KHOVANOV HOMOLOGY FOR ALMOST ALTERNATING KNOTS

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The large quantity of almost alternating knots gives rise to an important category in knot classification. We thus establish a result, previously given for the span of the bracket polynomial for almost alternating knots, in terms of the Jones polynomial. The Khovanov complex of a given knot K is generated by considering a planar projection of the knot with 2^n states, each of which consists of a collection of simple closed curves in the plane. Following results in leading papers, we find which specific knots differ from others satisfying an equation and we present an alternative proof of a theorem related to the span of the Jones polynomial of an almost alternating knot; finally, keeping up our idea of finding invariants, we study their Khovanov homology.

Resumen de Tesis Presentada a Escuela Graduada

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KHOVANOV HOMOLOGY FOR ALMOST ALTERNATING KNOTS

Por

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2017

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Departamento: Ciencias Matemáticas

La gran cantidad de nudos casi alternantes da lugar a una importante categoría en la clasificación de los nudos. Así, se establece un resultado previamente dado para la diferencia entre las potencias mayor y menor que ocurren en el polinomio bracket de nudos casi alternantes, en términos del polinomio de Jones. El complejo de Khovanov para un nudo K se genera al considerar una proyeccion planar del nudo con 2^n estados, cada uno de los cuales consiste en una colección de curvas cerradas simples en el plano. Siguiendo resultados de artículos destacados, encontramos los nudos que difieren de otros al no satisfacer cierta ecuación y presentamos una prueba alternativa para un teorema relativo a la diferencia entre las potencias mayor y menor que ocurren en el polinomio de Jones para nudos casi alternantes. Por último y manteniendo nuestra idea de encontrar invariantes, estudiamos la homología de Khovanov para esos nudos.

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Contents

Abstract ii
Resumen iii
Acknowledgements v

1 Introduction **1**
1.1 Justification 1
1.2 Previous publications 2

2 Knots and links **3**
2.1 Equivalence 5
2.2 Types of knots 7
2.2.1 Almost alternating knots 7
2.3 Invariants of knots 9
2.3.1 Jones Polynomial 9
2.4 Tabulating knots 14
2.4.1 Conway's notation 14

3 Khovanov homology **17**
3.1 Khovanov complex of a link diagram 17
3.2 Constructing Khovanov homology 19

4 Kh for almost alternating knots **22**

4.1 Knots up to nine crossings 22

4.1.1 Identifying almost alternating knots 24

4.2 Remarks for 8_{19} and 9_{42} knots 26

5 Span of the Jones polynomial **28**

5.1 Span for almost alternating knots 30

6 Conclusions and Future Work **37**

6.1 Conclusions 37

6.2 Future work 38

List of Figures

2.1 Embeddings-Link with 2 components (Hopf link). 3

2.2 A not prime knot. 4

2.3 An oriented diagram. 5

2.4 Reidemeister moves. 6

2.5 The trefoil knot. 7

2.6 An almost alternating projection. 8

2.7 A 2-almost alternating projection. 9

2.8 $0-smoothing$ and $1-smoothing$. 10

2.9 +1 crossing and -1 crossing. 10

2.10 Hyper-cube. 11

2.11 Removable crossing. 13

2.12 Tangle 14

2.13 The ∞-tangle and the 0-tangle 15

2.14 The 2-tangle. 15

2.15 Tangle -3 -2 16

2.16 Tangle -3 -2 -1 16

3.1 Diagram for the Trefoil knot. 21

4.1 Almost alternating projection for the 8_{19}. 25

4.2 9_{42} and 9_{46} knots. 26

5.1 Extreme states. 28

5.2 Values in a crossing. 29

5.3 Adequate knot. 29

5.4 Components in the extreme states. 32

5.5 Virtual crossing. 36

1 Introduction

1.1 Justification

It was around the end of the nineteenth century when knot theory gained earnestness. However, some of the most exciting results in this area have occurred in the last thirty years. A knot (link) is an embedding of a circle (circles) into the Euclidean space $\mathbb{R}^3$ (or the unit sphere S^3), that is a *knot* (link) is a closed curve that does not intersect itself anywhere [1]. It is defined an alternating knot as a knot that admits a projection in which the crossings alternate between under and over, as the diagram is traversed. The main focus of this research is the study of invariants for almost alternating knots.

A projection K of a link L is said to be almost alternating if one crossing change makes the projection alternating. A link L is almost alternating if L has an almost alternating projection and L does not have an alternating projection. Further, we can also take this idea of almost alternating knots and extend it. Define a projection K of a link L to be m-almost alternating if m crossing changes produce an alternating projection. A link L is m-almost alternating if it has an m-almost alternating projection and no (m-1)-almost alternating projection. The number m is called the dealternating number of the projection.

Polynomials are associated to diagrams of knots in order to identify equivalence. The Khovanov bracket is an invariant stronger than the Jones polynomial [4]. Khovanov's method is to replace the Kauffman bracket of a link projection with the Khovanov bracket, generalizing it to an exact chain complex for graded vector spaces where the graded Euler characteristic is the Jones polynomial. In order to construct the Khovanov bracket, we calculate the states of a diagram of the knot obtained from resolving all the crossings in all possible ways. Then, each complete smoothing is identified as a vertex in a n-dimensional cube with a vectorial space associated to it conducive to obtain a chain group. By last, differential maps are defined in order to get the homology.

1.2 Previous publications

Being the almost alternating knots a particular kind in the sense that they are close of being alternating, many mathematicians have been trying to generalize results previously obtained for alternating knots. In [2] Adams *et al.* showed that all but three of the nonalternating knots up through eleven crossings and all but two of the nonalternating links up through ten crossings are almost alternating. They also generalized the fact that for an alternating link L in a n-crossings reduced connected alternating projection, the bracket polynomial has span equal to $4n$. In order to obtain the generalization of this result they define two new conditions for links: dealternator reduced and dealternator connected. We used this result for the especific case when $m = 1$.

In 2013 J. González and P. Manchón presented a general formula for the sum of the number of components in the extreme states for an m-almost alternating knot. They did it by constructing a surface S and a graph Γ_D associated to the diagram. By using this result, they also provided geometrical proofs for the results of Adams [2] and Zhu [15], related to the span of the Jones polynomial.

In this work, we followed results given in Adams' paper in order to identify the almost alternating knots with up to nine crossings, which do not hold an equality. Furthermore, we present proof of the main result obtained by J. González and P. Manchón related to the span of the Jones polynomial for m-almost alternating knots.

2 Knots and links

Intituively, a knot is the result from joining the two ends of a rope after knotting it. This is, a *knot* is a closed curve that does not intersect itself anywhere.

Definition 2.0.1 *Let $f : X \longrightarrow Y$ be an injective function. Suppose $f' : X \longrightarrow f(X) \subseteq Y$ defined as $f'(x) = f(x)$ is a homeomorphism, then we say that f is an embedding of the space X into the space Y.*

Definition 2.0.2 *A **knot** is an embedding of a circle S^1 into the Euclidean space $\mathbb{R}^3$ (or into the unit sphere S^3).*

Definition 2.0.3 *A **link** of m components is a subset of S^3, or of $\mathbb{R}^3$, that consists of m disjoint, piecewise linear, simple closed curves. Thus, we can define knots as links with only one component. Figure 2.1 shows an embedding of one and two circles into the Euclidean space.*

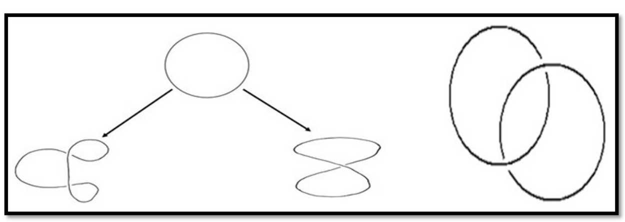


Figure 2.1: Embeddings-Link with 2 components (Hopf link).

Definition 2.0.4 *A knot K is said to be the **unknot** if it bounds an embedded piecewise linear disc in S^3.*

Projections into the plane $\mathbb{R}^2$ are made, in order to represent a knot. In such a projection, the **crossings** are identified as the areas in which one of the strands interrupts its way (understrand). In Mathematics we are always interested in defining operations among objects sharing certain characteristic, and indeed this is the case for knots.

Definition 2.0.5 *Two knots K_1 and K_2 can be added to obtain their sum (composition), $K_1 + K_2$, as follows. Suppose the knots to be in distinct copies of S^3, remove from each a ball that meets the knot in an unknotted spanning arc, and then identify together the resulting boundary spheres and their intersections with the knots, so that they match up.*

Definition 2.0.6 *A diagram $D \subset S^2$, of a link other than the unknot, is a prime diagram if any simple closed curve in S^2 that meets D transversely at two points bounds, on one side of it, a disc that intersects D in a diagram U of the unknotted ball-arc pair. D is strongly prime if such a U is always the trivial zero-crossing diagram.*

In other words, a diagram D is strongly prime if we can not find a circumference intersecting it transversely, such that the diagram has crossings in both sides of the circumference. As noted, in Figure 2.2 a knot that is not prime is shown.

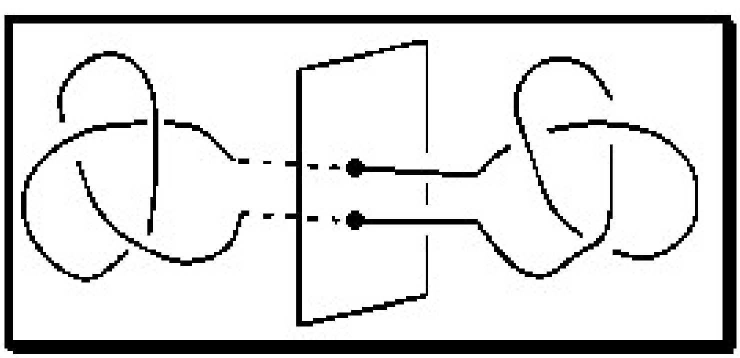


Figure 2.2: A not prime knot.

2.1 Equivalence

One of the most fundamental questions in the theory of knots is how to identify whether or not two projections are equivalent, that is if they represent the same knot.

Since a knot is a closed curve, an orientation can be assigned. As it is customary, an orientation is defined by the choice of a direction to go over the knot, and it is noted by arrows in the curve (Figure 2.3).

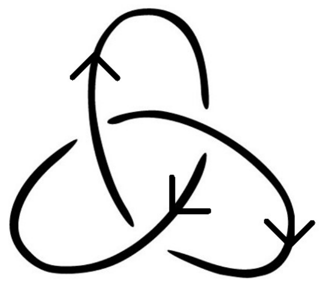


Figure 2.3: An oriented diagram.

Definition 2.1.1 *Links L_1 and L_2 in S^3 are equivalent if there is an orientation preserving piece-wise linear homeomorphism $h: S^3 \longrightarrow S^3$, such that $h(L_1) = L_2$.*

Definition 2.1.2 *A homotopy between two continuous functions $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ is a continuous function $F : X \times I \longrightarrow Y$, such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$.*

Definition 2.1.3 *A homotopy $F : X \times I \longrightarrow Y$ is called an* ***isotopy*** *if $F|_{X\times\{t\}}$ is a homeomorphism for all $t \in I$.*

The word isotopy refers to a deformation of a space X, which does not modify the topology of X. In knot theory there is particular interest in knowing when two embeddings of a space Y in another space X can be deformed to each other by an isotopy of the space X containing them.

Definition 2.1.4 *If $f : Y \longrightarrow X$ and $g : Y \longrightarrow X$ are embeddings of Y into X, then f and g are* ***ambient isotopic*** *if there is an isotopy $F : X \times I \longrightarrow X$ such that $F(x,0) = x$ for all $x \in X$ and $F(f(y),1) = g(y)$ for all $y \in Y$. The space X is called the ambient space and the function F is called an ambient isotopy.*

Definition 2.1.5 *Two knots $f, g : S^1 \longrightarrow \mathbb{R}^3$ are **equivalent** if they are ambient isotopic.*

Definition 2.1.6 *A **planar isotopy** is a piecewise-linear isotopy of the plane.*

Planar isotopy deforms a knot projection to another projection of the same knot without changing the topological structure of the knot. Although planar isotopy certainly modifies the distances between crossings, it neither can change the number of crossings nor which crossings are connected by which string of the projection.

In 1926, the German mathematician Kurt Reidemeister [1] proved that if we have two distinct projections of the same knot, we can get from one projection to the other by a series of moves.

Definition 2.1.7 *A Reidemeister Move is one of the three possible forms of changing a projection of a knot, by varying the relationship between the crossings. Two diagrams are said to be equivalent if one of them can be transformed into the other by following a finite sequence of Reidemeister Moves.*

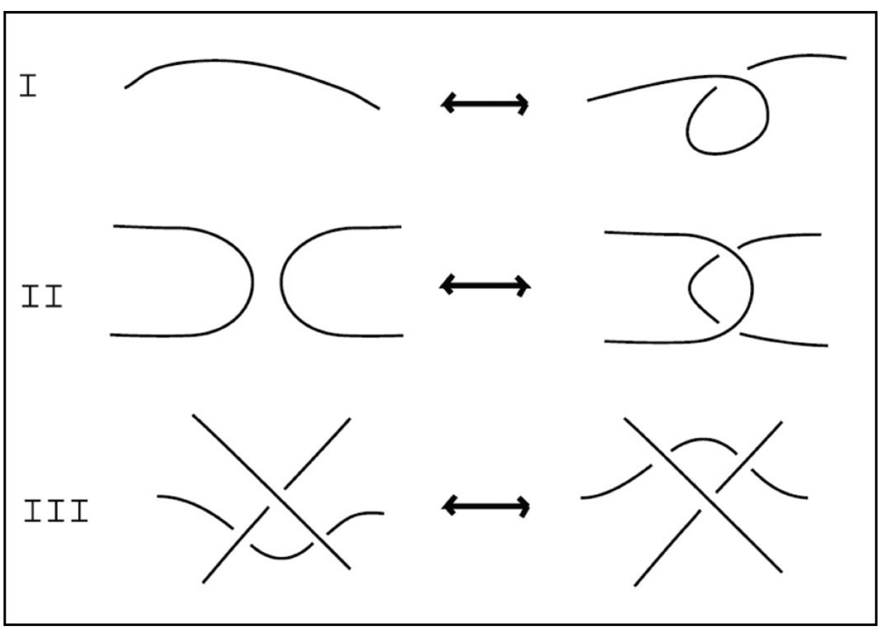


Figure 2.4: Reidemeister moves.

Theorem 2.1.8 ***Reidemeister's theorem*** *[5]. Two knots are equivalent if and only if there is a finite sequence of planar isotopies and Reidemeister moves taking a knot projection of one to a knot projection of the other.*

2.2 Types of knots

Several ways exist to classify knots. For instance based on the number of crossings or in the surface in which the knot lies.

It is our main criterion of classification the alternation of the cords in each crossing between under and over, as we go over the projection of the knot in a certain direction. Thus, if this alternation occurs the projection is said to be **alternating**. Thereby, a knot which admits a projection of this kind is called an alternating knot. Figure 2.5 shows an example, known as the trefoil knot.

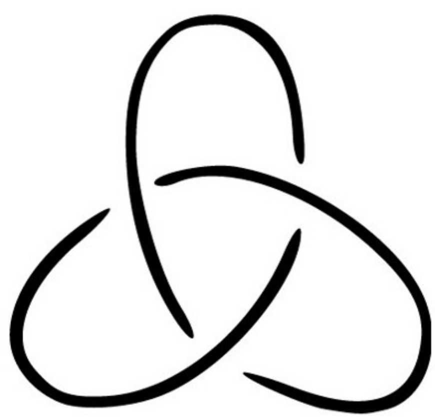


Figure 2.5: The trefoil knot.

2.2.1 Almost alternating knots

Maintaining the idea of alternation of the crossings, Adams *et al*. [2] introduced a new knot type, looking for a generalization for the concept of alternating knots.

Definition 2.2.1 *A projection of a knot is almost alternating if one crossing change makes the projection alternating. Hence, a knot K is said to be* ***almost alternating*** *if K has an almost alternating* *projection and K does not have an alternating projection. An example is presented in Figure 2.6.*

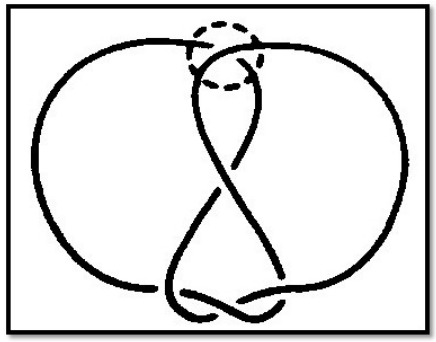


Figure 2.6: An almost alternating projection.

As a matter of fact, in [2] the authors showed that of the 393 nonalternating knots and links of eleven or fewer crossings, all but five, three knots and two links, are almost alternating. Then in 1999, H.Goda, M.Hirawasa and R.Yamamoto found almost alternating projections for one of each of the remaining knots and links. This means, of the nonalternating knots and links of eleven or fewer crossings, there are only two knots and one link which may not be almost alternating.

The concept of almost alternation can be taken beyond one crossing change needed to obtain an alternating projection. The following definition divides all knots into separate categories depending on the value of *m*.

Definition 2.2.2 *Define an **m-almost alternating knot** to be a knot that has a projection where m crossing (called the **dealternators**) changes would make the projection alternating, and the knot has no projection that could be made alternating in fewer crossing changes.*

The number *m* measures how far a knot is from being alternating. Following this definition, we consider alternating knots to be 0-almost alternating and almost alternating knots to be 1-almost alternating. An example of a 2-almost alternating knot is the white double of the trefoil knot, showed in Figure 2.7.

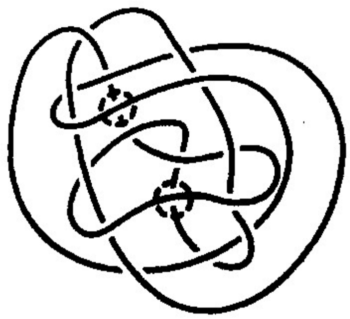


Figure 2.7: A 2-almost alternating projection.

2.3 Invariants of knots

Definition 2.3.1 *A knot invariant is a function*

$$K \longrightarrow f(K)$$

which assigns to each knot K an object $f(K)$ (i.e, a numerical value, polynomial, group, etc) in such a way that knots of the same type are assigned equivalent objects.

Hence, given that two projections determine different values for a specific invariant, we can state that the knots they represent are not the same.

2.3.1 Jones Polynomial

Among knot and link invariants, we are going to work on the polynomials associated to a diagram, especially with the Jones polynomial for an oriented link *L*, which construction is based on the bracket polynomial.

Let L be an oriented knot projection with n crossings, labeled arbitrarily. Each crossing can be solved (smoothed) in two different ways, as showed in Figure 2.8, using a $0-smoothing$ or a $1-smoothing$. A $0-smoothing$ is the way of solving the crossing connecting the two regions that the overstrand passes over, when rotated counter-clockwise until it reaches the understrand.

Similarly, a $1-smoothing$ connects the two regions that the overstrand passes over when rotated clockwise until it reaches the understrand. Solving a crossing is choosing between a 0- or 1-smoothing.

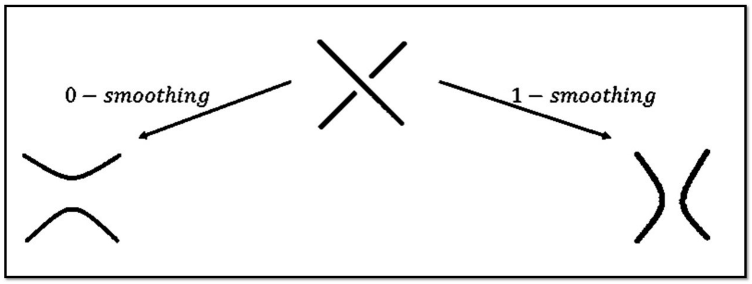


Figure 2.8: $0-smoothing$ and $1-smoothing$.

Definition 2.3.2 *A* ***state*** *is a choice of how to smooth all of the crossings in the projection of a link.*

Definition 2.3.3 *The Kauffman bracket is a function from unoriented link diagrams into the oriented plane (S^2) to Laurent polynomials with integer coefficients in an indeterminate variable q. It maps L to $\langle L\rangle \in \mathbb{Z}\left[q,q^{-1}\right]$ and follows the relations:*

$$\langle \bigcirc \rangle = 1$$
$$\langle L \sqcup \bigcirc \rangle = (q+q^{-1})\langle L \rangle$$
$$\langle \times \rangle = \langle \asymp \rangle - q\,\langle)(\rangle$$

At every crossing of an oriented projection, we have either a $+1$ or -1, as shown in Figure 2.9. Let n_+ and n_- denote the number of positive and negative crossings respectively. Then, $n = n_+ + n_-$.

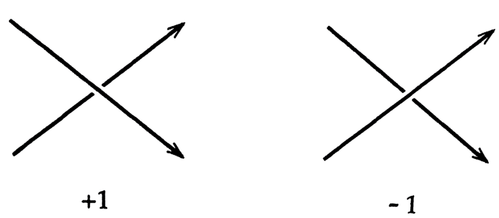


Figure 2.9: +1 crossing and -1 crossing.

The sum of those $+1$ and -1 is called the **writhe** of the projection. From the axioms above, we define the unnormalised Jones polynomial ($\tilde{J}$) as

$$\tilde{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$$

The Jones polynomial is obtained from $\tilde{J}$ by dividing it by the term $q + q^{-1}$, thus

$$J(L) = \frac{\tilde{J}(L)}{q + q^{-1}}$$

However, notice that if the orientation of each of the components of a link L is changed, the sign of the crossings are not modified. Thus, the Jones polynomial does not depend on the orientation chosen for the knot.

As we stated before, the crossings of the diagram L are ordered by $1, 2, 3, 4, ..., n$, then each of the 2^n states obtained by resolving each crossing either by a 0-smoothing or a 1-smoothing, can be indexed by a word of n zeros and ones, i.e an element of $\{0, 1\}^n$.

Following the crossings order, the set $\{0, 1\}^n$ is the vertex set of a hyper-cube as shown in Figure 2.10 (in this case, for a projection with three crossings), with an edge between words differing in exactly one place. We think of the resolution cube as increasing from the states with all the crossings solved by a 0-smoothing to the state with all the crossings solved with a 1-smoothing (also called the extreme states).

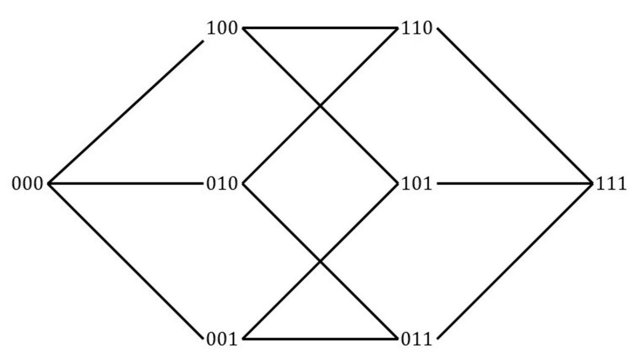
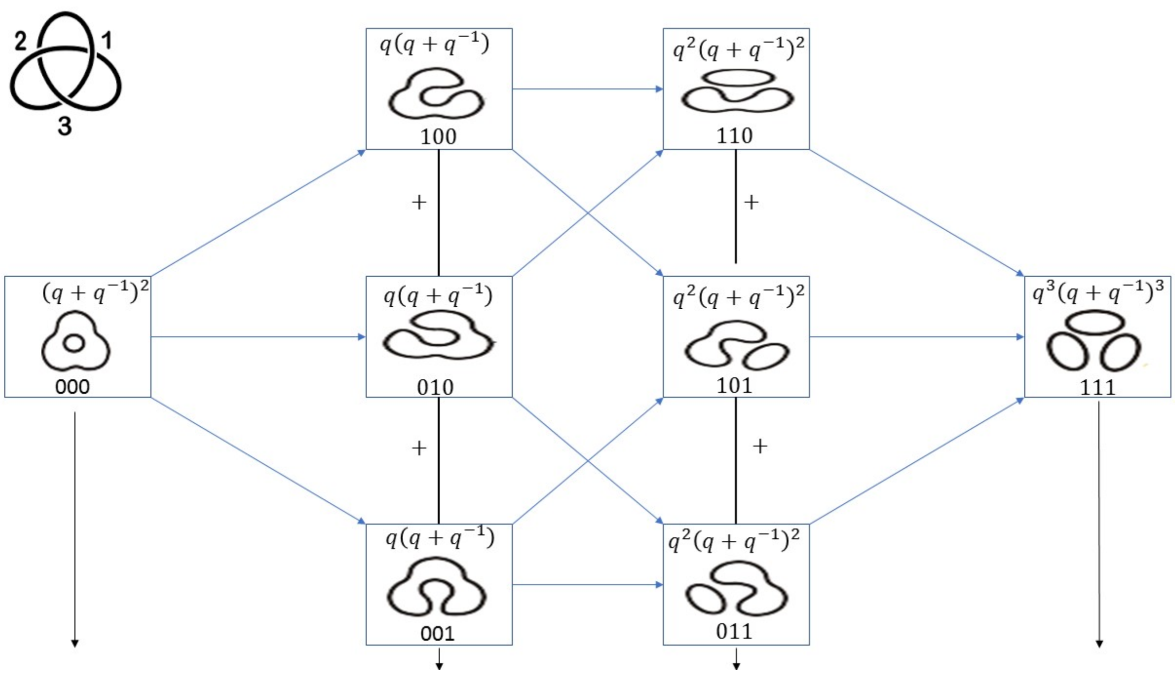


Figure 2.10: Hyper-cube.

To each vertex α of the hyper-cube it is assigned a state S_α (union of k planar cycles) of the diagram L. Then, to each S_α corresponds a term of the form $(-1)^r q^r \left(q+q^{-1}\right)^k$, where r is the number of 1-smoothings used in the state (the height of the smoothing). Now, the unnormalised Jones polynomial is obtained from the sum of all these terms over all $\alpha \in \{0,1\}^n$ and then, multiplying by the normalization term $(-1)^{n_-} q^{n_+-2n_-}$.

For the trefoil knot, the process can be depicted as follows



$$\left(q+q^{-1}\right)^2 \qquad -3q\left(q+q^{-1}\right) \qquad +3q^2\left(q+q^{-1}\right)^2 \qquad -q^3\left(q+q^{-1}\right)^3$$

$$= q^{-2}+1+q^2-q^6$$

then, $\tilde{J}(L) = \left(q^{-2}+1+q^2-q^6\right)\left((-1)^0 q^3\right) = q+q^3+q^5-q^9$ and thus,

$$J(L) = q^2+q^6-q^8$$

Definition 2.3.4 *The* ***span*** *of a polynomial is the difference between the highest power and the lowest power that occurs in the polynomial. For instance, the span of:* $A^3+A^2-1-A^{-2}$ *is*

$3-(-2)=5$

The bracket polynomial is not an invariant for knots since it is modified by a Type I Reidemeister move [1], in fact this move multiplies the bracket either by A^{-3} or A^{3}. Nevertheless, if we calculate the bracket polynomial from any projection of the knot and then calculate the span, we will always get the same number:

In effect, having K_1 and K_2 two different projections of the knot K, there exists a series of Reidemeister moves taking from K_1 to K_2. Reidemeister moves II and III do not modify the bracket polynomial, hence they do not change the span either. A Type I move increases (decreases) the highest (lowest) power in the polynomial by 3 (-3), such that the difference between them is unchanged. That is, the span of the bracket polynomial is an invariant for knots.

Some definitions and relevant results of alternating knots are presented. In [2] there were proven some generalizations of these results for almost alternating knots.

Definition 2.3.5 *We call an alternating projection* ***reduced*** *if there are no unnecessary crossings in the projection (as in Figure 2.11), this is, there is no obvious way to lower the number of crossings by a type I Reidemeister move.*

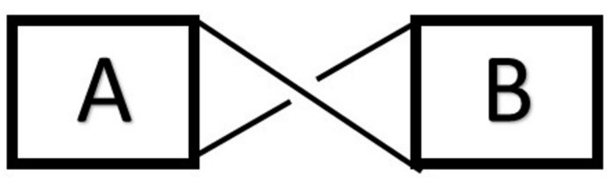


Figure 2.11: Removable crossing.

Next results were shown by Louis Kauffman, Morwen Thistlethwaite and Kunio Murasugi in 1986. Together they imply that we can determine the crossing number for any alternating knot. Proofs ar referred to [1,1,9] respectively.

Lemma 2.3.6 *If K has a reduced alternating projection of n crossings, then span$(< K >) = 4n$.*

Theorem 2.3.7 *Two reduced alternating projections of the same knot have the same number of crossings.*

Theorem 2.3.8 *A reduced alternating projection of a knot has the least number of crossings for any projection of that knot.*

2.4 Tabulating knots

2.4.1 Conway's notation

In this section, we introduce Conway's notation for knots [6]. This was the notation used to tabulate the prime knots through 11 crossings and prime links through 10 crossings in 1969. By studying this notation for knots, the authors in [2] were able to identify whether a knot is alternating or almost alternating.

Definition 2.4.1 *A tangle in a knot or link projection is a region surrounded by a circle such that the knot or link projection crosses the circle exactly four times. The four points where the knot or link projection crosses the circle are identified as occurring in the four compass directions NW, NE, SW, and SE.*

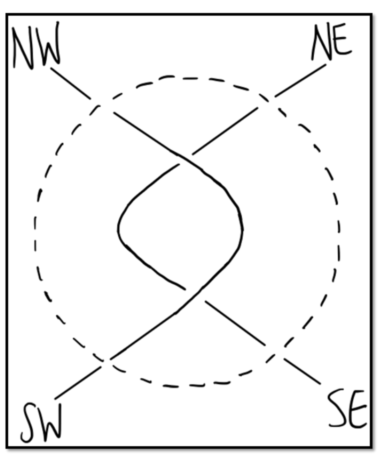


Figure 2.12: Tangle.

Constructing Tangles

Tangles are the elementary units of knot and link projections. As expected, two tangles are equivalent if we can get from one to the other by a sequence of Reidemeister moves. Furthermore the four endpoints must remain fixed and the strings of the tangle never journey outside the circle defining the tangle. Two of the simplest tangles are the one with two vertical strings, noted as the $\infty - tangle$ and the one with two horizontal strings, noted as the $0 - tangle$ (Figure 2.13).

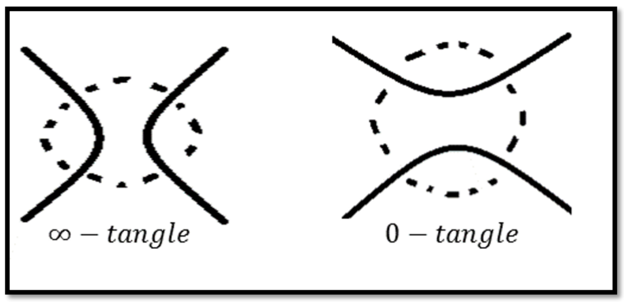


Figure 2.13: The ∞-tangle and the 0-tangle.

Those tangles define the starting point in the construction of more complex tangles. We could wind, for example, two horizontal strings around each other to get the next figure, and we denote this tangle by the number of left-handed twists we put in. In this case, the number is 2.

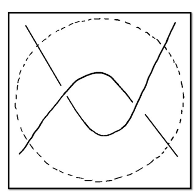


Figure 2.14: The 2-tangle.

If these two horizontal strings were twisted the other way around, we would have denoted the tangle by a -2. Note that for a positive-integer twist, the overstrand always has a positive slope.

The idea in this section is to understand Conway's notation for knots and links, so that we would be able to identify alternating and almost alternating knots. In order to accomplish this, a **rational** tangle is constructed.

Starting with, let us say a -3 tangle, first we reflect the tangle through the NW and SE diagonal line. Note that the two ends of the tangle along the diagonal are fixed when we perform the reflection and the two other that are not on the diagonal are switched. It is sometimes difficult to picture what happens to the crossings under the reflection but, we can figure out what happens to one crossing and then we can infer what must happen to the other crossings. Now we wind the two right-hand ends of the tangle around each other, let us say -2 times and we will obtain the tangle -3 -2 showed in Figure 2.15.

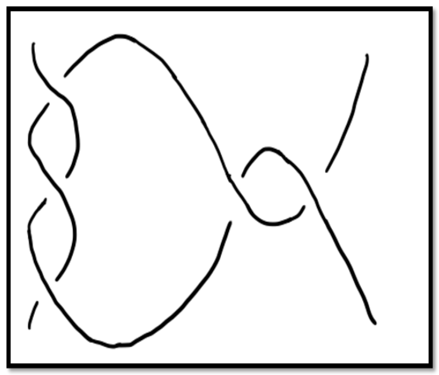


Figure 2.15: Tangle -3 -2

If we perform another reflection about the NW and SE diagonal, and then wind again the two right-hand ends, let us say -1 times, we will obtain the -3 -2 -1 tangle, as seen in Figure 2.16.

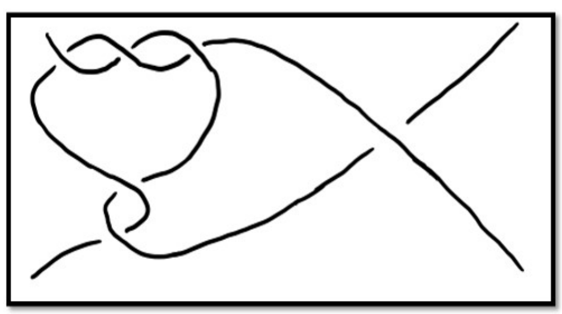


Figure 2.16: Tangle -3 -2 -1

Any tangle constructed in this way is called a rational tangle. If the ends of a rational tangle are closed off, a rational link results.

Definition 2.4.2 *This notation for rational links is what is called the* ***Conway's notation****.*

3
Khovanov homology

Khovanov's idea is to replace the Kauffman bracket $\langle D \rangle$ of a link L by what is called the Khovanov bracket, $[\![D]\!]$. This bracket is a chain complex of graded vector spaces whose graded Euler characteristic is $\langle D \rangle$.

3.1 Khovanov complex of a link diagram

Khovanov develops a way to take an oriented link diagram D into a $bi-graded$ chain complex, $C^{*,*}(D)$, the homology of which is the Khovanov homology of D. Let us take the diagram used to calculate the Jones polynomial of the trefoil knot as the starting point.

$$D \xrightarrow{Khovanov} C^{*,*}(D) \xrightarrow{homology} KH^{*,*}(D)$$

In essence, by replacing the variable q with this chain complex, an algebraic object, he is categorifying the unnormalised Jones polynomial. Moreover, the following properties are satisfied:

- If D is equivalent to another diagram D', then there is an isomorphism such that:

$$KH^{*,*}(D) \cong KH^{*,*}(D')$$

- The graded Euler characteristic is the unnormalised Jones polynomial, that is:

$$\sum_{i,j\in\mathbb{Z}} (-1)^i q^j dim(KH^{i,j}(D)) = \tilde{J}(D)$$

Now, it will be given some necessary definitions related to graded vector spaces.

Definition 3.1.1 *A graded vector space W is a vector space having a decomposition $W = \bigoplus W_m$, over $m \in \mathbb{Z}$. $\{W_m\}$ is the set of the homogeneous components.*

Definition 3.1.2 *The graded (or quantum) dimension, $qdim$, of a graded vector space $W = \bigoplus_m W^m$ is the polynomial in q defined by:*

$$qdim(W) = \sum_m q^m dim(W^m)$$

Definition 3.1.3 *For a graded vector space W and an integer l, it can be defined a new graded vector space $W\{l\}$ (a shifted version of W) by:*

$$W\{l\}^m = W^{m-l}$$

Notice that $qdim(W\{l\}) = q^l qdim(W)$.

Definition 3.1.4 *A chain complex is a sequence of homomorphisms of abelian groups with differential maps d:*

$$\cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} A_{n-2} \to \cdots \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-1}} A_{-2} \xrightarrow{d_{-2}} \cdots$$

where $d_n \circ d_{n+1} = 0$ for each n.

Definition 3.1.5 *Likewise, if $\bar{C}$ is a chain complex $\cdots \to \bar{C}^{r_\alpha} \xrightarrow{d^r_\alpha} \bar{C}^{r_\alpha+1} \cdots$ of vector spaces (possibly graded) and if $C = \bar{C}[s]$, then $C^{r_\alpha} = \bar{C}^{r_\alpha - s}$ (with all differentials shifted accordingly).*

Definition 3.1.6 *The homology of a chain complex is the set of modules, $H^n(C)$ given by*

$$H^n(C) = \frac{Ker\ d_n}{Im\ d_{n+1}}$$

Definition 3.1.7 *The elements of $Ker\ d_n$ are called cycles. The elements of $Im\ d_{n+1}$ are called boundaries.*

Definition 3.1.8 *The graded Euler characteristic of the chain complex $C^{*,*}(D)$ is given by*

$$\chi_q(C(D)) = \sum(-1)^i q^j dim(C^{i,j})$$

3.2 Constructing Khovanov homology

Consider a diagram L with crossing number n. Let V be a vector space spanned by the elements v_+ with degree $+1$, and v_- with degree -1. Thus V is a vector space with graded dimension $q+q^{-1}$.

In analogy to the process described in order to get the Jones polynomial, to each vertex α of the hyper-cube there is associated a graded vector space $V_\alpha = V^{\bigotimes k}\{r\}$, with k and r as in the previous section. In this way, the r^{th} chain group $[\![L]\!]^r$ with $r \in [0,n]$, is the direct sum of the vectors at the height r. That is,

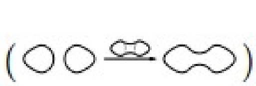
$$[\![L]\!] := \bigoplus_{\alpha : r = |\alpha|} V_\alpha(L)$$

It is needed now a differential turning $C(L) := [\![L]\!][-n_-]\{n_+ - 2n_-\}$ into a chain complex. To each edge of the cube we associate a **cobordism** (an orientable surface whose boundary is the union of the circles in the smoothings at either end). Edges of the cube can be labelled by a string of zeroes and ones with a star ($\star$) at the position that changes. For instance, the edge joining 0100 to 0110 is denoted as $01\star0$. We can also turn edges into arrows following the rule that, $\star = 0$ gives the tail and $\star = 1$ gives the head. The height of the arrow is the height of the tail. Thus, the maps with the same height, say d_ζ, are collapsed in

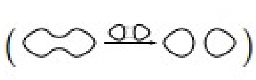
$$d^r := \sum_{|\zeta|=r} d_\zeta$$

For an arrow $\alpha \xrightarrow{\zeta} \alpha'$, note that the smoothings α and α' are identical except for a small disc (the changing disc) around the crossing that changes from $0-smoothing$ to a $1-smoothing$ (the one marked by a $\star$ in ζ). In other words, there are two options, either two circles merge into one or a circle splits in two.

Definition 3.2.1 *For any ζ, let d_ζ be the identity on the tensor factors not modified and let $m : V \bigotimes V \longmapsto V$ and $\Delta : V \longmapsto V \bigotimes V$ maps defined as*



$$\longrightarrow (V \otimes V \xrightarrow{m} V) \qquad m : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0 \end{cases}$$



$$\longrightarrow (V \xrightarrow{\Delta} V \otimes V) \qquad \Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

Recall that the Euler characteristic $\chi_q(C)$ of a chain complex C is the alternating sum of graded dimensions of the chain groups. Now, $C(L) := [\![L]\!]\,[-n_-]\,\{n_+ - 2n_-\}$ is in fact a chain complex.

Theorem 3.2.2 *The graded Euler characteristic of $C(L)$ is the unnormalised Jones polynomial of L:*

$$\chi_q(C(L)) = \tilde{J}(D)$$

Proposition 3.2.3 *The sequences $[\![D]\!]$ and $C(D)$ are chain complexes.*

Proofs:[4]

Define Kh(D) as the Poincaré polynomial of the complex $C(D)$ in the variable t where $H^r(D)$ is the r-th *cohomology* of the graded vector space $C(D)$:

$$Kh(D) := \sum_r t^r qdimH^r(D)$$

Theorem 3.2.4 *(Khovanov). The graded dimensions of the homology groups $H^r(D)$ are link invariants, and hence Kh(D), a polynomial in the variables t and q, is a link invariant that specializes the unnormalised Jones polynomial at $t = -1$.*

Proof:[4]

Figure 3.1 shows the process described above for the trefoil knot.

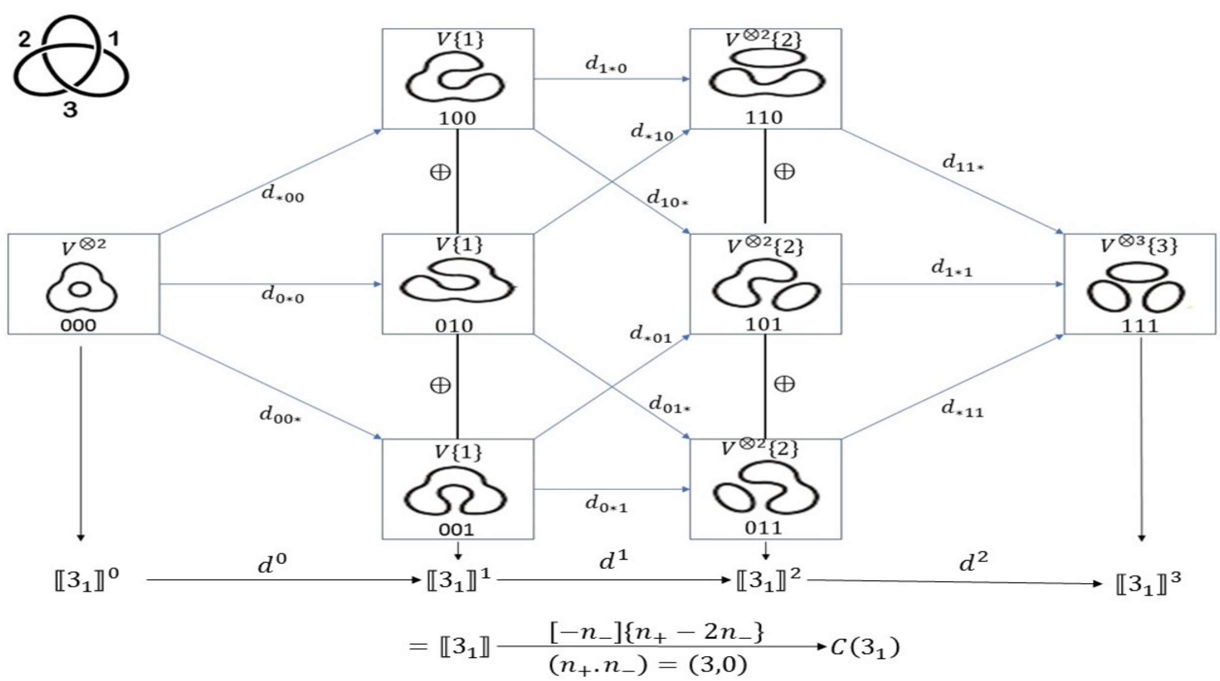


Figure 3.1: Diagram for the Trefoil knot.

4 Kh for almost alternating knots

4.1 Knots up to nine crossings

The generalization of the result proven by Kauffman, Thistlethwaite and Murasugi concerning the span of the bracket polynomial of an alternating link [1], is presented for the case of m-almost alternating links. It turns out that an equality holds for all but three of the almost alternating knots of nine or fewer crossings [2]. Those knots were identified and then their Khovanov bracket was studied, as to realize what aspects from these knots are different.

Let K_p be a reduced, connected m-almost alternating projection of a link K. Let $\mathbf{D}$ be the set of all 2^m alternating projections obtained by polynomial descomposition at the dealternators of K_p. Let $\mathbf{L}_i = \{L \in \mathbf{D} | L\ has\ i\ dealternators\ with\ A - channel\ splits\}$.

Definition 4.1.1 *K_p is said to be dealternator reduced if for all $L \in \boldsymbol{D}$, the projection L is reduced.*

In general terms, a diagram D is called dealternator reduced if there is no simple closed curve (called dealternator reducibility path) intersecting the projection of D in exactly one non-dealternator crossing and possibly in some dealternators.

Definition 4.1.2 *K_p is said to be dealternator connected if for each $L \in \boldsymbol{D}$, L is connected.*

In other words, a diagram D is called dealternator connected if there is no simple closed curve (called dealternator severing path) intersecting the projection D in a nonempty set of dealternators.

Note that if an m-almost alternating projection is not connected and reduced, it cannot be dealternator connected and reduced. We are assuming, however, that K_p is reduced and connected. The following lemma gives the tools needed for the main theorem in the paper.

Lemma 4.1.3 *If K_p is dealternator reduced and dealternator connected, then for links L and M $\in$ $\boldsymbol{L}_i$ and L' in $\boldsymbol{L}_{i+1}$, the following hold:*

- $maxdeg < L >= maxdeg < M >$ *and* $mindeg < L >= mindeg < M >$.
- $maxdeg < L' >= maxdeg < L > -2$ *and* $mindeg < L' >= mindeg < L > -2$.

Proof:[2]

Previously we stated a result for alternating knots developed in [1], related to the span of their bracket polynomial. The following theorem is the generalization of this result for m-almost alternating knots.

Theorem 4.1.4 *If a link K has n crossings in a dealternator reduced and dealternator connected m-almost alternating projection K_p, then $span(< K >) \leq 4(n-m-2)$.*

Proof:[2]

Corollary 4.1.5 *If an almost alternating link L has n crossings in an almost alternating projection L, then $span(< L >) \leq 4(n-3)$*

Proof:[2]

It is preferred to have the last results in terms of the Jones polynomial rather than the bracket polynomial, due to the fact that we are going to study the Khovanov homology of the given knots.

Suppose there is a knot K and $< K >= A^q + ... + A^p$, where q and p denote the highest and the lowest power of the bracket polynomial, respectively. Hence, $span(< K >) = q - p$. Let us construct the X polynomial [1] of K in order to obtain the Jones polynomial. Thus,

$$X(K) = (-A^3)^{-w(K)} < K >= (-A)^{-3w(K)} < K >$$

where $w(K)$ denotes the writhe of K. If we replace the bracket polynomial of K in the last equation, the following results:

$$X(K) = (-A^{-3w(K)}A^{q}) + \cdots + (-A^{-3w(K)}A^{pm})$$

$$= (-A^{-3w(K)+q}) + ... + (-A^{-3w(K)+p})$$

The Jones polynomial is obtained from the X polynomial by replacing the variable A by $t^{-\frac{1}{4}}$. Therefore,

$$V_K(t) = (-(t^{-\frac{1}{4}})^{-3w(K)+q}) + ... + (-(t^{-\frac{1}{4}})^{-3w(K)+p})$$

$$= (-t^{\frac{3}{4}w(K)-\frac{q}{4}}) + ... + (-t^{\frac{3}{4}w(K)-\frac{p}{4}})$$

Then,

$$span(V_K(t)) = (\frac{3}{4}w(K) - \frac{p}{4}) - (\frac{3}{4}w(K) - \frac{q}{4})$$

$$= \frac{q}{4} - \frac{p}{4} = \frac{1}{4}(q-p)$$

Hence, $4\ span(V_K(t)) = span(< K >)$. In this way, the following statement results: if a link K has n crossings in a dealternator reduced and dealternator connected m-almost alternating projection K_p, then $span(V_K(t)) \leq n-m-2$. Moreover, if an almost alternating link has n crossings in an almost alternating projection L_p, then $span(V_K(t)) \leq n-3$.

4.1.1 Identifying almost alternating knots

In [2], the authors state that in fact, all but three of the almost alternating knots of nine or fewer crossings satisfy the equality $span(V_K(t)) = n-3$. The following results constitute the key in the task of finding which of the knots up to nine crossings are almost alternating.

Theorem 4.1.6 *An algebraic knot with Conway notation containing no negative signs must be an alternating knot.*

Proof:[2]

Theorem 4.1.7 *A link has an almost alternating projection provided if its Conway notation includes only one negative sign.*

Proof:[2]

According to the Conway's notation presented in Rolfsen's table [14], we see that the almost alternating knots up to nine crossings are: 8_{19}, 8_{20}, 8_{21}, 9_{42}, 9_{43}, 9_{44}, 9_{45}, 9_{46}, 9_{47}, 9_{48} and 9_{49}. Now, we would like to state which knots do not hold the equality $span(V_t) = n-3$, where n represents the number of crossings. We study the possible number of crossings in an almost alternating projection of the knots and the results are the following.

- The 8_{19} knot has an almost alternating projection with nine crossings, presented in [1]. We are assuming, however, that no almost alternating projection of the 8_{19} is minimal, as suggested by [9]. Furthermore, $span(V_{8_{19}}(t)) = 5 < n-3$, with $n = 9$. Since $span(V_{8_{20}}(t)) = 6 = span(V_{8_{21}}(t))$, it is possible to find a nine-crossing almost alternating projection for these knots and in this way satisfy the statement.

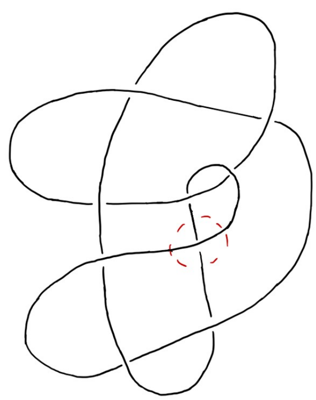


Figure 4.1: Almost alternating projection for the 8_{19}.

- For 9_{42} and 9_{46} knots, we have $span(V_{9_{42}}(t)) = 6 = span(V_{9_{46}}(t))$. Note that the minimum crossing number for a projection of these knots is $n = 9$, so that any almost alternating projection of 9_{42} and 9_{46} will have more than nine crossings, thus $6 < n-3$. For the others nine-crossing almost alternating knots, $span(V_K(t)) = 7$. That is, finding a ten-crossing almost alternating projection for those knots, make the equality hold.

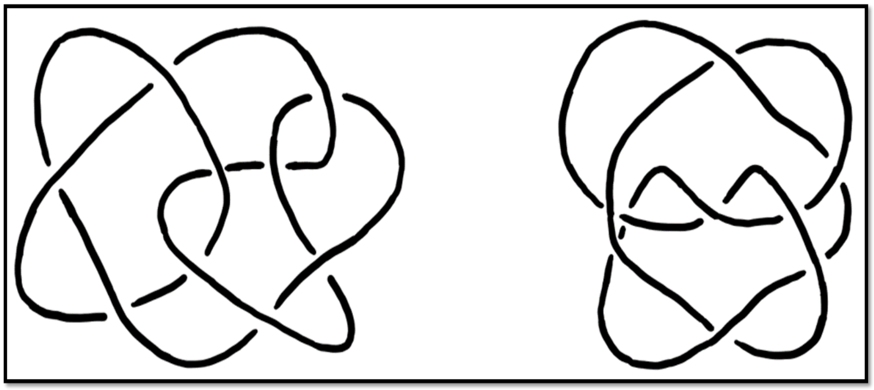


Figure 4.2: 9_{42} and 9_{46} knots.

4.2 Remarks for 8_{19} and 9_{42} knots

1. In [4], the author together with M.Khovanov and S.Garoufalidis formulated the following conjectures. First, they found out that $Kh_{\mathbb{Q}}(L)$ can be written in terms of another polynomial $Kh'(L)$, which contains less terms. Thereby is logic tabulate $Kh'(L)$ rather than $Kh_{\mathbb{Q}}(L)$.

$$Kh_{\mathbb{Q}}(L) = q^{s-1}(1+q^2+(1+tq^4)Kh'(L))$$

All prime knots with up to eleven crossings are in complete agreement with these conjectures. Actually, just a few non-alternating knots fail in holding the conjectures.

Conjecture 4.2.1 *For prime alternating L the integer $s(L)$ is equal to the signature of L and the polynomial $Kh'(L)$ contains only powers of tq^2.*

Analyzing the Khovanov homology tables, we can note that

- $s(L)$ is often equal to the signature of the knot $\sigma = \sigma(L)$.
- Most monomials in most $Kh'(L)$'s are of the form $t^r q^{2r}$ for some r.

Among the non-alternating knots of nine or fewer crossings the only two with exceptions are 8_{19} and 9_{42}.

In effect: for the 8_{19}, our first knot, we have $Kh'(8_{19}) = t^2q^4 + t^4q^6 + t^4q^8$. It is clear that this knot does not have the second property.

For 9_{42}, we have $Kh'(9_{42}) = \frac{1}{t^4q^6} + \frac{1}{t^2q^2} + \frac{1}{t} + tq^4$ and thus, we can note that this knot does not have the second property. Moreover, we can affirm that $s(9_{42}) = 0$ and $\sigma(9_{42}) = 2$. Hence, 9_{42} does not have the first property either. Actually, the 9_{42} knot among the knots with nine crossings or fewer, is the only one that has exception to the two properties stated from [4].

2. From [4] we see that for all but 12 of the 249 prime knots, the nontrivial cohomology groups lie on two adjacent diagonals. Those knots are called homologically thin (**H-thin**). As we may expect, knots that are not H-thin are called **H-thick**, this is homologically thick.

   As a matter of fact, there are 12 H-thick knots up to 10 crossings [10]: 8_{19}, 9_{42}, 10_{124}, 10_{128}, 10_{132}, 10_{136}, 10_{139}, 10_{145}, 10_{152}, 10_{153}, 10_{154}, and 10_{161}. Thus, 8_{19} and 9_{42} are the only knots up to nine crossings with off-diagonal elements in their Khovanov homology tables.

3. 8_{19} is the only non-hyperbolic knot with 8 crossings (it is the (3,4)-torus knot). 9_{42} has the second smallest volume (≈ 4.05686) among all 48 hyperbolic knots with 9 crossings. Moreover, 9_{42} has the smallest determinant (number calculated from the Seifert surface) among 9-crossing knots.

   Coincidentally, 9_{42} has the same volume as 10_{132} (another H-thick knot). This is the pair with the second smallest volume. The pair with the smallest volume is conformed by 5_2 and the (-2,3,7)-pretzel knot.

5 Span of the Jones polynomial

Let us denote as S_AD (S_BD) the state with all crossings solved with an 0-smoothing $(1 - smoothing)$. In this way, $|S_AD|$ is the number of components in the state S_AD. For instance, next picture shows a projection of the 8_{19} and the $S_A(8_{19})$ and $S_B(8_{19})$ states.

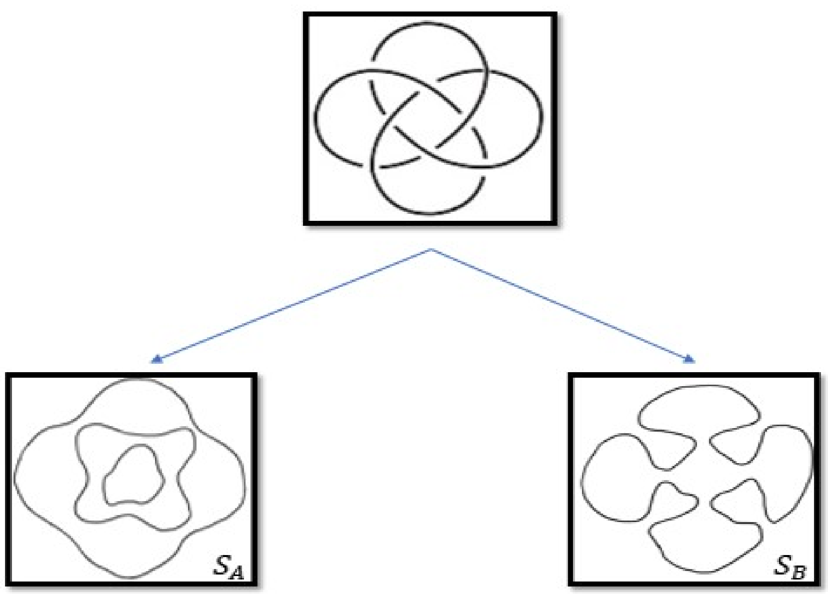


Figure 5.1: Extreme states.

It is desired to have a boundary for the span of the Jones polynomial for a link diagram D in terms of the number of components in the extreme states, this is S_AD and S_BD. If we denote a crossing by i, in a state s we assign a $+1$ or -1 depending on how the crossing is smoothed, as indicated by next figure.

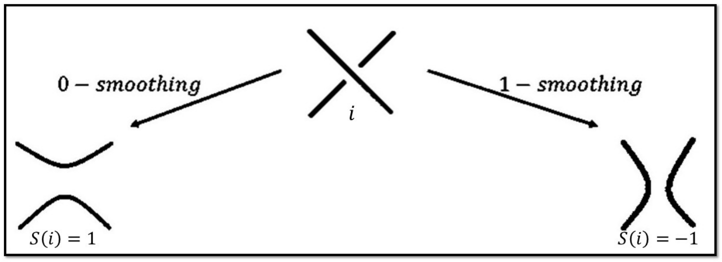


Figure 5.2: Values in a crossing.

Definition 5.0.1 *A diagram D is plus-adequate if* $|S_A D| > |sD|$ *for all state s with* $\sum_{i=1}^{n} S(i) = n-2$. *Likewise, D is called minus-adequate if* $|S_B D| > |sD|$ *for all state s with* $\sum_{i=1}^{n} S(i) = 2-n$. *If D holds both conditions, D is said to be adequate.*

There is another form of determining if a diagram is plus or minus adequate: If the two segments replacing any crossing in $S_A D$ never belong to the same component, the diagram is said to be plus adequate. Analogously for $S_B D$, detects minus adequacy. For instance, the trefoil is an adequate knot.

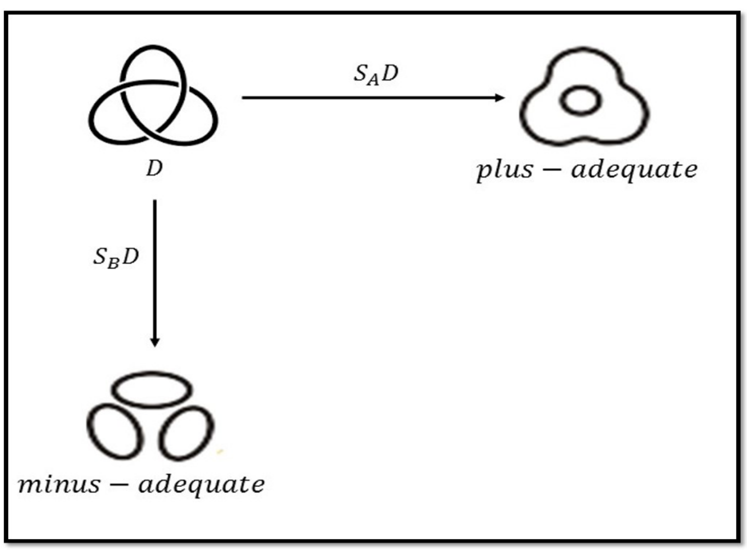


Figure 5.3: Adequate knot.

Lemma 5.0.2 (10, Lemma 5.4) *Let D be a link diagram with n crossings. Then*

1. $M\langle D\rangle \leq n+2|S_A D|-2$*, with equality when D is plus-adequate, and*

2. $m\langle D\rangle \geq -n-2|S_B D|+2$*, with equality when D is minus-adequate.*

Where $M\langle D\rangle$ $(m\langle D\rangle)$ *denotes the maximum (minimum) power in* $\langle D\rangle$.

So that, we have the following boundary for the span of the bracket polynomial, consequently for the Jones polynomial, for a link diagram with n crossings:

$$span\langle D\rangle \leq \ (n+2|S_A D|-2)-(-n-2|S_B D|+2)$$

$$\implies \qquad span\langle D\rangle \leq \ 2n+2\left(|S_A D|+|S_B D|\right)-4$$

Lemma 5.0.3 (10, Lemma 5.7) *Let D be a connected n-crossing diagram.*

1. *If D is alternating, then* $|S_A D|+|S_B D| = n+2$

2. *If D is non-alternating and strongly prime, then* $|S_A D|+|S_B D| < n+2$

5.1 Span for almost alternating knots

In [8] the authors stated that the number of components in the extreme states of a m-almost alternating diagram is given by $|S_A D|+|S_B D| = n+2-2m$, where n is the number of crossings in the diagram. We present a proof for the case of the almost alternating knots ($m=1$).

Theorem 5.1.1 *For a connected, strongly prime, almost alternating diagram D with n crossings*

$$|S_A D|+|S_B D| = n$$

.

Proof. Having an almost alternating diagram D, let D^* be the alternating diagram obtained from D by changing the dealternator crossing. Thus, D^* is connected and alternating. Hence

$$|S_A D^*| + |S_B D^*| = n + 2$$

Given that D and D^* are identical everywhere but in the dealternator crossing, by constructing the S_A and S_B states, the number of components (cycles) will differ just by one. Thus, we have the following possibilities:

1. $|S_A D| = |S_A D^*| + 1$ and $|S_B D| = |S_B D^*| + 1$
2. $|S_A D| = |S_A D^*| + 1$ and $|S_B D| = |S_B D^*| - 1$
3. $|S_A D| = |S_A D^*| - 1$ and $|S_B D| = |S_B D^*| + 1$
4. $|S_A D| = |S_A D^*| - 1$ and $|S_B D| = |S_B D^*| - 1$

Indeed the actual situation is the last one. Suppose the first option holds, then

$$|S_A D| + |S_B D| = |S_A D^*| + |S_B D^*| + 2$$

$$= n + 4 \, (\star)$$

Now suppose the second or third option hold, then

$$|S_A D| + |S_B D| = |S_A D^*| + |S_B D^*|$$

$$= n + 2 \, (\bullet)$$

By Lemma 5.7 Lickorish, we have that $|S_A D| + |S_B D| < n + 2$ for a connected, non-alternating, strongly prime diagram D with n crossings. Thus, $(\star)$ and $(\bullet)$ can not occur.

Notice that, the option 4 tells us that

$$|S_A D| + |S_B D| = |S_A D^*| + |S_B D^*| - 2$$

$$= n + 2 - 2 = n$$

□

Next graph shows an example of the result shown,

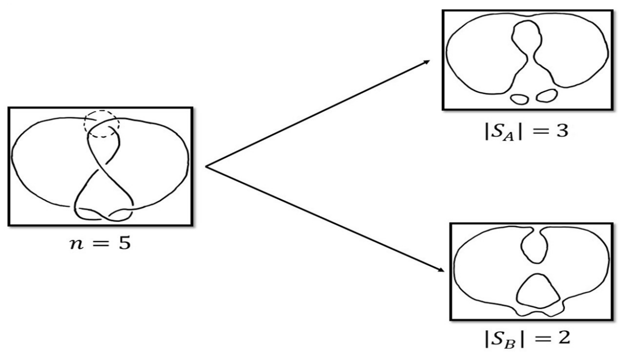


Figure 5.4: Components in the extreme states.

Corollary 5.1.2 [18, *Zhu*] *If D is a dealternator connected, almost alternating diagram with n crossings, then*

$$span\,(V_D(t)) \leq n - 1$$

Proof. Remember that the bracket polynomial is bounded by

$$span\langle D\rangle \leq\; 2n + 2\,(|S_A D| + |S_B D|) - 4$$

$$\implies \quad span\langle D\rangle \leq\; 2n + 2\,(n) - 4 = 4n - 4$$

$$\implies \quad span\langle D\rangle \leq 4\,(n-1)$$

$$\implies \quad 4span\,(V_D(t)) \leq 4(n-1) \qquad (span\langle D\rangle = 4span\,(V_D(t)))$$

and thus, $span\,(V_D(t)) \leq n - 1$ □

Adams *et al*. [2] established a boundary for the span of the bracket polynomial (and thus, for the Jones polynomial) by considering a diagram which is dealternator connected and dealternator reduced as well. So far, we have not considered the diagram to be dealternator reduced; in this section, we will. Let us introduce some results in order to obtain a proof of the result of Adams by calculating the number of components in S_AD and S_BD.

Theorem 5.1.3 (Lickorish) *Let D be an n-crossings adequate diagram. Then*

- *The term of the highest degree in the Kauffman bracket is* $(-1)^{|S_AD|-1}A^M$
- *The term of the lowest degree in the Kauffman bracket is* $(-1)^{|S_BD|-1}A^m$

where M and m are defined as before.

It is well known that each of the terms in the bracket polynomial is congruent module 4 with m and M [8]. Thus, the bracket of D can be explicitly written as

$$\langle D\rangle = a_mA^m + a_{m+4}A^{m+4} + \cdots + a_{M-4}A^{M-4} + a_MA^M \quad (*)$$

where some of the coefficient may be zero and a_M (a_m) denotes the maximal (minimal) coefficient of $\langle D\rangle$.

Lemma 5.1.4 *Suppose D is a dealternator connected almost alternating diagram with n crossings.*

- *Let D_1 (resp. D_2) be the diagram obtained by A-smoothing (resp. B-smoothing) the dealternator crossing. Then,*

  $$|S_AD_1| = |S_AD|, \quad |S_AD_2| = |S_AD|+1, \quad |S_BD_1| = |S_BD|+1, \quad \textit{and} \quad |S_BD_2| = |S_BD|$$

- *Let a_M (resp. a_{M_1}, a_{M_2}) be the hypothetic maximal coefficient of $\langle D\rangle$ (resp. $\langle D_1\rangle$, $\langle D_2\rangle$). Likewise, let a_m (resp. a_{m_1}, a_{m_2}) be the hypothetic minimal coefficient of $\langle D\rangle$ (resp. $\langle D_1\rangle$, $\langle D_2\rangle$). Then,*

  $$a_M = a_{M_1} + a_{M_2} \quad \textit{and} \quad a_m = a_{m_1} + a_{m_2}$$

Proof. Let us prove the first part of the lemma. Clearly S_AD is identical to S_AD_1 and S_BD is identical to S_BD_2.

For $|S_AD|$ and $|S_AD_2|$ we have the following possibilities:

$$|S_AD_2| = |S_AD| + 1 \quad or \quad |S_AD_2| = |S_AD| - 1$$

Suppose $|S_AD_2| = |S_AD| - 1$, then $|S_AD_2| + |S_BD_2| = |S_AD| + |S_BD| - 1$. Since the diagram D is almost alternating, D_2 is an alternating diagram with $n-1$ crossings. Thus, $|S_AD_2| + |S_BD_2| = (n-1)+2$ and $|S_AD| + |S_BD| = n$. Therefore,

$$(n-1)+2 = n-1 \qquad (\lightning)$$

Hence, $|S_AD_2| = |S_AD| + 1$.

Now, suppose $|S_BD_1| = |S_BD| - 1$. Then $|S_BD_1| + |S_AD_1| = |S_BD| + |S_AD| - 1$. Following the same reasoning as above, we obtain the equality $(n-1)+2 = n-1$, which is false. Hence $|S_BD_1| = |S_BD| + 1$.

In order to show the second part of the lemma, remember that by definiton

$$\langle D \rangle = A\langle D_1 \rangle + A^{-1}\langle D_2 \rangle$$

Since D_1 and D_2 are diagrams with $n-1$ crossings the highest terms of their bracket polynomials are given by $(-1)^{|S_AD_1|-1}A^{M_1}$ and $(-1)^{|S_AD_2|-1}A^{M_2}$ respectively. By the previous lemma we know that $|S_AD_1| = |S_AD|$ and $|S_AD_2| = |S_AD| + 1$, thus

$$(-1)^{|S_AD_1|-1}A^{M_1} = (-1)^{|S_AD_1|-1}A^{(n-1)+2|S_AD|-2} \quad and$$

$$(-1)^{|S_AD_2|-1}A^{M_2} = (-1)^{|S_AD_2|-1}A^{(n-1)+2|S_AD|+2-2}$$

therefore,

$$(-1)^{|S_AD_1|-1}A^{M_1} = (-1)^{|S_AD_1|-1}A^{M-1} \quad \Longrightarrow M_1 = M-1 \quad and$$

$$(-1)^{|S_AD_2|-1}A^{M_2} = (-1)^{|S_AD_2|-1}A^{M+1} \quad \Longrightarrow M_2 = M+1$$

Now, considering just the highest powers in $\langle D\rangle = A\langle D_1\rangle + A^{-1}\langle D_2\rangle$ we have

$$a_M A^M = A(a_{M_1})A^{M_1} + A^{-1}(a_{M_2})A^{M_2} \quad \Longrightarrow a_M A^M = a_{M_1}A^{M_1+1} + a_{M_2}A^{M_2-1}$$

$$\Longrightarrow a_M A^M = (a_{M_1} + a_{M_2})A^M$$

and thus, $a_M = a_{M_1} + a_{M_2}$. Analogously, $a_m = a_{m_1} + a_{m_2}$. □

Theorem 5.1.5 (*Adams et al.*) *If D is a dealternator connected and dealternator reduced almost alternating diagram with n crossings, then*

$$span(\langle D\rangle) \leq 4(n-3)$$

Proof. It is already known that the maximal value for the $span(\langle D\rangle)$ is

$$M - m = 2n + 2(|S_AD| + |S_BD|) - 4 = 4(n-1)$$

Recall that $\langle D\rangle$ can be written as in $(*)$. Suppose we have D_1 and D_2 as above. Since D_1 and D_2 are alternating and reduced diagrams, they are adequate as well. Then, we do know how the coefficients of their highest degrees look like,

$$a_{M_1} = (-1)^{|S_AD_1|-1} = (-1)^{|S_AD|-1} \quad and \quad a_{M_2} = (-1)^{|S_AD_2|-1} = (-1)^{|S_AD|}$$

respectively, and thus $a_M = a_{M_1} + a_{M_2} = 0$.

The same reasoning for a_{m_1} and a_{m_2} results in $a_m = 0$. Therefore, $span(\langle D\rangle)$ is at most

$(M-4)-(m+4) = M-m-8$, hence

$$span(\langle D\rangle) \leq 4(n-1)-8$$

$$\implies \quad span(\langle D\rangle) \leq 4(n-3)$$

$\square$

Notice that, since $span(\langle D\rangle) = 4span(V_D(t)$, the last result implies $span(V_D(t) \leq n-3$.

Remark 5.1.6 *A new type of knots,* ***virtual knots****, first introduced by Kauffman in 1996, shows the connection between having the boundaries for the span of the bracket polynomial and Khovanov homology. Virtual knot theory is a generalization of the classical knot theory, considering the embeddings into thickened orientable surfaces of genus not necessarily zero.*

The boundary for the $span(\langle D\rangle)$ allowed the author in [13] to state the conditions for a diagram be minimal by studying the thickness of the Khovanov complex.

A virtual link is defined as a link allowing a new type of crossing: a 4-valent vertex with a circle around it (called a virtual crossing).

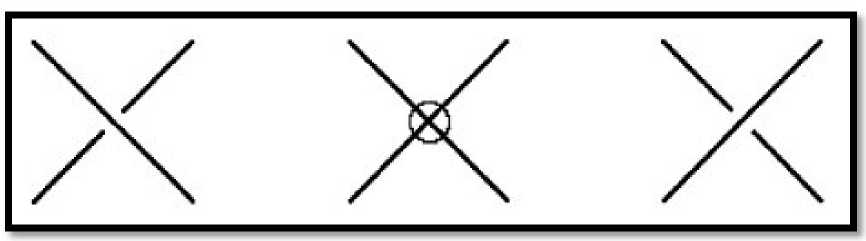


Figure 5.5: Virtual crossing.

The main result states that for an orientable virtual link which is $1-complete$ and $2-complete$, the diagram is minimal.

6 Conclusions and Future Work

6.1 Conclusions

Throughout the work, we have stated that our focus of study are the almost alternating knots. Using results referring to the Conway's notation and the Rolfsen table [14], we identified the three knots that do not hold the equality related to the span of the bracket polynomial presented in Adams' work [2]. By studying the Khovanov bracket there were found some interesting aspects for those three knots, in the sense of discrepancy with respect the others almost alternating knots up through nine crossings. Thus, by noting that the last height in the resolution cube were not contributing to the span of the polynomial, it was decided to look for a better boundary for this number just knowing the number of components in the extreme states.

We then established the inequality in terms of the Jones polynomial and presented proofs of the results of Adams' [2], and González and Manchón [8] for almost alternating knots, by studying the number of components in the extreme states. It turned out that the result from González and Manchón [8] for the case $m = 1$ yields the boundary found by Zhu [18], and then we were able to conclude the main bound considering a diagram which is both dealternator connected and dealternator reduced.

Although by using the Khovanov tables we were not able to identify a distiction for the 9_{46} knot, by studying surfaces we found that this knot was the only one with genus 1. Indeed, this fact highlights a difference between this knot and the others in question, since the 9_{46} is the only one for which it is possible to obtain a Seifert surface (an orientable surface that appears in the complement of any knot with one boundary component such that the boundary circle is that knot, [1]), that resembles a torus. Furthermore, 9_{44} and the 9_{46} are the only knots for which the **signature**, which is a topological invariant computed from the Seifert surface, is zero [4].

6.2 Future work

As future work, it is suggested:

- To study in a deeper way the Khovanov complex of the 8_{19} and the 9_{42}, looking for improvement of the boundary for the span of the Jones polynomial for knots with similar properties.
- To continue in the task of exploring the quantum properties of this knot in order to establish some rule that makes 9_{46} differ from the other almost alternating knots of nine or fewer crossings.
- To study the concept of the signature of a knot in order to find more information about 9_{46}.
- To find a relation between the $(3,k)$-torus knots and the almost alternating knots. We verified that for an almost alternating, (3,k)-torus knot D up to 10 crossings $span(V_t(D)) < n-3$.
- To study the relation between the boundaries for the span of the Jones polynomial and the concept of virtual links, in order to realize more aspects about the homological minimality of the diagram.

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