij} = -R_{jikl} = -R_{ijlk}, \qquad R_{ijkl} + R_{iklj} + R_{iljk} = 0. \tag{3.4}$$

Let π be a plane, that is a 2-dimensional subspace, in M_p and let X_1 and X_2 be an orthonormal basis for π. For each plane π in the tangent space M_p, the *sectional curvature* $K(\pi)$ for π is defined by

$$K(\pi) = R(X_1, X_2, X_1, X_2) = g(X_1, R(X_1, X_2) X_2).$$

$K(\pi)$ is independent on the choice of an orthonormal basis for π. In fact, if W_1 and W_2 form another orthonormal basis of π, then

$$W_1 = aX_1 + bX_2, \qquad W_2 = bX_1 - aX_2,$$

where a and b are real numbers such that $a^2 + b^2 = 1$. Using (3.1), we have that

$$R(X_1, X_2, X_1, X_2) = R(W_1, W_2, W_1, W_2).$$

Definition 12 *If $K(\pi)$ is a constant for all planes π in M_p and for all points $p \in M$, then M is called a* space of constant curvature.

Proposition 5 *If X_1, X_2 is a basis (not necessarily orthonormal) of a plane π in M_p, then*

$$K(\pi) = \frac{R(X_1, X_2, X_1, X_2)}{g(X_1, X_1)\, g(X_2, X_2) - (g(X_1, X_2))^2}.$$

Proof. We obtain the formula making use of the following orthonormal basis for π :

$$\frac{X_1}{(g(X_1,X_1))^{\frac{1}{2}}}, \qquad \frac{1}{a}\left[g(X_1,X_1)\, X_2 - g(X_1,X_2)\, X_1\right],$$

where $a = \left\{g(X_1,X_1)\left[g(X_1,X_1)\, g(X_2,X_2) - (g(X_1,X_2))^2\right]\right\}^{\frac{1}{2}}$. ■

Theorem 6 *(F. Schur.) Let M be a connected Riemannian manifold of dimension ≥ 3. If the sectional curvature $K(\pi)$, where π is a plane in M_p, depends only on $p \in M$, then M is a space of constant curvature.*

We can find a proof of this theorem in ([Kobayashi, 1963]) p.202

Remark 2 *It is important to note that Definition 12, Proposition 5 and Theorem 6 are valid only for Riemannian Manifolds. To see why, let N be a 4-dimensional pseudo-Riemannian manifold with metric tensor h and signature $-+++$. Let X_1, X_2, X_3, X_4 be a pseudo-orthonormal basis of M_p. Then*

$$\begin{aligned} h(X_1, X_1) &= h(X_2, X_2) = h(X_3, X_3) = -h(X_4, X_4) = 1 \\ \text{and } h(X_i, X_j) &= 0 \qquad \text{for } i \neq j. \end{aligned}$$

Let $Y_1 = X_1 + X_3$ and $Y_2 = X_2 - X_4$ be spanning the plane α, but we have that $h(Y_1, Y_2) = 0$ and $h(Y_2, Y_2) = 0$. So in this case we can not define the sectional curvature $K(\alpha)$ according to Proposition 5. Consequently, we cannot define the space of constant curvature by Definition 12. For pseudo-Riemannian manifolds we define the spaces of constant curvature by means of the following equation which is result of theorem of Schur for Riemannian spaces.

Corollary 7 *For a Riemannian space of constant curvature k, we have*

$$R(X, Y)Z = k\left[g(Z, Y)X - g(Z, X)Y\right]. \tag{3.5}$$

3.2 Ricci and Einstein Tensors

Definition 13 *Let M be a manifold with affine connection $\boldsymbol{\nabla}$. The* Ricci curvature, *denoted by Ric, is the symmetric tensor field of type $(0, 2)$ defined for all p in M as follows: For all X, $Y \in M_p$, $Ric(X, Y)$ is equal to the trace of the mapping $W \mapsto R(W, Y)X$ of M_p into itself, where $W \in M_p$.*

If M is a pseudo-Riemannian manifold and if $\{e_1, \ldots, e_m\}$ is a basis of M_p and $\{\omega^1, ..., \omega^m\}$ its dual, then,

$$Ric(X,Y) = \left\langle \omega^i, R\left(e_i, Y\right) X \right\rangle$$

One may express X and Y in the natural basis, $\mathbf{X} = X^j \left(\frac{\partial}{\partial u^j}\right)$, $\mathbf{Y} = Y^j \left(\frac{\partial}{\partial u^j}\right)$, and then write

$$Ric(X,Y) = R_{ij} X^i Y^j$$

where

$$R_{ij} = R^a{}_{iaj}. \qquad \text{(Ricci curvature components)}$$

The *Ricci Tensor* is the $(1,1)$ -tensor field which corresponds to the Ricci curvature. The components of the Ricci tensor may be obtained by raising one index of the Ricci curvature. It does not matter which index is raised since the Ricci tensor is symmetric. Thus,

$$R^i_j = g^{ai} R_{aj} = g^{ai} R_{ja} \qquad \text{(Ricci tensor components)}$$

The trace of the Ricci curvature is said to be the *scalar curvature* τ. Traditionally, τ has been denoted by R. That is,

$$\tau = R = R^a_a = g^{ij} R_{ij.}$$

From the second Bianchi identity for the Riemann Tensor we obtain:

$$g^{lj}\left[R_{lijh;k} + R_{likj;h} + R_{lihk;j}\right] = 0,$$

and consequently

$$R^j{}_{ijh;k} + R^j{}_{ikj;h} + R^j{}_{ihk;j} = 0,$$

since $g^{ik}_{;j} = 0$ and $g_{ik;j} = 0$ (we can take g^{lj} in and out of covariant derivative at will). Using the antisymmetry on the indices j and k we get:

$$R^j{}_{ijh;k} - R^j{}_{ijk;h} + R^j{}_{ihk;j} = 0$$

so

$$R_{ih;k} - R_{ik;h} + R^{j}{}_{ihk;j} = 0 \tag{3.6}$$

These equations are called the contracted second Bianchi identity. By a similar process applied to (3.6), it follows that.

$$\begin{aligned} g^{ih}\left[R_{ih;k} - R_{ik;h} + R^{j}{}_{ihk;j}\right] &= 0 \\ R_{;k} - R^{h}{}_{k;h} - R^{j}{}_{k;j} &= 0 \\ R_{;k} - 2R^{h}{}_{k;h} &= 0 \\ \text{or} \qquad 2R^{h}{}_{k;h} - R_{;k} &= 0 \end{aligned}$$

However, since $R_{;k} = \delta^h_k R_{;h}$.

$$\left[R^h_k - \frac{1}{2}\delta^h_k R\right]_{;h} = 0.$$

Raising the index k with g^{kl}, we arrive at

$$\left[R^{hl} - \frac{1}{2}g^{hl} R\right]_{;h} = 0.$$

We define *the Einstein tensor* by

$$G^{hl} = R^{hl} - \frac{1}{2}g^{hl} R$$

and we infer that

$$G^{hl}{}_{;l} = 0.$$

Notice that the tensor G^{hl} was constructed only from the Riemann tensor and the metric.

3.3 Weyl Tensor

The components of the *Weyl's conformal tensor* C_{abcd} are given by

$$\begin{aligned} C^{i}{}_{jkl} &= R^{i}{}_{jkl} + \frac{\tau}{(m-1)(m-2)}\left(g_{jk}\delta^i_l - g_{jl}\delta^i_k\right) \\ &\quad - \frac{1}{m-2}\left(R_{jl}\delta^i_k - R_{jk}\delta^i_l - g_{jk}R^i_l - g_{jl}R^i_k\right), \end{aligned} \tag{3.7}$$

which has the symmetries

$$C_{abcd} = -C_{bacd} = -C_{abdc} = C_{cdab}, \quad C_{a[bcd]} = 0$$

The case $m = 3$ is of special interest. Indeed, by choosing a pseudo-orthonormal coordinate system, the diagonal of (g_{ij}) is ± 1 at a point , and it is readily shown that the Weyl conformal curvature tensor vanishes.

Proposition 8 *The Weyl tensor satisfies that*

$$C^i_{jil} = 0 \tag{3.8}$$

Proof. *From the equation (3.7), we have*

$$\begin{aligned} C^i{}_{jil} &= R^i{}_{jil} + \frac{\tau}{(m-1)(m-2)}\left(g_{ji}\delta^i_l - g_{jl}\delta^i_i\right) \\ &\quad -\frac{1}{m-2}\left(R_{jl}\delta^i_i - R_{ji}\delta^i_l - g_{ji}R^i_l - g_{jl}R^i_i\right) \end{aligned}$$

then,

$$\begin{aligned} C^i{}_{jil} &= R_{jl} + \frac{\tau}{(m-1)(m-2)}\left(g_{jl} - g_{jl}m\right) \\ &\quad -\frac{1}{m-2}\left(R_{jl}m - 2R_{jl} - g_{jl}\tau\right) = 0 \end{aligned}$$

■

The Weyl tensor is completely traceless, i.e., the contraction with respect to each pair of indices vanishes.

Chapter 4

Killing Structures and Related Themes

4.1 Mappings of Manifolds

Let M and N be manifolds (not necessarily of the same dimension) and let

$$\Phi : M \to N$$

be a C^{∞}mapping. In a natural manner, we can compose Φ with a function $f : N \to \mathbb{R}$ and define $\Phi_* f = f \circ \Phi : M \to \mathbb{R}$. Similarly, in a natural way, Φ "carries" along tangent vectors at $p \in M$ to tangent vectors at $\Phi(p) \in N-$ i.e., it defines a mapping

$$\Phi^* : M_p \to N_{\Phi(p)}$$

as follows: for $A \in M_p$ we define $\Phi^* A \in N_{\Phi(p)}$ by

$$(\Phi^* A)(h) = A(h \circ \Phi)$$

for all smooth functions $h : N \to \mathbb{R}$. Note that Φ^* is linear and may be viewed as the "*differential* of Φ at p". By the known implicit function theorem [1] $\Phi : M \to N$

[1] [Dieudonné, 1970] p.270

will be one-to-one in a neighborhood of p if $\Phi^* : M_p \to N_{\Phi(p)}$ is one-to-one.

Remark 3 *The matrix of Φ^* in the coordinate basis of a coordinate system $\{u^i\}$ at p and a coordinate system $\{v^j\}$ at $\Phi(p)$ equals the Jacobian matrix of the mapping Φ, expressed in local coordinates i.e.,*

$$(\Phi^*)_i^j = \left(\frac{\partial v^j}{\partial u^i}\right). \tag{4.1}$$

In the same way, we can use Φ to "pull back " covectors at $\Phi(p)$. We define the mapping

$$\Phi_* : N^*_{\Phi(p)} \to M^*_p$$

by requiring that for all $A \in M_p$ and $\mu \in N^*_{\Phi(p)}$,

$$(\Phi_*\mu)(A) = \mu(\Phi^* A).$$

We can extend the action of Φ_* to a mapping of tensors of type $(0, s)$ at $\Phi(p)$ to tensors of type $(0, s)$ at p by:

$$(\Phi_* T)(A_1, \cdots, A_s) = T((\Phi^* A_1), \cdots, (\Phi^* A_s)),$$

for $A_i \in M_p$, $1 \leq i \leq m$. Similarly, we can extend the action of Φ^* to a mapping of tensors of type $(r, 0)$ at p to tensor of type $(r, 0)$ at $\Phi(p)$ by

$$(\Phi^* T)(\mu_1, \cdots, \mu_r) = T((\Phi_*\mu_1), \cdots, (\Phi_*\mu_r)),$$

for $\mu_i \in N^*_{\Phi(p)}, 1 \leq i \leq n$. However, in general we can not extend Φ^* or Φ_* to mixed tensors since Φ^* does not deal with lower index tensors, while Φ_* does not with upper index tensors.

Definition 14 *A $C^\infty-$mapping $\Phi : M \to N$ is called a* diffeomorphism *of M onto N if Φ is a one-to-one mapping of M onto N and Φ^{-1} is C^∞. A diffeomorphism of M onto itself is called a* transformation *of M.*

If Φ is a diffeomorphism (which necessarily implies $\dim M = \dim N$), then we can use Φ^{-1} to extend the definition of Φ^* to tensors of all of types by using the fact that $(\Phi^{-1})^*$ goes from $N_{\Phi(p)}$ to M_p. If $T^{i_1 \cdots i_r}_{\quad j_1 \cdots j_s}$ are the components of a tensor T of type (r, s) at p, we define the components of the tensor $(\Phi^* T)$ at $\Phi(p)$ by,

$$(\Phi^* T)(\mu_1, \cdots, \mu_r, A_1, \cdots, A_s) = T\left((\Phi_* \mu_1), \cdots, \left(\left[\Phi^{-1}\right]^* A_s\right)\right).$$

In the same way we can extend the mapping Φ_* to all of tensors. However, it is not difficult to show that $\Phi_* = (\Phi^{-1})^*$, so we need only Φ^* and $(\Phi^{-1})^*$.

A transformation Φ of M induces an automorphism Φ^* of the algebra $\wedge\mathfrak{D}_1$ of differential forms on M and, in particular, an automorphism of the algebra $\mathfrak{F}(M)$ of functions on M :

$$(\Phi_* f)(p) = f(\Phi(p)), \qquad f \in \mathfrak{F}(M), \text{ and } p \in M.$$

It induces also an automorphism Φ^* of $\mathfrak{D}^1$ by

$$(\Phi^* X)_p = (\Phi^*)_q (X_q),$$

where $\Phi(q) = p$, $X \in \mathfrak{D}^1$. They are related by

$$\Phi_*((\Phi^* X) f) = X(\Phi_* f)^{2} \quad \text{for } X \in \mathfrak{D}^1 \text{ and } f \in \mathfrak{F}(M). \tag{4.1}$$

Although any mapping Φ of M into N carries differential forms ω on N into a differential form $\Phi_*(\omega)$ on M, in general, Φ does not send a vector field on M

[2]

$$\begin{aligned} \Phi_*[(\Phi^* X) f] &= \Phi_*[\Phi^*(X_{\Phi^{-1}}) f] \\ &= \Phi_*[X_{\Phi^{-1}}(f \circ \Phi)] \\ &= \Phi_*(X f) \\ &= X f \circ \Phi \\ &= X(\Phi_* f) \end{aligned}$$

into a vector field on N. We say that a vector field X on M is Φ *-related* to vector field Y on N if

$$(\Phi_*)_p X_p = Y_{\Phi(p)},$$

for all $p \in M$.

If Φ is a transformation of M, its differential Φ^* gives us a linear isomorphism of the tangent space $M_{\Phi^{-1}(p)}$ onto the tangent space $M_{p.}$ This linear isomorphism can be extended to an isomorphism of the tensor algebra $\mathfrak{D}\left(\Phi^{-1}(p)\right)$ onto the tensor algebra $\mathfrak{D}(p)$ [3], which we denote by $\tilde{\Phi}$. Given a tensor field T we can define a tensor field $\tilde{\Phi}T$ by

$$\left(\tilde{\Phi}T\right)_p = \tilde{\Phi}\left(T_{\Phi^{-1}(p)}\right),$$

for $p \in M$. In this way, every transformation Φ of M induces an algebra automorphism of $\mathfrak{D}(M)$ which preserves types and commutes with contractions.

Remark 4 *If Φ is a transformation of M and T is a tensor field on M, we can compare T with $\tilde{\Phi}T$. If $\tilde{\Phi}T = T$, then even though we have "moved T" via Φ, it has "stayed the same." In other words, Φ is a* symmetry transformation for the tensor T. *In the case of the metric g, a symmetry transformation, i.e., a diffeomorphism Φ such that $(\Phi^* g) = g$ is called an* isometry.

4.2 Vector Fields and One-Parameter Groups of Transformations

Definition 15 *A* global 1-parameter group of differentiable transformations *of M denoted by Φ_t $(-\infty < t < \infty)$ is a differentiable mapping of $\mathbb{R}\times M$ into M,*

$$\begin{aligned} \mathbb{R}\times M &\rightarrow M \\ (t,p) &\longmapsto \Phi_t(p), \end{aligned}$$

[3]We used Proposition 2.12. of [Kobayashi, 1963].

which satisfies the following conditions:

(i) For each $t \in \mathbb{R}$, $\Phi_t : p \longmapsto \Phi_t(p)$ is a *transformation* of M.

(ii) For all $t,\ s \in \mathbb{R}$, and $p \in M$,

$$\Phi_{t+s}(p) = \Phi_t(\Phi_s(p))$$

(iii)

$$\Phi_o = id_M. \tag{4.2}$$

The 1-parameter group of transformations Φ_t induces a vector field X on M defined by the equation

$$(Xf)(p) = \lim_{t \to 0} \frac{f(\Phi_t(p)) - f(p)}{t} \tag{4.3}$$

where f is an arbitrary differentiable function. The limit is assured by the differentiability of the mapping $(t,p) \longmapsto \Phi_t(p)$.

On the other hand, a vector field X on M does not necessarily induce a global 1-parameter group of transformations Φ_t on M. However, associated with a point p of M there is a neighborhood U of p and a constant $\varepsilon > 0$ such that for $|t| < \varepsilon$ there is a (local) 1-parameter group of transformations Φ_t i.e., a differentiable mapping

$$\begin{array}{rcl} (-\varepsilon, \varepsilon) \times U & \to & U \\ (t,p) & \longmapsto & \Phi_t(p), \end{array}$$

satisfying the conditions:

$(i)'$ For each $t \in (-\varepsilon, \varepsilon)$, $\Phi_t : p \longmapsto \Phi_t(p)$ is a *transformation* of U onto $\Phi_t(U)$;

$(ii)'$ For all $|t|, |s|, |s+t| < \varepsilon$ and $p \in U$, is that if $\Phi_s(p) \in U$ then,

$$\Phi_{t+s}(p) = \Phi_t(\Phi_s(p)) \tag{4.4}$$

$(iii)'$

$$\Phi_o = id_U \tag{4.5}$$

Moreover, Φ_t induces the local vector field X, that is equation (4.3) is satisfied for each $p \in U$ and differentiable function f. Indeed, let $p \in U$. Then $(-\varepsilon, \varepsilon) \ni t \longmapsto \Phi_t(p)$ is a curve passing through p for $t = 0$. From (4.3)

$$X_p = \frac{d}{dt}\Phi_t(p)\,|_{t=0}.$$

We can summarize it in the following theorem.

Theorem 9 *Let X be a vector field on a manifold M. Then for each point $p \in M$ there exist its open neighborhood U and exactly one local 1-parameter group of transformations Φ_t in U such that the vector field induced by that group coincides on U with X. (for a proof see [Kobayashi, 1963], p.13)*

If a vector field determines a (global)1-parameter group of transformations, then the vector field X is said to be *complete.*

Proposition 10 [4] *On a compact manifold M, every vector field X is complete.*

Proposition 11 *Let Φ be a transformation of M. If a vector field X generates a local 1-parameter group of transformations Φ_t, then the vector field $\Phi^* X$ generates the group $\Phi \circ \Phi_t \circ \Phi^{-1}$*

Proof. It is clear that $\Phi \circ \Phi_t \circ \Phi^{-1}$ is a local 1-parameter group of transformations. To show that it induces the vector field $\Phi^* X$, let p be an arbitrary point of M and $q = \Phi^{-1}(p)$. Since Φ_t induces X, the vector $X_q \in M_q$ is tangent to the curve $x(t) = \Phi_t(q)$ at $q = x(0)$. It follows that the vector

$$(\Phi^* X)_p = \Phi^*(X_q) \in M_p \tag{4.6}$$

[4] [Kobayashi, 1963] p.14

is tangent to the curve

$$y(t) = \Phi \circ \Phi_t(q) = \Phi \circ \Phi_t \circ \Phi^{-1}(p).$$

$\blacksquare$

Corollary 12 *A vector field X is invariant by Φ, that is, $\Phi^* X = X$, if and only if Φ commutes with Φ_t.*

Proposition 13 [5]*Let X and Y be vector fields on M. If X generates a local 1-parameter group of transformations Φ_t, then*

$$[X, Y] = \lim_{t \to 0} \frac{1}{t} [Y - (\Phi_t)^* Y].$$

More precisely,

$$[X, Y]_p = \lim_{t \to 0} \frac{1}{t} \left[Y_p - ((\Phi_t)^* Y)_p \right], \qquad p \in M.$$

The limit on the right hand side is taken with respect to the natural topology of the tangent vector space M_p.

Corollary 14 *With same notations as in Proposition 13 we have more generally*

$$(\Phi_s)^* [X, Y] = \lim_{t \to 0} \frac{1}{t} [(\Phi_s)^* Y - (\Phi_{s+t})^* Y]$$

for any value s.

Proof. For fixed value of s, consider the vector field $(\Phi_s)^* Y$ and apply the Proposition 13. Then we have

$$\begin{aligned} [X, (\Phi_s)^* Y] &= \lim_{t \to 0} \frac{1}{t} [(\Phi_s)^* Y - (\Phi_t)^* \circ (\Phi_s)^* Y] \\ &= \lim_{t \to 0} \frac{1}{t} [(\Phi_s)^* Y - (\Phi_{s+t})^* Y], \end{aligned}$$

[5]For a proof see [Goldberg, 1970] p. 101

since, $\Phi_s \circ \Phi_t = \Phi_{s+t}$. On the other hand, $(\Phi_s)^* X = X$ by the Corollary 12. Since $(\Phi_s)^*$ preserves the bracket, we obtain

$$(\Phi_s)^* [X, Y] = [X, (\Phi_s)^* Y]$$

■

Remark 5 *The conclusion of Corollary 14 can be written as*

$$\left(\frac{(d (\Phi_t)^* Y)}{dt}\right)_{t=s} = -(\Phi_t)^* [X, Y]$$

Corollary 15 *Suppose X and Y generate 1-parameter groups of transformations Φ_t and ψ_s, respectively. Then $\Phi_t \circ \psi_s = \psi_s \circ \Phi_t$ for every s and t if only if $[X, Y] = 0$.*

Proof. If $\Phi_t \circ \psi_s = \psi_s \circ \Phi_t$ for every s and t, Y is invariant by every Φ_t by Corollary 12. By Proposition 13, $[X, Y] = 0$.

Conversely if $[X, Y] = 0$, then $\frac{(d(\Phi_t)^* Y)}{dt} = 0$ for every t by the remarks of Corollary 14. Therefore $(\Phi_t)^* Y$ is a constant vector at each point p so that Y is invariant by every Φ_t. By Corollary 12, every ψ_s commutes with every Φ_t ■

4.3 Lie Derivative

Let X be a vector field on M and Φ_t a local 1-parameter group of transformations generated by X. We shall define the *Lie derivative* $\pounds_X T$ of a tensor field T with respect to a vector field X as follows. For simplicity, we assume that Φ_t is a global 1-parameter group of transformations of M. For each t, let $\tilde{\Phi}_t$ be an automorphism of the tensor algebra $\mathfrak{D}(M)$. Then for any tensor field T on M, we set

$$\pounds_X T = \lim_{t \to 0} \frac{1}{t} \left[T - \tilde{\Phi}_t T \right], \tag{4.7}$$

where all tensors appearing in Equation 4.7 are evaluated at the same point p. The mapping $\pounds_X$ of $\mathfrak{D}(M)$ into itself, which sends T into $\pounds_X T$, is called the *Lie Differentiation with respect to X.*

Proposition 16 [6]*Lie differentiation $\pounds_X$ with respect to a vector field X satisfies the following conditions:*

(a) $\pounds_X$ is a derivation of $\mathfrak{D}(M)$, that is, it is $\mathbb{R}-$ linear and satisfies

$$\pounds_X (T \otimes T') = (\pounds_X T) \otimes T' + T \otimes (\pounds_X T'),$$

for all T, $T' \in \mathfrak{D}(M)$;

(b) $\pounds_X$ is type-preserving: $\pounds_X (\mathfrak{D}_s^r(M)) \subset \mathfrak{D}_s^r(M)$;

(c) $\pounds_X$ commutes with every contraction;

(d) $\pounds_X f = Xf$ for every function f;

(e) $\pounds_X Y = [X, Y]$ for every vector field Y.

Proposition 17

$$(\pounds_X \omega)(Y) = X(\omega(Y)) - \omega([X, Y]), \quad \textit{for} \;\; \omega \in \mathfrak{D}_1. \tag{4.8}$$

.

Proof. Let $\omega \in \mathfrak{D}_1$, X, $Y \in \mathfrak{D}^1$. By (a) we have that,

$$\pounds_X (\omega \otimes Y) = (\pounds_X \omega) \otimes Y + \omega \otimes (\pounds_X Y).$$

[6]For a proof see [Kobayashi, 1963], p.29

Now, we apply the respective contractions and by (b) and (c), we get,

$$\begin{aligned} \pounds_X (\omega (Y)) &= C_1^1 [(\pounds_X \omega) \otimes Y] + C_1^1 [\omega \otimes (\pounds_X Y)]; \\ X (\omega (Y)) &= (\pounds_X \omega)(Y) + \omega (\pounds_X Y), \end{aligned} \tag{4.9}$$

by (e) we have,

$$(\pounds_X \omega)(Y) = X (\omega (Y)) - \omega ([X, Y]).$$

■

Now, we work out the components of the Lie derivatives. From (2.1), we obtain that

$$\pounds_X Y^i = (\pounds_X Y)^i = [X, Y]^i = \xi^k \eta^i_{,k} - \eta^k \xi^i_{,k}. \tag{4.10}$$

Similarly, If $\omega = \omega_i du^i$, then

$$\pounds_X \omega_j = (\pounds_X \omega)_j = \xi^k \omega_{j,k} + \omega_k \xi^k_{,j}. \tag{4.11}$$

Finally, we obtain

$$\pounds_X T^{i_1 \cdots i_r}_{\;\; j_1 \cdots j_s} = \xi^k T^{i_1 \cdots i_r}_{\;\; j_1 \cdots j_s, k} - \sum_l T^{i_1 \cdots k \cdots i_r}_{\;\; j_1 \cdots j_s} \xi^{i_l}_{,k} + \sum_m T^{i_1 \cdots i_r}_{\;\; j_1 \cdots k \cdots j_s} \xi^k_{,j_m} \tag{4.12}$$

where $l = 1, ... r$ and $m = 1, ... s$[7].

The result of applying the procedure used in (4.8) gives us its generalization,

$$(\pounds_X T)(Y_1, ..., Y_r) = [X, T (Y_1, ..., Y_r)] - T (Y_1, ..., [X, Y_i], ..., Y_r),$$

for T being a tensor field of type $(1, r)$, $X, Y_1, ..., Y_r \in \mathfrak{D}^1$,

and,

$$(\pounds_X \omega)(Y_1, ..., Y_r) = [X, \omega (Y_1, ..., Y_r)] - \omega (Y_1, ..., [X, Y_i], ..., Y_r),$$

for $\omega \in \mathfrak{D}_r$.

We provide an important lemma about derivations. For a proof see [Kobayashi, 1963] p.30.

[7] We can find a detailed explanation of (4.10), (4.11) and (4.12) in [Yano, 1955] Chapter I §3

Lemma 18 *Two derivations D_1 and D_2 of $\mathfrak{D}(M)$ coincide if they coincide on $\mathfrak{F}(M)$ and $\mathfrak{D}^1(M)$.*

This Lemma 18 permits us to infer the following propositions.

Proposition 19 *For any vector fields X and Y we have that*

$$\pounds_{[X,Y]} = [\pounds_X, \pounds_Y].$$

Proof. By virtue of Lemma 18, it is sufficient to show that $[\pounds_X, \pounds_Y]$ has the same effect as $\pounds_{[X,Y]}$ on $\mathfrak{F}(M)$ and $\mathfrak{D}^1(M)$. For $f \in \mathfrak{F}(M)$, we have

$$[\pounds_X, \pounds_Y] f = XYf - YXf = [X,Y] f = \pounds_{[X,Y]} f.$$

For $Z \in \mathfrak{D}^1(M)$, we have

$$\begin{aligned} [\pounds_X, \pounds_Y] Z &= [X,[Y,Z]] - [Y,[X,Z]] \\ &= [[X,Y], Z] = \pounds_{[X,Y]} Z \end{aligned}$$

by the Jacobi identity. ■

Proposition 20 *Let Φ_t be a local 1- parameter group of local transformations generated by a vector field X. For any tensor T, we have*

$$\tilde{\Phi}_s(\pounds_X T) = -\left[\frac{d\left(\tilde{\Phi}_t(T)\right)}{dt}\right]_{t=s}.$$

Proof. By definition,

$$\pounds_X T = \lim_{t \to 0} \frac{1}{t}\left[T - (\tilde{\Phi}_t T)\right]$$

Replacing T by $\tilde{\Phi}_s(T)$, we obtain

$$\begin{aligned} \pounds_X\left(\tilde{\Phi}_s(T)\right) &= \lim_{t \to 0} \frac{1}{t}\left[\tilde{\Phi}_s(T) - (\tilde{\Phi}_t\left(\tilde{\Phi}_s(T)\right)\right] \\ &= \lim_{t \to 0} \frac{1}{t}\left[\tilde{\Phi}_s(T) - (\tilde{\Phi}_{t+s}(T)\right] \\ &= -\left[\frac{d\left(\tilde{\Phi}_t(T)\right)}{dt}\right]_{t=s}. \end{aligned}$$

Our problem is therefore to prove that $\tilde{\Phi}_s(\pounds_X T) = \pounds_X\left(\tilde{\Phi}_s(T)\right)$, i.e., $\pounds_X T = \tilde{\Phi}_s^{-1} \circ \pounds_X \circ \tilde{\Phi}_s(T)$ for all tensor fields T. It is a straightforward verification to see that $\tilde{\Phi}_s^{-1} \circ \pounds_X \circ \tilde{\Phi}_s$ is a derivation of $\mathfrak{D}(M)$. By lemma 18, it is sufficient to prove that $\pounds_X$ and $\tilde{\Phi}_s^{-1} \circ \pounds_X \circ \tilde{\Phi}_s$ coincide on $\mathfrak{F}(M)$ and $\mathfrak{D}^1(M)$. We already noted in the proof of Corollary 14 that they coincide on $\mathfrak{D}^1(M)$. The fact that they coincide on $\mathfrak{F}(M)$ follows from the following formulas

$$\begin{aligned} \Phi_*\left[(\Phi^* X) f\right] &= X(\Phi_* f) \\ \tilde{\Phi}^{-1} f &= \Phi_* f, \end{aligned}$$

which hold for any transformation Φ of M and from $(\Phi_s)^* X = X$. (See(Corollary 12)). ■

Corollary 21 *A tensor field T is invariant by Φ_t for every t if and only if $\pounds_X T = 0$.*

Remark 6 *To study the action of $\pounds_X$ on an arbitrary tensor field, it is helpful to introduce a coordinate system on M where the parameter t along the integral curves of X is chosen as one of the coordinates. Without loss of generality we may consider it as u^1, so $X = \frac{\partial}{\partial u^1}$. This always can be done locally in a neighborhood where $X \neq 0$. The action of Φ_t corresponds then to $u^1 \to u^1 + t$, with $u^2, \ldots, u^m$ held fixed. From Equation 4.1, we have $(\Phi^*)^i_j = \delta^i_j$ and hence, the coordinates basis components of $\Phi_t^* T$ at the point p whose coordinate are $(u^1, \ldots, u^m)$ are*

$$\left(\Phi_t^* T\right)^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}\left(u^1, \ldots, u^m\right) = T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}\left(u^1 - t, \ldots, u^m\right).$$

Consequently, the components of the Lie derivative of T in a coordinate system adapted to X are simply

$$\pounds_X T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} = \frac{\partial T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}}{\partial u^1}.$$

Thus, in particular, Φ_t will be a symmetry transformation of T if and only if the components $T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}$ in a coordinate system adapted to X are independent of coordinate u^1.

4.4 Killing Vector Fields and Killing Equations

Definition 16 *Let g be the Riemannian or pseudo-Riemannian metric on M and let g_{ij} be its components in a local chart $\{u^1, ..., u^m\}$ of M. A local 1-parameter group of transformations Φ_s is called a local 1-parameter group of isometries if*

$$(\Phi_t)^* g = g$$

Definition 17 *A vector field X induced by the local 1-parameter group of isometries is called* Killing vector field.

This obviously means that X leaves g invariant. In other words, X is a Killing vector if and only if $\pounds_X g \equiv 0$. (See Corollary 21).

Proposition 22 *For any vector field X*

$$\pounds_X g_{ij} = K_{i;j} + K_{j;i}$$

where $K = X^\flat (= g(X, \cdot))$ is the 1-form dual to X.

Proof. Let U be a coordinate neighborhood with local coordinates $\{u^1, \ldots, u^m\}$ .Then $g = g_{ij} du^i \otimes du^j$ in U. Applying the derivation $\pounds_X$ to g and (4.11) we obtain,

$$\left(\pounds_X du^i\right)_k = \left(X du^i\right)_k + \xi^j_{,k} \delta^i_j = \xi^i_{,k}$$

and

$$\begin{aligned} (\pounds_X g) &= (X g_{ij})\ du^i \otimes du^j + g_{ij}\xi^i_{,l}\ du^l \otimes du^j + g_{ij}\xi^j_{,l}\ du^i \otimes du^l \\ &= \xi^k g_{ij,k}\ du^i \otimes du^j + g_{ij}\xi^i_{,l}\ du^l \otimes du^j + g_{ij}\xi^j_{,l}\ du^i \otimes du^l \\ &= \left(\xi^k g_{ij,k} + g_{kj}\xi^k_{,i} + g_{ik}\xi^k_{,j}\right) du^i \otimes du^j. \end{aligned}$$

It follows that

$$\pounds_X g_{ij} = \xi^k g_{ij,k} + g_{kj}\xi^k_{,i} + g_{ik}\xi^k_{,j} \tag{4.13}$$

On the another hand, we have that the right hand side of (4.13) is equal to $K_{i;j} + K_{j;i}$. Indeed, we can write

$$\pounds_X g_{ij} = \xi^k g_{ij;k} + g_{kj}\xi^k_{;i} + g_{ik}\xi^k_{;j}$$

By natural isomorphism $\flat$, we have that

$$\left(X^\flat\right)_i = g_{ik}\xi^k \equiv K_i.$$

So, since, $g_{ij;k} = 0$,

$$K_{i;j} = \left(g_{ik}\xi^k\right)_{;j} = \xi^k g_{ik;j} + g_{ik}\xi^k_{;j} = g_{ik}\xi^k_{;j}.$$

Hence

$$\pounds_X g_{ij} = K_{i;j} + K_{j;i}. \tag{4.14}$$

■

In Proposition 22 we worked out an expression for $\pounds_X g_{ij}$ in terms of covariant derivatives instead of partial derivatives. The advantage of (4.14) over (4.13) is that the terms in the right hand side of (4.14) are components of tensor fields which is not true for the corresponding terms of (4.13).

In addition, we have that

$$\pounds_X g_{ij} \equiv 0, \tag{4.15}$$

from (4.13)

$$\xi^k g_{ij,k} + g_{kj}\xi^k_{,i} + g_{ik}\xi^k_{,j} = 0, \tag{4.16}$$

if and only if

$$K_{i;j} + K_{j;i} = 0, \tag{4.17}$$

so,

$$\pounds_X g_{ij} \equiv K_{i;j} + K_{j;i} = 0, \tag{Killing equations}$$

where

$$K_{i;j} = K_{i,j} - \Gamma^r_{ji}K_r. \tag{4.18}$$

Thus,

$$K_{i,j} + K_{j,i} - 2\Gamma^r_{ij} K_r = 0. \tag{4.19}$$

(4.17) or (4.19) is a system of $\frac{m(m+1)}{2}$ scalar, linear, homogeneous differential equations on m scalar functions $K_1, \ldots, K_m$. To put it into a standard form we introduce additional dependent variables K_{ij} by,

$$K_{ij} = K_{i;j}$$

which are subject to

$$K_{ij} + K_{ji} = 0. \tag{4.20}$$

Consequently,

$$K_{ij;k} = K_{i;jk}.$$

Finally, we arrive at the following system of equations

$$K_{i;j} = K_{ij}. \qquad \text{In other words, } \frac{\partial K_i}{\partial u^j} + ... \tag{4.21a}$$

$$K_{ij;k} = -K_r R^r_{kij}. \quad \text{In other words, } \frac{\partial K_{ij}}{\partial u^k} + ... \tag{4.21b}$$

To prove (4.21b) we differentiate covariantly (4.20). We obtain

$$K_{ij;k} + K_{ji;k} = 0. \tag{4.22}$$

If those equations are satisfied, so also are

$$K_{ij;k} + K_{ji;k} + K_{ik;j} + K_{ki;j} - (K_{jk;i} + K_{kj;i}) = 0.$$

From the Ricci identity it follows that

$$K_{ij;k} - K_{ik;j} = K_h R^h_{ijk}. \tag{4.23}$$

Using the preceding equations we have

$$\begin{aligned} 2K_{ij;k} &= K_{ij;k} - K_{ji;k} \\ &= (K_{ij;k} - K_{ik;j}) + (K_{jk;i} - K_{ji;k}) + (K_{kj;i} - K_{ki;j}) \\ &= -K_h \left(R^h{}_{ikj} + R^h{}_{jik} + R^h{}_{kij} \right) \end{aligned}$$

So

$$2K_{ij;k} + K_h \left(R^h{}_{ikj} + R^h{}_{jik} + R^h{}_{kij} \right) = 0.$$

In consequence of the first Bianchi identity these equations reduce to

$$K_{ij;k} = -K_h R^h{}_{kij}, \tag{4.24}$$

Thus (4.21b) holds.

Remark 7 [8]*If* $\pounds_X g = 0$ *and* $\pounds_Y g = 0$ *then* $\pounds_{[X,Y]} g = 0$. *So the set of Killing vector fields is a Lie algebra over* $\mathbb{R}$.

Remark 8 *An important consequence of Equation (4.24) is that the Killing vector field,* X, *is completely determined by the values of* X^i *(or* K_i*) and* K_{ij} *at any point* $p \in M$; *if we are given* X^i *(or* K_i*) and* K_{ij} *at* p, *then* X^i *and* K_{ij} *at any other point* q *are determined by integration of the system of ordinary differential equations*

$$X_i K_{i;j} = X_i K_{ij}$$
$$X_k K_{ij;k} = -K_h R^h_{kij} v^k$$

along any curve connecting p *and* q, *where* X_i *denotes the tangent vector to the curve. Two important consequences of this result are:*

- *If a Killing vector field* X *has* $X^i = 0$ *and* $K_{ij} = 0$ *at a point* p, *then* $X \equiv 0$ *everywhere.*[9]
- *On a manifold of dimension* m, *there can be at most* $\frac{m(m+1)}{2}$ *linearly independent Killing vector fields.*

[8] See [Kobayashi, 1963] p.238

[9] There is more detail in [Kramer et al., 1980] p.100

Now, we consider under what conditions equations (4.16), or their equivalent (4.17), admit one or more solutions. The following theorem helps us in this direction.

Theorem 23 [10] *When and only when M is a space of constant curvature, the equations of Killing admit solution involving $\frac{m(m+1)}{2}$ parameters; in all other cases there are fewer parameters.*

Proof. We assume that M is a space of constant curvature. (See Definition 12 and Remark 2)

From the Ricci identity (2.23) for two times covariant tensor K_{ij} we have

$$K_{ij;kl} - K_{ij;lk} = K_{hj}R^h{}_{ikl} + K_{ih}R^h{}_{jkl}. \tag{4.25}$$

On the other hand, differentiating (4.24) covariantly,

$$K_{ij;kl} = -K_{hl}R^h{}_{kij} - K_h R^h{}_{kij;l},$$

and

$$K_{ij;lk} = -K_{hk}R^h{}_{kij} - K_h R^h{}_{lij;k}$$

so

$$K_{ij;kl} - K_{ij;lk} = -K_{hl}R^h{}_{kij} + K_{hk}R^h{}_{lij} + K_h \left(R^h{}_{lij;k} - R^h{}_{kij;l}\right), \tag{4.26}$$

subtracting by sides (4.25) and (4.26) we get

$$K_h \left(R^h{}_{kij;l} - R^h{}_{lij;k}\right) + K_{hj}R^h{}_{ikl} + K_{ih}R^h{}_{jkl} + K_{hl}R^h{}_{kij} - K_{hk}R^h{}_{lij} = 0. \tag{4.27}$$

Since the space has constant curvature, the equations (4.27) which are the conditions of integrability of (4.21a) and (4.21b) are satisfied.

[10] [Eisenhart, 1933] p.215

If we write (4.24) in the form

$$\frac{\partial K_{ij}}{\partial u^k} = K_{hi}\Gamma^h_{ik} + K_{ih}\Gamma^h_{jk} - K_h R^h{}_{kij}, \tag{4.28}$$

and observe that by definition

$$\frac{\partial K_i}{\partial u^j} = K_h \Gamma^h_{ij} + K_{ij} \tag{4.29}$$

we see that a solution of (4.21a) and (4.21b) is a solution of the system of equations (4.28) and (4.29) in the $m\,(m+1)$ quantities K_i and K_{ij}. Since (4.20) are holding the general solution of above system involves $\frac{m(m+1)}{2}$ parameters.

Conversely, from (4.27) we have that

$$R^h{}_{kij;l} - R^h{}_{lij;k} = 0 \tag{4.30}$$

and from the other terms of (4.27) we have

$$K_{hp}\left(\delta^p_j R^h{}_{ikl} - \delta^p_i R^h{}_{jkl} + \delta^p_l R^h{}_{kij} - \delta^p_k R^h{}_{lij}\right) = 0,$$

from which, because of (4.20) it follows that

$$\delta^p_l R^h{}_{kij} - \delta^h_l R^p{}_{kij} + \delta^p_j R^h{}_{ikl} - \delta^h_j R^p{}_{ikl} - \delta^p_i R^h{}_{jkl} + \delta^h_i R^p{}_{jkl} - \delta^p_k R^h{}_{lij} + \delta^h_k R^p{}_{lij} = 0. \tag{4.31}$$

Contracting the indices for l and p, we have in consequence of (3.4) and the definition of Ricci curvature components,

$$\begin{aligned} mR^h{}_{kij} - R^h{}_{kij} - R^h{}_{kij} - \delta^h_j R_{ik} + R^h{}_{jik} + \delta^h_i R_{jk} + R^h{}_{ikj} &= 0 \\ (m-2)\,R^h{}_{kij} + \left(\delta^h_i R_{jk} - \delta^h_j R_{ik}\right) + R^h{}_{jik} + R^h{}_{ikj} &= 0 \end{aligned}$$

By Bianchi identity we have that $R^h{}_{jik} + R^h{}_{ikj} = R^h{}_{kij}$. Then,

$$R^h{}_{kij} = \frac{1}{(m-1)}\left(\delta^h_j R_{ik} - \delta^h_i R_{jk}\right) \tag{4.32}$$

Multiplying (4.32) by g_{hl} and summing over h we obtain

$$R_{lkij} = \frac{1}{(m-1)}\left(g_{jl}R_{ik} - g_{il}R_{jk}\right). \tag{4.33}$$

Multiplying (4.33) by g^{ki} and summing over k and i; results in

$$mR_{jl} = Rg_{jl},$$

so that from (4.33) we obtain

$$R_{lkij} = \frac{R}{m\,(m-1)}\left(g_{jl}g_{ik} - g_{il}g_{jk}\right),$$

where $R = g^{ij}R_{ij}$. Hence M is a space of constant curvature[11]. From (4.32) we get

$$R^{h}{}_{kij} = \frac{R}{m\,(m-1)}\left(\delta^h_j g_{ik} - \delta^h_i g_{jk}\right). \tag{4.34}$$

Then (4.31) is satisfied identically, as well as (4.30) a consequence of the fact that g_{ij} and g^{ij} are behave like constants under covariant differentiations[12]. Then the theorem follows. ■

From (4.27), using the natural isomorphisms $\flat$ and $\#$ for vectors and covectors, we get

$$K^r\left(R_{rkij;l} - R_{rlij;k}\right) + K^r_{;j}R_{rikl} + K^r_{;i}R_{rjkl} + K^r_{;l}R_{rkij} - K^r_{;k}R_{rlij} = 0. \tag{4.35}$$

We have that $R_{rkij;l} - R_{rlij;k} = R_{ijrk;l} - R_{ijrl;k}$ and from the Bianchi identity $R_{ijrk;l} + R_{ijlr;k} = -R_{ijkl;r}$. Then raising the index i we obtain

$$K^r R^i_{jlk;r} + K^r_{;j}R^i_{rkl} + K^{ri}_{;}R_{rjkl} + K^r_{;l}R^i_{jrk} + K^r_{;k}R^i_{jlr} = 0 \tag{4.36}$$

This equation (4.36) is nothing else than (see(4.12))

$$\pounds_{K^\#} R^i_{jlk} = 0, \tag{4.37}$$

where $K^\# = X$ is a contravariant vector resulting from applying the isomorphism $\#$ to K. Equation (4.37) is the first set of integrability conditions for (4.21a) and (4.21b). The others can be obtained in a similar form.

[11]When $R_{hijk} = \rho\left(g_{hk}g_{ij} - g_{hj}g_{ik}\right)$, ρ must to be a constant and M is called a space of constant curvature. See [Okubo, 1987] p.215

[12]See [Eisenhart, 1926] p.29

Now, we consider more closely the system (4.28) and (4.29) whose integrability conditions are equations (4.37). If these equations are satisfied identically with respect to K_i and K_{ij}, the system (4.29) is completely integrable (Frobenius' theorem, see Appendix C). Otherwise, we have a sequence of equations which must be compatible, if equations (4.24) are to have solutions. The following theorem due to L. P. Eisenhart gives us the guide to find solution of (4.21a) and (4.21b) providing us with the complete list of integrability conditions for the systems (4.21a) and (4.21b).

Theorem 24 *In order that a pseudo-Riemannian manifold M with metric $g = g_{ij}du^i \otimes du^j$ may admit a group of isometries, it is necessary and sufficient that there exists a positive integer N such that the first N sets of the equations*

$$\begin{aligned} \pounds_X g_{ij} &= 0, \\ \pounds_X R^i{}_{jlk} &= 0, \\ \pounds_X \boldsymbol{\nabla}_{m_1} R^i{}_{jlk} &= 0, \\ \pounds_X \boldsymbol{\nabla}_{m_2} \boldsymbol{\nabla}_{m_1} R^i{}_{jlk} &= 0, \end{aligned}$$

and so on are compatible in X^i and K_{ij}, and that X^i and K_{ij} satisfying these equations satisfy the $(N+1)-$ equations identically. If there are $\frac{m(m+1)}{2} - r$ linearly independent solutions in the first N set of equations except the first set of equations, then the solution of Killing equations depends on r parameters. (The dimension of the algebra of Killing is r.)

4.4.1 Examples

Flat Spaces

In the context of Riemannian or pseudo-Riemannian spaces, we say that the space is *flat* when the curvature tensor is zero. In $\mathbb{R}^N$, we define an inner product by

$(x \cdot y) = g_{ij}x^iy^j$. $(x \cdot y)$ is not necessarily positive definite, but it is nondegenerate. In spaces of constant curvature and in particular when the curvature is zero, the Killing equations are completely integrable and we can use a coordinate system $\{x^i\}$ in which g, the metric, is constant, i.e,

$$g = g_{ij}dx^idx^j,$$

where (g_{ij}) is equal to a constant. Then, it follows from (2.16) and (2.17) that ${R^h}_{ijk} = 0$. So it is zero in every coordinate system[13]. The covariant derivative, denoted by ";" and the partial derivative denoted by "," have in such coordinate system the same meaning, and the Killing equations read:

$$K_{i,j} + K_{j,i} = 0. \tag{4.38}$$

Consequently

$$K_{i,jk} = 0.$$

Therefore, K_i are linear functions of $x^{i\prime}s$. Since

$$K_{i,j} = A_{ij}$$

and from (4.38) we have

$$A_{ij} + A_{ji} = 0.$$

Therefore,

$$A_{ij} = -A_{ji} \quad \text{(antisymetry)}. \tag{4.39}$$

We obtain,

$$K_i = A_{ik}x^k + B_i,$$

where A_{ik} and B_i are arbitrary constants, restricted by (4.39). The vector field $X = K^\#$ is given by

$$\begin{aligned} X^i &= g^{il}K_l \\ X^i &= A^i_kx^k + B^i, \end{aligned}$$

[13][Eisenhart, 1933] p.188

where ${A^i}_k = g^{il} A_{lk}$, and $B^i = g^{il} B_l$, (remember that g^{il} are constants).

On the other hand, since Remark 7, the Killing vector fields form a Lie algebra over $\mathbb{R}$. We construct a basis for that algebra. We see that B^i and A^i_k are independent sets of constants. If we put $B^1 = 1$, $B^i = 0$ for all $i \neq 1$, and $A^i_k = 0$ we obtain $X_1 = \frac{\partial}{\partial x^1}$. By a similar method we obtain the other Killing vector fields,

$$X_2 = \frac{\partial}{\partial x^2}, \ldots, X_N = \frac{\partial}{\partial x^N}. \tag{4.40}$$

Next, because of (4.39) we construct first a basis for the vector space of $N \times N$ skew-symmetric matrices.

Since

$$A_{jk} = g_{jl} {A^l}_k,$$

then

$$g_{jl} A^l_k + g_{kl} {A^l}_j = 0$$

so, the basis can be chosen according to

$$A^{(\alpha,\beta)}_{jk} = \left(\delta^\alpha_j \delta^\beta_k - \delta^\alpha_k \delta^\beta_j \right), \text{ for } 1 \leq \beta < \alpha \leq N$$

that is one matrix $A^{(\alpha,\beta)}_{jk}$ for each ordered pair (α, β). Then each skew symmetric matrix can be written in the following form

$$A_{jk} = \sum_{1 \leq \beta < \alpha \leq N} A_{\alpha\beta} \left(\delta^\alpha_j \delta^\beta_k - \delta^\alpha_k \delta^\beta_j \right).$$

Further,

$$A^{i\ (\alpha,\beta)}_k = g^{i\alpha} \delta^\beta_k - g^{i\beta} \delta^\alpha_k.$$

For each ordered pair (α, β), and $B^i = 0$, we obtain the following Killing vector field.

$$\begin{aligned} X_{(\alpha,\beta)} &= x^k \left(g^{i\alpha} \delta^\beta_k - g^{i\beta} \delta^\alpha_k \right) \frac{\partial}{\partial x^i} \\ X_{(\alpha,\beta)} &= \left(x^\beta g^{i\alpha} - x^\alpha g^{i\beta} \right) \frac{\partial}{\partial x^i}, \quad \text{where } 1 \leq \beta < \alpha \leq N. \end{aligned} \tag{4.41}$$

Therefore, the Killing vector fields which form a basis of the Lie algebra of Killing vector fields have the form presented in (4.40) and (4.41). There are $\frac{N(N+1)}{2}$ of them.

Example 4 *If $N = 3$, and g is positive definite, then $g = dx^2 + dy^2 + dz^2$, in the global cartesian coordinates $\{x, y, z\}$. Thus,*

$$(g_{ij}) = \left(g^{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

According to (4.40) and (4.41) the Killing vector fields are:

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}. \tag{4.42}$$

and

$$\left(x^\beta g^{i\alpha} - x^\alpha g^{i\beta}\right) \frac{\partial}{\partial x^i}, \quad \textit{where } 1 \leq \beta < \alpha \leq 3.$$

If

$$\begin{aligned} \beta &= 1,\ \alpha = 2 \quad \textit{then} \quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \\ \beta &= 1,\ \alpha = 3 \quad \textit{then} \quad x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} \\ \beta &= 2,\ \alpha = 3 \quad \textit{then} \quad y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \end{aligned} \tag{4.43}$$

Now for those Killing vector fields it is interesting to see the one parameter groups of transformations generated by them. For example:

$$\text{for } \frac{\partial}{\partial x},$$

we obtain,

$$\begin{aligned} \frac{dx}{d\lambda} &= 1, \text{ then } x = \lambda + x_o \\ \frac{dy}{d\lambda} &= 0, \text{ then } y = y_o, \\ \frac{dz}{d\lambda} &= 0, \text{ then } z = z_o. \end{aligned}$$

where x_o, y_o and z_o are the coordinates of p and lambda is an affine parameter. The equations for x, y, z are called *the integral curves* of $\frac{\partial}{\partial x}$ passing through p. Also such equations represent a one parameter group of transformations, which assign to each value of λ a point transformation of $\mathbb{R}^3$. Next,

$$\text{for } x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

$$\frac{dx}{d\lambda} = -y, \quad \frac{dy}{d\lambda} = x,$$

so,

$$\frac{d^2x}{d\lambda^2} = -\frac{dy}{d\lambda} = -x.$$

Thus, the solution for this differential equation is

$$x = A\cos(\lambda) - B\sin(\lambda),$$

and

$$y = -\frac{dx}{d\lambda} = A\sin(\lambda) + B\cos(\lambda).$$

Still, we must express x and y in agreement with the initial conditions. We are looking for integral curves which for $\lambda = 0$ cross the point p. Then

$$x_o = A, \quad y_o = B.$$

Therefore the integral curves for $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and passing through p are

$$\begin{aligned} x &= x_o\cos(\lambda) - \ y_o\sin(\lambda), \\ y &= x_o\sin(\lambda) + y_o\cos(\lambda), \\ z &= z_o. \end{aligned}$$

So, for each value of λ we have a $(x(\lambda), y(\lambda), z(\lambda))$ which represent a point transformation of $\mathbb{R}^3$. Therefore, the Killing vector fields in (4.42) generate the well known *translations* and those in (4.43) generate the *rotations.* All of the one parameter groups of transformations are global.

Minkowski Space-Time

Minkowski space-time[14] is the manifold $M = \mathbb{R}^N$, $N > 1$ together with the metric

$$g = ds^2 = -{dx_1}^2 + \sum_{i=2}^{N} {dx_i}^2,$$

which is globally of hyperbolic signature. For that reason M is called a pseudo-Euclidean space. The geodesics of Minkowski space-time are the straight lines of the pseudo-Euclidean space $\mathbb{R}^N$.

We study the Killing vector fields and the one parameter groups of transformations that they generate in a Minkowski space-time of dimension 4, with coordinates $\{x, y, z, t\} \in \mathbb{R}^4$, and $g = dx^2 + dy^2 + dz^2 - c^2dt^2$. Therefore,

$$(g_{ij}) = \left(g^{ij}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c^2 \end{pmatrix}.$$

We obtain the Killing vector fields using the same arguments as in Section 4.4.1. Thus, the Killing vector fields forming a basis of the Lie algebra of Killing vector fields are:

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}. \tag{4.44}$$

and

$$\left(x^\beta g^{i\alpha} - x^\alpha g^{i\beta}\right) \frac{\partial}{\partial x^i}.$$

[14] [Beem et al., 1996], p. 174-179

where $1 \leq \alpha < \beta \leq 4$ and $(x^1, x^2, x^3, x^4) = (x, y, z, t)$. We obtain,

$$\begin{aligned}
\alpha &= 1,\ \beta = 2 \quad \text{then} \quad y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} && (4.45)\\
\alpha &= 1,\ \beta = 3 \quad \text{then} \quad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} && (4.46)\\
\alpha &= 2,\ \beta = 3 \quad \text{then} \quad z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} && (4.47)\\
\alpha &= 1,\ \beta = 4 \quad \text{then} \quad x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} && (4.48)\\
\alpha &= 2,\ \beta = 4 \quad \text{then} \quad y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y} && (4.49)\\
\alpha &= 3,\ \beta = 4 \quad \text{then} \quad z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} && (4.50)
\end{aligned}$$

The one parameter groups of transformations for Killing vector fields generated by (4.44), (4.45)-(4.47) have similar presentation as in Example 4. The only difference is in additional equations $t = t_o$. But, a one parameter groups generated by (4.48)-(4.50) have different form.

$$\text{For} \quad x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x},$$

we obtain,

$$\begin{aligned}
\frac{dx}{d\lambda} &= t, \qquad \frac{dt}{d\lambda} = x && (4.51)\\
\frac{dy}{d\lambda} &= 0,\ \text{then } y = y_o, && (4.52)\\
\frac{dz}{d\lambda} &= 0,\ \text{then } z = z_o. && (4.53)
\end{aligned}$$

The solutions are represented by means of hyperbolic functions. From (4.51) we obtain

$$\frac{d^2x}{d\lambda^2} = \frac{dt}{d\lambda} = x,$$

then we find that

$$\begin{aligned}
x &= x_o \cosh(\lambda) + t_o \sinh(\lambda), && (4.54)\\
t &= x_o \sinh(\lambda) + t_o \cosh(\lambda). && (4.55)
\end{aligned}$$

Thus the corresponding integral curves are represented by equations (4.52)-(4.55).

A special case of transformations in Minkowski space-time are Lorentz transformations. We recall hyperbolic trigonometric identity

$$\cosh^2(\lambda) - \sinh^2(\lambda) = 1.$$

Let

$$\beta = \tanh(\lambda).$$

Now if $\lambda \to \infty$ then $\beta \to -1$, and if $\lambda \to -\infty$ then $\beta \to 1$. So, since $\tanh(\lambda)$ is a decreasing function of λ, $-1 < \beta < 1$.

Next,

$$\begin{aligned} \beta^2 &= \frac{\sinh^2(\lambda)}{\cosh^2(\lambda)} = \frac{\cosh^2(\lambda) - 1}{\cosh^2(\lambda)}, \\ \text{and, } \cosh(\lambda) &= \frac{1}{\sqrt{1-\beta^2}}, \sinh(\lambda) = \frac{\beta}{\sqrt{1-\beta^2}} \end{aligned}$$

Then (4.54) and (4.55) become the so called *Lorentz transformations.*

$$\begin{aligned} x &= \frac{x_o + \beta t_o}{\sqrt{1-\beta^2}} = \gamma(x_o + \beta t_o) \\ t &= \frac{t_o + x_o\beta}{\sqrt{1-\beta^2}} = \gamma(t_o + x_o\beta). \end{aligned}$$

4.4.2 Killing Vector Fields and Geodesics

To infer an explicit form of Killing vectors we have to integrate Killing equations. Some simplification of that process can be obtained if we know nontrivial geodesic vector fields. The following proposition is an useful tool for that.

Proposition 25 *Let X be a Killing vector field and let Z be a geodesic vector field, i.e. $\nabla_Z Z = 0$. Then*

$$Zg(Z, X) = 0, \tag{4.56}$$

that is, $g(Z, X)$ is constant along any geodesic γ which is an integral curve of Z.

Proof. *Indeed, using the components of Z and X we have*

$$\begin{aligned}
Z^i \boldsymbol{\nabla}_i \left(Z^m X_m\right) &= \left(Z^i \boldsymbol{\nabla}_i Z^m\right) X_m + Z^m Z^i \boldsymbol{\nabla}_i X_m \\
&= Z^m Z^i \boldsymbol{\nabla}_i X_m \\
&= -Z^i Z^m \boldsymbol{\nabla}_m X_i \\
&= -Z^m \boldsymbol{\nabla}_m \left(Z^i X_i\right) + \left(Z^m \boldsymbol{\nabla}_m Z^i\right) X_i \\
&= -Z^m \boldsymbol{\nabla}_m \left(Z^i X_i\right).
\end{aligned}$$

Hence

$$Zg\left(Z, X\right) = Z^i \boldsymbol{\nabla}_i \left(Z^m X_m\right) = 0$$

■

Remark 9 *We have even stronger assertion that a vector field X on M defines an "infinitesimal isometry" if and only if the inner product of X and a unit vector tangent to a geodesic in M is constant along the geodesic. ([Okubo, 1987], p. 573)*

That fact has an important application in physics. Then in the global coordinate system $\{x, y, z, t\}$

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 - c^2 (dt)^2.$$

Now we consider a timelike geodesic. Such a geodesic can be parameterized by t or by the proper time τ, which is defined by the equation

$$c^2 d\tau^2 = -ds^2.$$

Along the geodesic,

$$ds^2 = \left(\frac{dx}{dt}\right)^2 dt^2 + \left(\frac{dy}{dt}\right)^2 dt^2 + \left(\frac{dz}{dt}\right)^2 dt^2 - c^2 dt^2.$$

But $\mathbf{v}^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$ and then,

$$\begin{aligned} ds^2 &= \left(-c^2 + \mathbf{v}^2\right)(dt)^2 \\ ds^2 &= -c^2\left(1 - \frac{\mathbf{v}^2}{c^2}\right)(dt)^2 = -c^2 d\tau^2. \\ \text{Hence,} \qquad d\tau &= \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\, dt, \end{aligned}$$

if we require $d\tau > 0$ for $dt > 0$. Let $X = \frac{\partial}{\partial x^4}$ and let Z be a vector tangent to that particular timelike geodesic. Then along this geodesic

$$g(X, Z) = \frac{dt}{c d\tau} = A_1$$

where A_1 is a constant. Therefore,

$$\frac{dt}{d\tau} = A_1 c,$$

so,

$$\begin{aligned} \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} &= A_1 c \\ \frac{mc^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} &= A_1 mc^3 \end{aligned} \tag{4.57}$$

where c is the velocity of light. Then (4.57) tells us that the total energy for a particle of mass m is conserved

Now, let $X = \frac{\partial}{\partial x^1}$ and let Z be as before. Then

$$g(X, Z) = \frac{dx}{c d\tau} = A_2$$

where A_2 is a constant. Then,

$$\frac{dx}{d\tau} = A_2 c.$$

So,

$$\begin{aligned} \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \frac{dx}{dt} &= A_2 c \\ \frac{m\mathbf{v}^1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} &= A_2 mc \end{aligned} \tag{4.58}$$

where c is the velocity of light. Equation (4.58) tells us that the momentum of a particle of mass m in the x direction is conserved. Next, we can see that for $X = \frac{\partial}{\partial x^2}$ or $\frac{\partial}{\partial x^3}$, the equations are similar. So $g\left(\frac{\partial}{\partial x^1}, X\right)$, $g\left(\frac{\partial}{\partial x^2}, X\right)$, $g\left(\frac{\partial}{\partial x^3}, X\right)$ tell us that the momentum of a particle of mass m which moves at velocity $\mathbf{v}$ is conserved. In this way, the inner product of each Killing vector field with a vector tangent to a geodesic defines a physical quantity of a particle, which is conserved in time.

4.5 Null Tetrad formalism

Now, we consider the case for a four-dimensional pseudo-Riemannian manifold, with matric tensor g of hyperbolic signature $+++-$. That fact implies that locally there exist four pointwise linearly independent vector fields $\{e_1, e_2, e_3, e_4\}$,where e_3, e_4 are real vectors and e_2 is a complex conjugate of e_1.

Definition 18 *The null tetrad* $\{e_a\}, a = 1, ..., 4$ *are four vector fields in which*

$$g(e_a, e_b) = g_{ab},$$

$$(g_{ab}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The tetrad vectors

$$e_b = e_b^\nu \frac{\partial}{\partial u^\nu} \tag{4.59}$$

determine the linear differential forms

$$e^a = e_\mu^a du^\mu \tag{4.60}$$

where $\mu = 1, ..., 4$. $\{e^1, e^2, e^3, e^4\}$ is an 1-forms basis, and $\{e_1, e_2, e_3, e_4\}$ is its dual basis. Their members are called *the null tetrad 1-forms* and *the null tetrad vector fields* respectively. They possess this property:

$$\mathsf{g}(e_1, e_2) = \mathsf{g}(e_2, e_1) = 1 = \mathsf{g}(e_3, e_4) = \mathsf{g}(e_4, e_3)$$

and otherwise

$$\mathsf{g}(e_i, e_j) = 0, \ i, j = 1, ..., 4$$

The tensor g, is sometimes called the line element ds^2. In terms of the null tetrad 1-forms the metric is given by:

$$ds^2 = 2\left(e^1 e^2 + e^3 e^4\right).$$

The scalar product of two vectors X, $Y \in M_p$ is given by

$$X \cdot Y = g_{ab} \xi^a \eta^b,$$

where ξ^a, η^b are the components of X and Y respectively. Two vectors are *orthogonal* if their scalar product vanishes. A non-zero vector X is said to be *spacelike, timelike,* or *null,* respectively, when the product $X \cdot X = g_{ab}\xi^a\xi^b$ is positive, negative or zero. In a coordinate basis, we write the line element ds^2 as

$$ds^2 = g_{ij} du^i du^j.$$

In space-time[15], an *orthonormal basis* $\{E_\alpha\}$ consists of three spacelike vectors E_a and one timelike vector $E_4 \equiv t$, such that

$$\{E_\alpha\} = \{E_a, t\} = \{E_1, \ E_2, \ E_3, \ t\},$$

$$\begin{aligned} g_{ab} \quad &= \quad E_{1a}E_{1b} + E_{2a}E_{2b} + E_{3a}E_{3b} - t_{4a}t_{4b} \\ &\Longleftrightarrow \quad E_a \cdot E_b = \delta_{ab}, \ t \cdot t = -1, \ E_a \cdot t = 0. \end{aligned}$$

[15] This is a common term used by physicist to call a Pseudo Riemannian four- dimensional differentiable(C^∞, Hausdorff) manifold (M, g) of signature $(1,3)$ [i.e., diag$(- + ++)$]. See [Beem et al., 1996] p.25

We also illustrate the action of the natural isomorphism $\flat$ on the vector fields e_i. We have that

$$(e_j)^\flat = g(\cdot, e_j).$$

So,

$$\left\langle (e_1)^\flat, e_i \right\rangle = g(e_i, e_1) = g_{i1} = \begin{cases} 1, & : \text{ if } i = 2 \\ 0, & : \text{ otherwise.} \end{cases}$$

Thus,

$$(e_1)^\flat = e^2.$$

In the same way we obtain

$$(e_2)^\flat = e^1, \quad (e_3)^\flat = e^4, \quad (e_4)^\flat = e^3.$$

We can get the reciprocal equalities using $\#$, the inverse of $\flat$. In tensor components, we have that

$$e^3_\mu = e^\nu_4 g_{\mu\nu}\ , \quad e^\nu_4 = g^{\mu\nu} e^3_\mu,$$

and so on.

Now we present the components of Ricci and Weyl tensors in terms of null tetrad. We can contract $(Ric \otimes e_a \otimes e_b)$ and the tetrad components of the Ricci tensor R_{ab} are

$$R_{ab} = R_{\mu\nu} e^\mu_a e^\nu_b \tag{4.61}$$

If we do the same with the Weyl tensor, we obtain

$$C_{abcd} = C_{\alpha\beta\mu\nu} e^\alpha_a e^\beta_b e^\mu_c e^\nu_d, \tag{4.62}$$

$$C_{\alpha\beta\mu\nu} = C_{abcd} e^a_\alpha e^b_\beta e^c_\mu e^d_\nu. \tag{4.63}$$

4.6 Algebraic Properties of the Weyl Tensor

The analysis of an algebraic structure of the Riemann tensor as a linear mapping was first developed by Petrov [Kramer et al., 1980]. It was reduced to certain eigenvalue problem so he could give invariant characterization of the Weyl tensor at a point. We shall not present the details of his work here. Instead we call some of the conclusions. We provide them in terms of (tetrad or tensor) components of the corresponding object. In general, there exist precisely four distinct null directions (i.e., nontrivial null vectors l^a, defined up to scaling $l^a \rightarrow \lambda l^a$) which satisfy the relation

$$l^b l^c l_{[f} C_{a]bc[d} l_{g]} = 0, \tag{4.64}$$

where C_{abcd} is the Weyl tensor defined by equation of (3.7). Those null directions are called *principal null directions* of the Weyl tensor. In fact, every nontrivial tensor satisfying the algebraic conditions of the Weyl tensor possesses, in general, four principal null directions.

In special cases some of these null directions coincide (in which case they satisfy stronger relations than (4.64), resulting in fewer than four principal null directions. Weyl tensor is said to be *algebraically special* if it admits at least one *multiple* principal null direction.

Algebraic Classification of Weyl Tensor C$_{abcd} \neq 0$			
Type	**Description**	**Conditions satisfied by multiple principal null direction l^a**	
I	Algebraically general; four distinct principal null directions	$l^b l^c l_{[f} C_{a]bc[d} l_{g]} = 0$	$\nwarrow \nearrow$ $\swarrow \searrow$
II	Two of principal null directions coincide	$l^b l^c C_{abc[d} l_{f]} = 0$	$\uparrow\uparrow$ $\swarrow \searrow$
D	Two pairs of principal null directions coincide	$l^b l^c C_{abc[d} l_{f]} = 0$ (two solutions)	$\uparrow\uparrow$ $\searrow\searrow$
III	Three principal null directions coincide	$l^c C_{abc[d} l_{f]} = 0$	$\uparrow\uparrow\uparrow$ $\searrow$
N	All four principal null directions coincide	$l^c C_{abcd} = 0$	$\uparrow\uparrow\uparrow\uparrow$

Chapter 5

Special Class of Metrics and their Killing Vector Fields

5.1 Introduction

We present a method of studying the nontrivial solutions of Killing equations. The method is applied to a class of pseudo-Riemannian structures that depends on two arbitrary holomorphic[1] functions of one complex variable. Some constraints on these functions arise as a consequence of the existence of nontrivial Killing vector fields. First of all we present the pseudo-Riemannian structure, i.e., the metric tensor, and some characterization of the Ricci and the Weyl tensors, then, we present and apply the method.

[1]They are functions defined on an open subset of the complex number plane $\mathbb{C}$ with values in $\mathbb{C}$ that are complex-differentiable at every point. This condition implies that the function are infinitely often differentiable and can be described by their Taylor series.

5.2 Metric Tensor

The class of metric used in this work is a generalization of the metrics presented in [Plebański and Rózga, 2002]. It is a class of pseudo-Riemannian metrics g on a space-time manifold M with signature $+++-$. M is given by

$$M = \left\{(t,\ r,\ y) \in \mathbb{R}^2 \times \mathbb{C},\ y \in \mathcal{O} \subset \mathbb{C},\ W > 0\right\},$$

where

$$W = \left\{\left|r + \bar{A}_{,\bar{y}}\right|^2 - |H|^2 \exp\left[-\frac{1}{C}\left(2t + \bar{y}A + y\bar{A}\right)\right]\right\}. \tag{5.1}$$

and $\mathcal{O} \subset \mathbb{C}$ is a domain of two holomorphic functions H and A. It has to be added that, g belongs to a general class detailed in [Plebański, 1978].

In [Plebański and Rózga, 2002] there is employed a coordinate system $\{u^1, u^2, u^3, u^4\} = \{r, t, y, \overline{y}\}$ where $\overline{y}$ is the complex conjugate of y. The metric tensor is

$$\begin{aligned} \mathsf{g} &= 2\left(e^1 e^2 + e^3 e^4\right) && (5.2)\\ \text{or}\quad \mathsf{g} &= \mathsf{g}_{\mu\nu} du^\mu du^\nu, && (5.3) \end{aligned}$$

where

$$\begin{aligned} e^1 &:= (r + A_{,y})\, dy + \left[\overline{H} \exp\left(-\frac{t + \overline{A}}{C}\right)\right] d\overline{y} \\ e^2 &:= \overline{e^1} \\ e^3 &:= dt + A d\overline{y} + \overline{A} dy \\ e^4 &:= -dr \end{aligned} \tag{5.4}$$

A, H are holomorphic function of the complex variable $y \in \mathcal{O}$ and $\overline{A}$, $\overline{H}$ their complex conjugate functions respectively. $C \neq 0$ is a real parameter and e^2 is the complex conjugate of e^1. (Complex conjugation is to be denoted by a bar).

The range of coordinates $\{r, t, y, \bar{y}\}$ is restricted by the condition:

$$W > 0, \tag{5.5}$$

where W is defined by (5.1). This is a physical condition[2]. The two additional conditions we impose are,

$$e^3 \wedge de^3 \neq 0, \tag{5.6}$$

and

$$H \neq 0, \text{ everywhere.} \tag{5.7}$$

The former condition (5.6) tells us that e^3 can not be expressed as a multiple of the differential. It is equivalent to

$$\bar{A}_{,\bar{y}} - A_{,y} \neq 0, \text{ for all } y \in \mathcal{O}. \tag{5.8}$$

And the latter condition (5.7) guarantees that the curvature tensor and in particular the Weyl tensor are everywhere nontrivial.

5.2.1 Geometric Properties

One of the properties of the null tetrad vector fields $\{e_1, e_2, e_3, e_4\}$ is the geometric one related to the covariant derivatives. We verify it using Maple 7®,

$$\boldsymbol{\nabla}_{e_3} e_i = 0 = \boldsymbol{\nabla}_{e_4} e_i, \tag{5.9}$$

for $1 \leq i \leq 4$. As a result of (5.9), the vector fields e_3 and e_4 are geodesic and the family of tangent vectors corresponding to each e_1, e_2, e_3, e_4 is parallel along the integral curves of e_3 and e_4 (see 2.7). In particular, the vector fields e_3 and e_4 span the tangent space of a totally geodesic two dimensional surface. Also, in the coordinate system $\{r, t, y, \overline{y}\}$

$$\begin{aligned} e_3 &= \frac{\partial}{\partial t}, \\ e_4 &= -\frac{\partial}{dr}. \end{aligned}$$

[2]See [Plebański and Rózga, 2002] pg. 6037

If X is a Killing vector field, then $e_3 g(e_3, X) = 0$ and $e_4 g(e_4, X) = 0$, see (4.56). Thus $g(e_3, X)$ and $g(e_4, X)$ represent physical quantities which are conserved, when a particle moves along the corresponding geodesic.

5.2.2 Ricci and Weyl Tensors

In addition, this metric possesses the properties related to the curvature, to be specific, the properties of the Ricci and Weyl tensors.

We find , using Maple 7® that the Ricci curvature can be written in a form of

$$Ric = -\rho e^3 \otimes e^3. \tag{5.10}$$

Its components in local coordinates are,

$$R_{\mu\nu} = -\rho e^3_\mu e^3_\nu,$$

and its components with respect to $\{e_1, e_2, e_3, e_4\}$ are

$$\begin{aligned} R_{ab} &= Ric(e_a, e_b) \\ &= -\rho e^3(e_a) e^3(e_b) \\ &= -\rho \delta^3_a \delta^3_b. \\ R_{33} &= -\rho \end{aligned}$$

(The only nonzero null tetrad component is $R_{33} = -\rho$). Thus the 1-form $e^3 = e^3_\mu du^\mu$ (or equivalently the vector field $e_4 = e^\mu_4 \frac{\partial}{\partial u^\mu}$) is distinguished by the Ricci curvature.

Also, we have that the Weyl tensor is of an algebraic type N, i.e., the Weyl tensor is not trivial and fulfills the condition

$$C(X, Y, Z, e_4) = 0, \qquad \text{for all } X, Y, Z \in M_p. \tag{5.11}$$

The only algebraically independent components of the Weyl tensor and of the Ricci tensor, with respect to the basis $\{e_1, e_2, e_3, e_4\}$, which is dual of the one-forms basis$\{e^1, e^2, e^3, e^4\}$, are respectively[3]

$$C_{3131} = -\frac{H\left(r + \bar{A}_{,\bar{y}}\right)\exp\left[-\frac{1}{C}\left(t + \bar{y}A\right)\right]}{C^2W},$$

and

$$R_{33} = \frac{-2H\bar{H}\exp\left[-\frac{1}{C}\left(2t + y\bar{A} + \bar{y}A\right)\right]}{C^2W}.$$

5.3 Integration of Killing Equations

We apply the Proposition 25 to the metric tensor (5.2), which have the property that the vector field Z of the form

$$Z = a\frac{\partial}{\partial r} + b\frac{\partial}{\partial t},$$

where a, b are arbitrary constants, is geodesic.

We obtain the following equations on the first two components of the Killing one-form K, where $K = g(X, \cdot)$, for X a Killing vector field,

$$\begin{aligned} \frac{\partial K_1}{\partial r} &= 0 \\ \frac{\partial K_2}{\partial t} &= 0 \\ \frac{\partial K_2}{\partial r} + \frac{\partial K_1}{\partial t} &= 0. \end{aligned}$$

That is equivalent to,

$$K_1 = t\ q(y, \bar{y}) + \varphi_1(y, \bar{y}) \tag{5.12}$$

$$K_2 = -r\ q(y, \bar{y}) + \varphi_2(y, \bar{y}) \tag{5.13}$$

[3]For a detailed explanation about components of the Weyl tensor and the uniqueness of the direction of e_4 in Equation 5.11 see Appendix B.

where the functions q, φ_1, φ_2 are real-valued functions of y, $\bar{y}$. There are no restrictions on K_3, K_4 which are complex-valued functions of all four variables and K_4 is the complex conjugate of K_3. Now, after we have established the dependency on r and t of K_1 and K_2, we look at the remaining seven out of the ten Killing equations. First we concentrate on the four Killing equations which involve the derivatives of K_3 and K_4 with respect to r and t. They have the form:

$$K_{3;1} = -K_{1;3}, \tag{5.14}$$

$$K_{3;2} = -K_{2;3}, \tag{5.15}$$

$$K_{4;1} = -K_{1;4}, \tag{5.16}$$

$$K_{4;2} = -K_{2;4}. \tag{5.17}$$

In addition, (5.16) and (5.17) are complex conjugates of (5.14) and (5.15) respectively. From (4.18) we obtain an explicit form of those equations. So, (5.14) and (5.15) read

$$\begin{aligned}
\frac{\partial K_3}{\partial r} - 2\Gamma^3_{13}K_3 - 2\Gamma^4_{13}K_4 &= 2\Gamma^1_{31}K_1 + 2\Gamma^2_{13}K_2 - \frac{\partial K_1}{\partial y}, \\
\frac{\partial K_3}{\partial t} - 2\Gamma^3_{23}K_3 - 2\Gamma^4_{23}K_4 &= 2\Gamma^1_{23}K_1 + 2\Gamma^2_{23}K_2 - \frac{\partial K_2}{\partial y}.
\end{aligned}$$

Where K_1 and K_2 can be substituted from (5.12) and (5.13).

On the other hand, we use the Ricci Identity (2.23) to obtain integrability conditions of the system (5.14)-(5.17) and the result is the following system of linear algebraic equations on K_3 and K_4 :

$$\begin{aligned}
R^3_{312}K_3 + R^4_{312}K_4 = -R^1_{312}K_1 - R^2_{312}K_2 \\
+K_{1;32} - K_{2;31},
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
R^3_{412}K_3 + R^4_{412}K_4 = -R^1_{412}K_1 - R^2_{412}K_2 \\
+K_{1;42} - K_{2;41}.
\end{aligned} \tag{5.19}$$

The matrix of the above system turns out to be nonsingular. Therefore we can obtain from it a unique expressions for K_3 and K_4. In that way one gets an explicit dependence of $K_1, ..., K_4$ on r and t. Still however, we have to make sure that those functions satisfy (5.14)-(5.17). And so, we substitute them back into (5.14)-(5.17), and we get a system of equations involving the functions of y and $\bar{y}$: q, φ_1, φ_2.

Also we obtain that (5.14) and (5.16) are identically satisfied while (5.15) and (5.17) are not. The numerators of Killing equations involving $K_{2;3}$ and $K_{2;4}$ are polynomial functions of degree ≤ 4 in r, whose coefficients are expressed in terms of q and φ_2. Exploring that fact we infer that φ_2 and q must be constants, and these are the only conditions from (5.15) and (5.17).

Now, we pass through a similar process with the remaining three Killing equations

$$\begin{aligned} K_{3;3} &= 0 \\ K_{4;3} + K_{3;4} &= 0 \\ K_{4;4} &= 0. \end{aligned}$$

These equations result in some conditions on φ_1, H and A.

We can obtain and integrate the constraints on q, φ_1, H and A applying a separation of variables arguments, due to the simple dependencies of Killing equations on r and t. We have used the symbolic computation program Maple$^{\circledR}$7 to execute a quite long computation.

The results are discussed in the next section.

5.4 Results

Theorem 26 *The nontrivial Killing vector fields for the metric (5.2) exist in the following cases. In each of these cases the corresponding Lie algebra of Killing vectors fields is one-dimensional.*

5.4.1 Case 1

$$A,_{yyy} = -i\mu\,(A,_{yy})^2$$

and

$$A,_{yy} \neq 0 \neq \mu.$$

Then,

$$\begin{aligned}
q &= 0\\
A &= \frac{-i\left[(\mu y - i\mathsf{v})ln(\mu y - i\mathsf{v}) + y\mu(\mathsf{a}-1) + \mathsf{b}\mu\right]}{\mu^2}\\
H,_y &= \frac{(y\mathsf{s} - 2i\mu C + A\bar{\mathsf{v}} + \beta)}{C\ i\ (\mu y - i\mathsf{v})}H,\\
\varphi_1 &= \alpha\Psi \qquad \text{where,}\\
\Psi &= \left[\bar{y}y + (i\mu A - \mathsf{s})\ \bar{y} - \left(i\mu\ \bar{A} + \bar{\mathsf{s}}\right) y - \mathsf{v}\bar{A} - \bar{\mathsf{v}}A - \beta\right]\\
\varphi_2 &= \alpha, \qquad \text{and} \quad \alpha \neq 0
\end{aligned}$$

where α, β, μ are real and s , v, a, b are complex constants.

- For this case we get the following contravariant components for the Killing vector field:

$$\begin{aligned}
K^1 &= -\alpha\\
K^2 &= \frac{\alpha\left[\left(A,_y - \bar{A},_{\bar{y}}\right)\Psi + \bar{A}\Psi,_{\bar{y}} - A\Psi,_y\right]}{\bar{A},_{\bar{y}} - A,_y}\\
K^3 &= \frac{\alpha\Psi,_{\bar{y}}}{A,_y - \bar{A},_{\bar{y}}}\\
K^4 &= \frac{\alpha\Psi,_y}{\bar{A},_{\bar{y}} - A,_y}
\end{aligned}$$

5.4.2 Case 2

$$\mu = 0$$

and

$$A_{,yy} \neq 0$$

then,

$$\begin{aligned}
A_{,yyy} &= 0 \\
q &= 0 \\
A &= \frac{y^2}{2\mathsf{v}} + 2\ \mathsf{a}_2\ y + \mathsf{a}_1 \\
H_{,y} &= H\left(\frac{y\bar{\mathsf{s}} + A\bar{\mathsf{v}} + \beta}{C\mathsf{v}}\right) \\
H &:= \mathsf{h}\exp\left(\frac{y^2\bar{\mathsf{s}} + 2\bar{\mathsf{v}}\int A dy + 2\beta y}{2\mathsf{v}C}\right) \\
Where,\ \mathsf{s} &= -2\mathsf{a}_2\mathsf{v} \ \text{ and } \int A dy = \frac{y^3}{6\mathsf{v}} + \mathsf{a}_2\ y^2 + \mathsf{a}_1 y \\
\varphi_1 &= \alpha\ \left(\bar{y}\ y - \mathsf{s}\bar{y}\ - \bar{\mathsf{s}}y - \ \mathsf{v}\bar{A} - \bar{\mathsf{v}}A - \beta\right) \\
\varphi_2 &= \alpha, \qquad \alpha \neq 0
\end{aligned}$$

where α, β is real a_2, a_1,v, h are arbitrary complex constants.

- For this case we get the following contravariant components to Killing vector

$$\begin{aligned}
K^1 &= -\alpha \\
K^2 &= -\alpha(y\bar{y} + 2y\bar{\mathsf{a}}_2\bar{\mathsf{v}} + 2\bar{y}\mathsf{a}_2\mathsf{v} - \beta) \\
K^3 &= \mathsf{v}\alpha \\
K^4 &= \bar{\mathsf{v}}\alpha
\end{aligned}$$

5.4.3 Case 3

$$\begin{aligned}
A_{,yy} &= 0, \quad \text{and} \\
\bar{A}_{,\bar{y}} - A_{,y} &\neq 0,
\end{aligned}$$

then,

$$
\begin{aligned}
q &= 0 \\
A &= \mathsf{a}_2\ y + \mathsf{a}_1 .\text{Where, } \ \mathsf{a}_2 \neq \bar{\mathsf{a}}_2 \\
H_{,y} &= \frac{H\left\{\mathsf{k}y + \mathsf{s}\right\}}{C(y + \mathsf{b})}, \\
H &= \mathsf{h}\exp\left\{\frac{\mathsf{k}y + (\mathsf{k} - \mathsf{s}\mathsf{b})\ln(y + \mathsf{b})}{C}\right\}, \\
\text{where, } \mathsf{s} &= (\bar{\mathsf{a}}_2 - \mathsf{a}_2)\,\omega - \bar{\mathsf{a}}_1\mathsf{b} - 2C, \\
\mathsf{k} &= \bar{\mathsf{b}}(\bar{\mathsf{a}}_2 - \mathsf{a}_2) - \bar{\mathsf{a}}_1 . \\
\varphi_{1,yy} &= 0,\ \ \varphi_{1,\bar{y}\bar{y}} = 0, \quad \varphi_{1,y\bar{y}} = \alpha, \quad \alpha \neq 0 \\
\varphi_1 &= \alpha(\bar{y}\ y + \bar{\mathsf{b}}y + \mathsf{b}\bar{y} + \omega), \\
\varphi_2 &= 0\ .
\end{aligned}
$$

where α, ω are real constants, a_2, a_1,s, h, b, k are complex constants.

- For this case we get the following contravariant components to Killing vector

$$
\begin{aligned}
K^1 &= 0 \\
K^2 &= \frac{\alpha\left[(\bar{\mathsf{a}}_1 - \bar{\mathsf{a}}_2\bar{\mathsf{b}})y + (\mathsf{a}_2\mathsf{b} - \mathsf{a}_1)\bar{y} + \bar{\mathsf{a}}_1\mathsf{b} - \mathsf{a}_1\bar{\mathsf{b}}\right]}{(\bar{\mathsf{a}}_2 - \mathsf{a}_2)} - \alpha\omega \\
K^3 &= \frac{\alpha\,(y + \mathsf{b})}{(\mathsf{a}_2 - \bar{\mathsf{a}}_2)} \\
K^4 &= \frac{\alpha\left(\bar{y} + \bar{\mathsf{b}}\right)}{(\bar{\mathsf{a}}_2 - \mathsf{a}_2)}
\end{aligned}
$$

5.4.4 Case 4

$$\varphi_{1,y\bar{y}} = 0$$

then,

$$\begin{aligned}
q &= 0 \\
A &= \mathsf{a}_2\, y + \mathsf{a}_1 \\
H_{,y} &= \frac{H\,[\mathsf{v}y + \mathsf{w}]}{C\,(1+i\beta)} \\
H &= \mathsf{h}\exp\left\{\frac{y\,(\mathsf{v}y + 2\mathsf{w})}{2C\,(1+i\beta)}\right\} \\
\text{where,}\;\; \mathsf{w} &= \eta\,(\bar{\mathsf{a}}_2 - \mathsf{a}_2) - (1+i\beta)\,\bar{\mathsf{a}}_1 \\
\mathsf{v} &= (1-i\beta)\,(\bar{\mathsf{a}}_2 - \mathsf{a}_2) \\
\varphi_{1,yy} &= 0,\;\; \varphi_{1,\bar{y}\bar{y}} = 0. \\
\varphi_1 &= \alpha\,[(1+i\beta)\,\bar{y} + (1-i\beta)\,y + \eta]\,, \quad \alpha \neq 0 \\
\varphi_2 &= 0.
\end{aligned}$$

where α, β, η are real constants, v, w, a_2, a_1, h, are complex constants.

- For this case we get the following contravariant components to Killing vector

$$\begin{aligned}
K^1 &= 0 \\
K^2 &= \alpha\frac{\mathsf{a}_2\,(1+i\beta)\,\bar{y} - \bar{\mathsf{a}}_2\,(1-i\beta)\,y + \eta(\mathsf{a}_2 - \bar{\mathsf{a}}_2)}{(\bar{\mathsf{a}}_2 - \mathsf{a}_2)} \\
&\quad +\alpha\frac{\bar{\mathsf{a}}_1\,(1+i\beta) - \mathsf{a}_1\,(1-i\beta)}{(\bar{\mathsf{a}}_2 - \mathsf{a}_2)} \\
K^3 &= \frac{\alpha\,(1+i\beta)}{(\mathsf{a}_2 - \bar{\mathsf{a}}_2)} \\
K^4 &= \frac{\alpha\,(1-i\beta)}{(\bar{\mathsf{a}}_2 - \mathsf{a}_2)}
\end{aligned}$$

5.4.5 Case 5

$$\varphi_{1,y\bar{y}} = 0$$

then,

$$
\begin{aligned}
q &= 0 \\
A &= \mathsf{a}_2\, y + \mathsf{a}_1 \\
H_{,y} &= \frac{H\,[\mathsf{v}y + \mathsf{w}]}{iC} \\
H &= \mathsf{h}\exp\left\{\frac{y\,(\mathsf{v}y + 2\mathsf{w})}{2iC}\right\} \\
\text{where, }\ \mathsf{w} &= \eta\,(\bar{\mathsf{a}}_2 - \mathsf{a}_2) - i\bar{\mathsf{a}}_1 \\
\mathsf{v} &= i\,(\mathsf{a}_2 - \bar{\mathsf{a}}_2) \\
\varphi_{1,yy} &= 0,\ \ \varphi_{1,\bar{y}\bar{y}} = 0. \\
\varphi_1 &= \alpha\,[i\bar{y} - iy + \eta]\,,\ \ \alpha \neq 0 \\
\varphi_2 &= 0
\end{aligned}
$$

where α, η are real constants, v, w, a_2, a_1, h, are complex constants.

- For this case we get the following contravariant components to Killing vector

$$
\begin{aligned}
K^1 &= 0 \\
K^2 &= \frac{\alpha\,[i\mathsf{a}_2\bar{y} + i\bar{\mathsf{a}}_2 y + \eta(\mathsf{a}_2 - \bar{\mathsf{a}}_2) + i\bar{\mathsf{a}}_1 + i\mathsf{a}_1]}{(\bar{\mathsf{a}}_2 - \mathsf{a}_2)} \\
K^3 &= \frac{i\alpha}{(\mathsf{a}_2 - \bar{\mathsf{a}}_2)} \\
K^4 &= \frac{i\alpha}{(\bar{\mathsf{a}}_2 - \mathsf{a}_2)}
\end{aligned}
$$

5.4.6 Some Local One -Parameter Groups of Transformations

We can obtain the local one parameter group of transformations for Case 2 integrating the contravariant components of the Killing vector fields K^i, $i = 1, ...4$ with respect to affine parameter λ, we obtain,

$$
\begin{aligned}
r &= -\alpha\lambda + r_o \\
t &= -\alpha\left[\frac{\lambda^3\,|\mathsf{v}|\,\alpha^2}{3} + \frac{\lambda^2\alpha}{2}\left[y_o\mathsf{v} + \bar{y}_o\bar{\mathsf{v}} + 2\,|\mathsf{v}|\,(\bar{\mathsf{a}}_2 + \mathsf{a}_2)\right]\right] \\
&\quad -\alpha\lambda\left[|y_o| + 2y_o\bar{\mathsf{v}}\bar{\mathsf{a}}_2 + 2\bar{y}_o\mathsf{v}\mathsf{a}_2 - \beta\right] + t_o \\
y &= \mathsf{v}\alpha\lambda + y_o \\
\bar{y} &= \bar{\mathsf{v}}\alpha\lambda + \bar{y}_o
\end{aligned}
$$

5.5 Discussion

Explicit expressions for nontrivial Killing vector fields have been found based on the properties of metric (5.2), under conditions (5.5)-(5.7). In some cases we have found explicit formulae of a local one parameter group of transformations, in special cases when the dependency of K^i, $i = 1, ..., 4$ on y and $\bar{y}$ is at most linear.

The expressions for the Killing vector fields have been worked out for the specific coordinate system, without an attempt to change it as in Remark 6.

All of the results are local. To make global statements would require taking into account other properties of the manifold, such as topological, analytical and so on. A paper by Nomizu [Nomizu, 1960] would be helpful to move in that direction.

The assumption of existence of Killing vector fields and the properties of the class of metrics let us find conditions on functions φ_1, A and H, and then explicit expressions of those functions. The metric is independent of φ_1, while components of the Killing vector field, K^i, $i = 1, ..., 4$, are not, due this fact, we find the presence of α in all K^i expressions, i.e., the Lie algebra is one-dimensional.

One can be tempted to ask why we had to apply this particular method. In fact often, given a metric we can find Killing vector fields just by inspection. For this purpose we look for a coordinate on which the metric is independent. For example given the following metric,

$$g = -\left(1 - \frac{2M}{r}\right) c^2 dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 \left[d\vartheta^2 + \left(\sin^2 \vartheta\right) d\varphi^2\right]$$

where M is a positive constant. In a coordinate system $\{t, r, \vartheta, \varphi\}$, we can see that it is independent on t and φ, consequently $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$ are Killing vector fields. However, they are not the only ones. Indeed, there are more. The point is, when we work with a non flat manifold with a metric not necessarily "pretty", it is difficult to find Killing vector fields without studying the integrability conditions and the restrictions which arise as consequence of those.

Chapter 6

Conclusions

1. Only under conditions presented in Case 1 to 5 nontrivial Killing vector fields exist for the metric (5.2).

2. The Lie algebra of Killing vector fields is one dimensional

3. The geometric property (5.9) is very important to find the dependency of covariant components K_1, K_2, K_3, K_4 on two real variables r, t instead of four; r, t, y, $\bar{y}$. This fact, is decisive to obtain explicit expressions for functions φ_1, A, and H, and further explicit expressions for the contravariant components of Killing vectors fields.

4. For the metric (5.2), the 1-form e^3 is distinguished by the Ricci curvature, and the Weyl tensor, which is of algebraic type N. As a consequence of that the tetrad components C_{3131} of Weyl tensor and R_{33} of Ricci curvature are the only algebraically independent components of those objects.

5. We found explicit expression for local one parameter groups of transformations in the case when the dependency of K^i, $i = 1, ..., 4$ on y and $\bar{y}$ is at most linear. In general, it is difficult to find "nice" expressions for a local

one parameter group of transformations, as in the most general Case 1, because three of four components of contravariant Killing vector field depend on Ψ and A and their first derivatives.

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Appendix A

Conformal Transformations

Definition 19 *Let M be an m-dimensional Pseudo-Riemannian manifold with metric g. A metric g^* on M is said to be* conformally related *to g if it is proportional to g, that is, if there is a function $\rho > 0$ on M such that $g^* = \rho^2 g$, which does not change the angle between two vectors at a point. By a* conformal transformation *of M is meant a transformation Φ of M with the property that*

$$\Phi^* g = \rho^2 g$$

where ρ is a positive function on M. If ρ is a constant function, Φ is said to be homothetic transformation. *If ρ is identically equal to 1, Φ is nothing but an* isometry.

If the Pseudo-Riemannian metric g is conformally related to a Pseudo-Riemannian metric g^* which is locally Pseudo-Euclidean, the Riemannian manifold M with the metric g is said to be *conformally flat.* Clearly then, the Weyl conformal curvature tensor of M vanishes.

Proposition 27 *A necessary and sufficient condition for a Pseudo-Riemannian manifold of dimension $m > 3$ be conformally flat is that its Weyl conformal curvature tensor vanish.*

Appendix B

Related with Weyl Tensor for the Metric g

Proposition 28 *The Weyl tensor satisfying the condition (5.11) is of the form*

$$C = 4C_{3131}e^3 \wedge e^1 \otimes e^3 \wedge e^1 + 4C_{3232}e^3 \wedge e^2 \otimes e^3 \wedge e^2.$$

Thus, it is nontrivial if and only if $C_{3131} \neq 0$

Proof. We assume that the Weyl tensor is not trivial. In components (5.11) reads

$$C_{\alpha\beta\mu\nu}e_4^v = 0$$

From (4.63) we have

$$\begin{aligned} C_{abcd}e_\alpha^a e_\beta^b e_\mu^c e_v^d e_4^v &= 0 \\ C_{abcd}e_\alpha^a e_\beta^b e_\mu^c \delta_4^d &= 0 \\ C_{abc4}e_\alpha^a e_\beta^b e_\mu^c &= 0. \end{aligned}$$

Therefore,$C_{abc4} = 0$, for all a, b, c. Because of symmetries of the Weyl tensor, $C_{a4cd} = 0$, $C_{4bcd} = 0$ and $C_{ab4d} = 0$ for all a, b, c, d.

On the other hand, from (4.62) we get

$$g^{ab}C_{abcd} = 0.$$

So, if $c = 4$ or $a = b$ we have that $C_{abcd} = 0$ and by symmetries the only non trivial are $g^{bc}C_{abcd}$, but they vanish. So we have to verify what happens when each a, b, c, d are 1, 2, 3. We show that if $ab = 12$ then $C_{12cd} = 0$.

$$\begin{aligned}
\text{if} \quad g^{bc}C_{1bc2} &= 0 \\
C_{1122} + C_{1212} + C_{1342} + C_{1432} &= 0 \\
\text{so,} \quad C_{1212} &= 0,
\end{aligned}$$

$$\begin{aligned}
\text{if} \quad g^{bc}C_{1bc3} &= 0 \\
C_{1123} + C_{1213} + C_{1343} + C_{1433} &= 0 \\
\text{so,} \quad C_{1213} &= 0,
\end{aligned}$$

$$\begin{aligned}
\text{if} \quad g^{bc}C_{2bc3} &= 0 \\
C_{2123} + C_{2213} + C_{2343} + C_{2433} &= 0 \\
\text{so,} \quad C_{1223} &= 0.
\end{aligned}$$

Hence,

$$C_{12cd} = 0.$$

Now we check what happens when $ab = 13$. The only components that remain to investigate are $C_{1332,}$ $C_{1323,}$ $C_{1313,}$ and C_{2323}

Since,

$$g^{bc}C_{3bc3} = 0$$

then,

$$\begin{aligned}
C_{3123} + C_{3213} &= 0 \\
-C_{1323} - C_{1323} &= 0
\end{aligned}$$

so,

$$C_{1323} = 0.$$

Consequently,

$$\begin{aligned}
C &= C_{abcd} e^a \otimes e^b \otimes e^c \otimes e^d \\
C &= C_{1313} e^1 \otimes e^3 \otimes e^1 \otimes e^3 - C_{1331} e^1 \otimes e^3 \otimes e^3 \otimes e^1 \\
&\quad + C_{3131} e^3 \otimes e^1 \otimes e^3 \otimes e^1 - C_{3113} e^1 \otimes e^3 \otimes e^1 \otimes e^3 + (\text{similar terms for } C_{2323}) \\
&= C_{1313} e^1 \otimes e^3 \otimes \left(e^1 \otimes e^3 - e^3 \otimes e^1\right) + C_{3131} e^3 \otimes e^1 \otimes \left(e^3 \otimes e^1 - e^1 \otimes e^3\right) \\
&\quad + (\text{similar terms for } C_{2323}) \\
&= C_{3131} \left(e^3 \otimes e^1 - e^1 \otimes e^3\right) \otimes \left(e^3 \otimes e^1 - e^1 \otimes e^3\right) + (\text{similar terms for } C_{2323}) \\
&= 4 C_{3131} e^3 \wedge e^1 \otimes e^3 \wedge e^1 + 4 C_{3232} e^3 \wedge e^2 \otimes e^3 \wedge e^2.
\end{aligned}$$

Hence, if $C \neq 0$ then $C_{3131} \neq 0$, since $e^2 = \overline{e^1}$, and $C_{3232} = \overline{C_{3131}}$.

The converse is trivial. ■

Proposition 29 *If K is a null non trivial vector and $C(X, Y, Z, K) = 0$, for all X, Y, $Z \in M_p$, where $C \neq 0$, then the direction of K is unique, i.e., if there is another non trivial vector L such that $C(X, Y, Z, L) = 0$ for $X, Y, Z \in M_p$ then L is proportional to K.*

Proof. We can choose a null tetrad $\{e_1, e_2, e_3, e_4\}$ so that $K = e_4$. Then,

$$L = Ae_1 + Be_2 + De_3 + Ee_4.$$

Next,

$$\begin{aligned}
C(X, Y, Z, L) &= 0 \\
AC(X, Y, Z, e_1) + BC(X, Y, Z, e_2) + DC(X, Y, Z, e_3) + EC(X, Y, Z, e_4) &= 0 \\
AC(X, Y, Z, e_1) + BC(X, Y, Z, e_2) + DC(X, Y, Z, e_3) &= 0
\end{aligned}$$

We recall that $C_{abcd} = C(e_a, e_b, e_c, e_d)$. Let $(X, Y, Z) = (e_3, e_1, e_3)$, then

$$\begin{aligned} AC(e_3, e_1, e_3, e_1) + BC(e_3, e_1, e_3, e_2) + DC(e_3, e_1, e_3, e_3) &= 0 \\ AC_{3131} + BC_{3132} + DC_{3133} &= 0 \\ AC_{3131} &= 0 \end{aligned}$$

and consequently,

$$A = 0$$

since,

$$C_{3131} \neq 0.$$

Similarly with $(X, Y, Z) = (e_3, e_2, e_3)$ we obtain $B = 0$, because $C_{3232} \neq 0$ and if we set $(X, Y, Z) = (e_1, e_3, e_1)$ we infer $D = 0$. So $E \neq 0$ since L is not trivial. Therefore L is proportional to e_4. ■

Appendix C

The Theorem of Frobenius

Definition 20 [1] *Let E, F be two Banach spaces over K, A, (resp.B) an open subset of E (resp. F) , U a mapping of $A \times B$ into the Banach space $L(E;F)$ of linear and continuous transformations from E to F. A differentiable mapping u of A into B is a* solution of the total differential equation

$$y' = U(x, y) \tag{C.1}$$

if, for any $x \in A$, we have

$$u'(x) = U(x, u(x)) . \tag{C.2}$$

Remark 10 *When $E = K$, $L(E;F)$ is identified to F, and a total differential equation is thus an ordinary differential equation*

$$x' = f(t, x) .$$

When $E = K^m$ is finite dimensional, a linear mapping U of E into F is defined by its value at each of the m basis vector of E, and, by definition, C.2 is thus equivalent to the system of m "partial differential equations"

$$D_i y = f_i\left(x^1, ..., x^m, y\right) \qquad (1 \le i \le m) \tag{C.3}$$

[1] This section is taken from [Dieudonné, 1970], p.307 - 311

Definition 21 *Let U be a mapping of $A \times B$ into the Banach space $L(E; F)$, where A, B are open subsets of E, F respectively. Equation (C.1) is* completely integrable *in $A \times B$ if, for every point $(x_o, y_o) \in A \times B$, there is an open neighborhood S of x_o in A such that there is a unique solution u of (C.1), defined in S, with values in B, and such that $u(x_o) = y_o$.*

Theorem 30 *(Frobenius) Suppose U is continuously differentiable in $A \times B$. In order that (C.1) be completely integrable in $A \times B$, it is necessary and sufficient that, for each $(x, y) \in A \times B$, the relation*

$$\begin{aligned} D_1U(x,y) \cdot (s_1, s_2) + D_2U(x,y) \cdot (U(x,y) \cdot s_1, s_2) = \\ D_1U(x,y) \cdot (s_2, s_1) + D_2U(x,y) \cdot (U(x,y) \cdot s_2, s_1) \end{aligned} \tag{C.4}$$

holds for any pair (s_1, s_2) in $E \times E$.

Remark 11 *When $E = K^m$, the Frobenius condition (C.4) of complete integrability is equivalent, for the system (C.3), to the relations*

$$\begin{aligned} \frac{\partial}{\partial x^j} f_i\left(x^1, ..., x^m, y\right) + \frac{\partial}{\partial y} f_i\left(x^1, ..., x^m, y\right) f_j\left(x^1, ..., x^m, y\right) = \\ \frac{\partial}{\partial x^i} f_j\left(x^1, ..., x^m, y\right) + \frac{\partial}{\partial y} f_j\left(x^1, ..., x^m, y\right) f_i\left(x^1, ..., x^m, y\right) \end{aligned}$$

(where it must be remembered that $\frac{\partial}{\partial y} f_i(x^1, ..., x^m, y)$ is an element of $L(F; F)$ (a matrix if F is finite dimensional), and $f_j(x^1, ..., x^m, y)$ an element of F).