KHOVANOV HOMOLOGY FOR $(3, k)$-TORUS KNOTS

By

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2016

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This thesis studies the construction of Khovanov homology for $(3, k)$-torus knots by using combinatorial topology and skein theory, identifying common characteristics of Khovanov Bracket for $(3, k)$-torus knots. The *r-th* homology, $\mathcal{H}^r$, of the complex $\mathcal{C}$ is calculated explicitly for $r = 0, 1, 2k-1$ and $2k$, it allows to obtain some exponents of the variables q and t in the graded Poincaré polynomial of the complex $\mathcal{C}$, which is called the Khovanov bracket.

Resumen de Disertación Presentado a Escuela Graduada
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Requerimientos para el grado de Maestría en Ciencias

HOMOLOGIA DE KHOVANOV PARA NUDOS TOROIDALES $(3, k)$

Por

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2016

Consejero: Juan A. Ortiz-Navarro
Departamento: Matemáticas

En esta tesis se analiza la construcción de la homología de Khovanov para nudos toroidales $(3, k)$ mediante el uso de la topología combinatorial y teoría de skein, identificando características comunes del bracket de Khovanov para nudos toroidales $(3, k)$. La r-ésima homología, $\mathcal{H}^r$, del complejo $\mathcal{C}$ se calcula de forma explícita para $r = 0, 1, 2k - 1$ y $2k$, lo que permite obtener algunos exponentes de las variables q y t en el polinomio de Poincaré del complejo $\mathcal{C}$, el cual es llamado Khovanov bracket.

To my husband, my parents and my sister.

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CHAPTER 1
INTRODUCTION

1.1 Justification

In knot theory some invariants have been introduced for the study and classification of knots. One of these is the Jones polynomial and strictly stronger than this: the Khovanov homology, which is a categorification of the Jones polynomial. Khovanov associates to each knot a chain complex of graded vector spaces whose Euler characteristic is the Jones polynomial.

To compute the Khovanov bracket which is the graded Poincaré polynomial of the complex, the states of a diagram of the knot are calculated, which are obtained from the different complete smoothings for the diagram. Each state is a collection of disjoint simple closed curves on the plane. The states are organized in the vertices of a $n-$dimensional cube and graded vector spaces are associated to these to get the chain groups. Finally, a differential is defined on the edges of the cube to construct the homology.

Every knot in S^3 is either a torus knot, a satellite knot or a hyperbolic knot. A torus knot is a knot which can be embedded on the torus as a simple closed curve. A satellite knot is a knot that contains an incompressible, non-boundary parallel torus in its complement and a hyperbolic knot is a knot that has a complement that is a hyperbolic 3-manifold.

Torus knots have a particular form in their projection with minimal number of crossings, as a result the complete smooothings give similar and predictable states, which give rise to a specific homology that can be determined.

The $(2, k)$-torus knots are alternating and their homology is well known. It occupies exactly two diagonals, [2].

Our interest is to analyze what is happening in the cube of smoothings and through the maps on the edges to collapse the cube to a complex; the homology derived from this chain complex and the common characteristics between types of knots, such as torus knots and particularly $(3, k)$-torus knots.

1.2 Previous publications

From the Jones polynomial arises Khovanov homology, which categorifies the Jones polynomial by constructing a chain complex of graded vector spaces such that the homology of this chain complex, which is called the Khovanov homology, is a knot invariant and the graded Euler characteristic of this complex is the Jones polynomial. Hence M. Khovanov constructed a homological theory that generalizes the Jones polynomial, it replaces polynomials with graded vector spaces to turn the Jones polynomial into a homological object.

Dror Bar Natan [2] describes in a more accessible way the Khovanov bracket and shows that this is strictly stronger than the Jones polynomial.

An interesting type of knots are torus knots, that is, knots that lie on an unknotted torus, without crossing over or under themselves as they lie on the torus [1]. To name a torus knot, differentiate two types of curves on a torus: meridian curve and longitude curve. Depending on how many times the knot crosses these curves will be called a (p, q)-torus knot for some pair of integers p and q. Mathematicians such as Kunio Murasugi have worked with these type of knots. He proved that the least number of crossings that occurs in any projection, for a (p, q)-torus knot is exactly the minimum of $p(q-1)$, or $q(p-1)$, [7].

1.3 Objectives

1.3.1 Main objective

- The main objective of this thesis is to find formulas for the Khovanov homology and terms of the Khovanov bracket for $(3, k)$-torus knots, for integers k, relatively prime and greater than 3.

1.3.2 Secondary objectives

- To analize the Khovanov homology and construction of Khovanov bracket and Jones polynomial.
- To study the Khovanov Bracket for $(3, k)$-torus knots .
- To compute the Khovanov bracket and the Khovanov bracket module 2 for some $(3, k)$-torus knots and compare them.
- To identify common characteristics of Khovanov Bracket for $(3, k)$-torus knots.
- To expose the calculation of Khovanov bracket for $(3, k)$-torus knots by considering the projection with least number of crossings.

CHAPTER 2
PRELIMINARES

2.1 Knots

Definition 2.1. *A knot is an embedding of* S^1 *into in the* 3*-dimensional Euclidean space* $\mathbb{R}^3$ *or in the* 3*-sphere* S^3*. A link is an embedding of a disjoint union of* n *circles into* $\mathbb{R}^3$ *or* S^3*. A link of one component is a knot.*

In order to visualize and manipulate knots and links, **projections** in $\mathbb{R}^2$ can be made. The places where the knot or link crosses itself in the picture are called **crossings** and a distinction is made between the strand that passes over and the strand passing below drawing an interruption or discontinuity in the second one. There are many projections of the same knot or link.

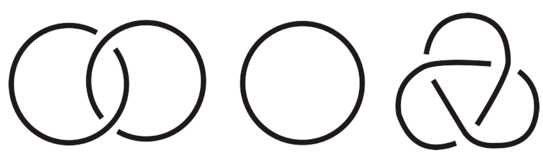


Figure 2–1: Hopf link, unknot and trefoil

A projection of a knot or link is called generic or regular if it has no triple intersections, no tangencies and no cusps. In what follows it will be referred interchangeably to regular o generic projection as projection.

Figure 2–2: A triple intersection, a tangency and a cusp

An **oriented knot (link)** is a knot (link) with an orientation defined, i.e. choosing a direction to travel around it, placing directed arrows along the projection of the knot (link) in the chosen direction.

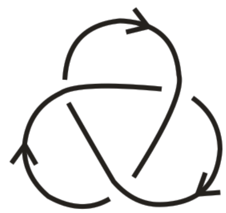


Figure 2–3: An oriented knot

Suppose two projections are taken, whether or not they represent the same knot or link is what it is wanted to know; it is expected that somehow one can get to the other, or say they do not represent the same knot or link. This is a fundamental problem in knot theory and it will be discussed in section (2.2)

2.1.1 Torus knots

There are many types of knots. In this thesis, torus knots are the focus point.

Intuitively speaking, a torus looks like the surface of a doughnut. Topologicaly, a torus is a closed surface defined as the product: $S^1 \times S^1$.

Definition 2.2. *A torus knot is a knot that lies on an unknotted (standard) torus in* $\mathbb{R}^3$*, without crossing over or under itself as it lies on the torus.*

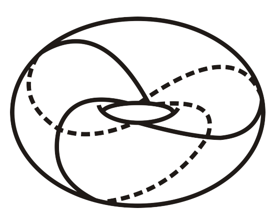


Figure 2–4: A trefoil on a torus

Definition 2.3. *On a torus, a meridian curve is a closed curve that goes once around the shorter way of the torus. A longitude curve is a closed curve that goes once around the long way of the torus. A longitude curve intersects a meridian exactly once.*

To describe a torus knot an associated ordered pair of nonnegative integers (p, q), where p and q are relatively prime is used such that p represents how many times the knot goes around the meridian, and q is the number of times the knot goes around the longitude.

For example, the trefoil knot in figure (2–4) goes three times meridionally around the torus and twice longitudinally, so that $p = 3$, $q = 2$, hence the trefoil knot is called a $(3, 2)-$ torus knot.

Theorem 1. *A $(p, q)-$ torus knot is equivalent to a $(q, p)-$ torus knot, [1].*

A (p, q) torus knot has one projection with $p(q-1)$ crossings and another with $q(p-1)$ crossings. Murasugi [7] proved that the least number of crossings that occurs in any projection, for a (p, q) torus knot is exactly the minimum of $p(q-1)$, or $q(p-1)$.

2.2 Equivalence of knots

Definition 2.4. *Let $f, g : X \longrightarrow Y$ be continuous functions. A continuous function $F : X \times [0, 1] \longrightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$ is called an isotopy if $F|_{X\times\{t\}}$ is a homeomorphism for all $t \in [0, 1]$.*

Definition 2.5. *Let $f, g : Y \longrightarrow X$ be embeddings of Y into X. f and g are ambient isotopic if there is an isotopy $F : X \times [0, 1] \longrightarrow Y$ such that $F(x, 0) = x$ for all $x \in X$ and $F(f(y), 1) = g(y)$ for all $y \in Y$.*

Definition 2.6. *Two knots $f, g : S^1 \longrightarrow \mathbb{R}^3$ are equivalent if they are ambient isotopic.*

Definition 2.7. *A planar isotopy of a knot projection is a continuous deformation of the projection.*

Then, two knots are equivalent if it is possible deform one to the other by ambient or planar isotopy.

2.3 Reidemeister Moves

Suppose two projections of equivalent knots, then ambient or planar isotopy have to exist. However finding such isotopy can be difficult. In 1927, Kurt Reidemeister proved that the existence of an ambient isotopy between knots projections is equivalent to the existence of a sequence of moves, called Reidemeister moves. These are three types of Reidemeister moves:

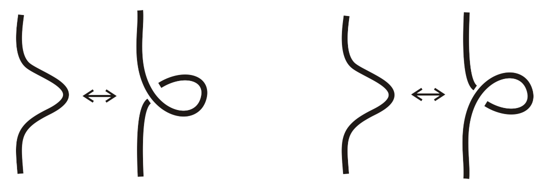


Figure 2–5: Type I Reidemeister move

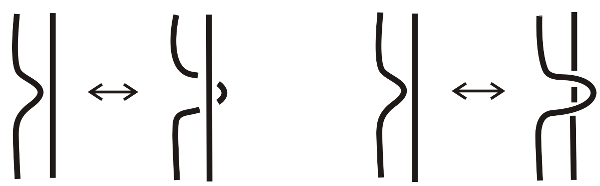


Figure 2–6: Type II Reidemeister move

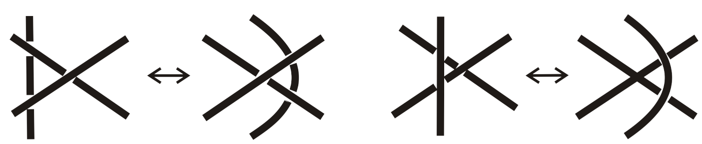


Figure 2–7: Type III Reidemeister move

Definition 2.8. *Two projections are regularly isotopic if one can be obtained from the other by a sequence of Reidemeister moves of type II and III or ambient isotopic by a sequence of Reidemeister moves type I, II and III.*

Theorem 2. *Two knots are equivalent if and only if there is a finite sequence of planar isotopies and Reidemeister moves taking a knot projection of one to a knot projection of the other.*

See [8]

2.4 Knot invariants

As there are many projections for the same knot it is useful to find properties associated to given knot such that they do not depend on the projection.

Definition 2.9. *A knot invariant is a property of a knot that does not change under ambient isotopy.*

Hence, if two knots have different values for any knot invariant they are not equivalent. A consequence of this definition and the Reidemeister's theorem is that to prove a given property of a knot is a knot invariant is sufficient to show that it is invariant under the three types of Reidemeister moves and planar isotopy.

An invariant can be a mathematical entity such as a numerical value, a polynomial or an algebraic group.

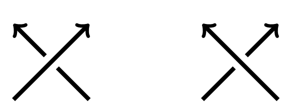
2.5 Polynomial invariants

Associating polynomials to each knot is a useful and interesting way to study knots. Polynomials invariants are computed from a projection but any two projections of the same knot generate the same polynomial.

2.5.1 Jones polynomial

In 1984, V. Jones was working with operator algebras and he discovered a Laurent polynomial with integer coefficients, i.e. a finite polynomial expression that can have both positive and negative powers of the indeterminate.

Let L be an oriented knot projection with n crossings labeled from 1 to n in some arbitrary way, let $\mathcal{X}$ be the set of crossings, let n_+, n_- be the number of right handed (positive) and left handed (negative) crossings in $\mathcal{X}$, respectively, as in figure (2–8). So $n = n_+ + n_-$.



$+1$ $\qquad$ -1

Figure 2–8: Possitive and negative crossings

Each crossing can be smoothed in two different ways, either by $0 - \textit{smoothing}$ or $1 - \textit{smoothing}$ according to figure (2–9).

Figure 2–9: Smoothings for a crossing

The Kauffman bracket is a function that maps L to an element of the ring of Laurent polynomials with integers coeficients in an indeterminate q, $\langle L\rangle \in \mathbb{Z}\left[q, q^{-1}\right]$, which satisfies:

1. $\langle \bigcirc \rangle = 1$
2. $\langle L \sqcup \bigcirc \rangle = (q + q^{-1})\langle L\rangle$
3. $\langle \times \rangle = \langle \asymp \rangle - q\,\langle)(\rangle$

From these axioms, the unormalized Jones Polynomial is

$$\widehat{J}\,(L) = (-1)^{n_-}\, q^{n_+ - 2n_-}\, \langle L\rangle$$

and the Jones polynomials is

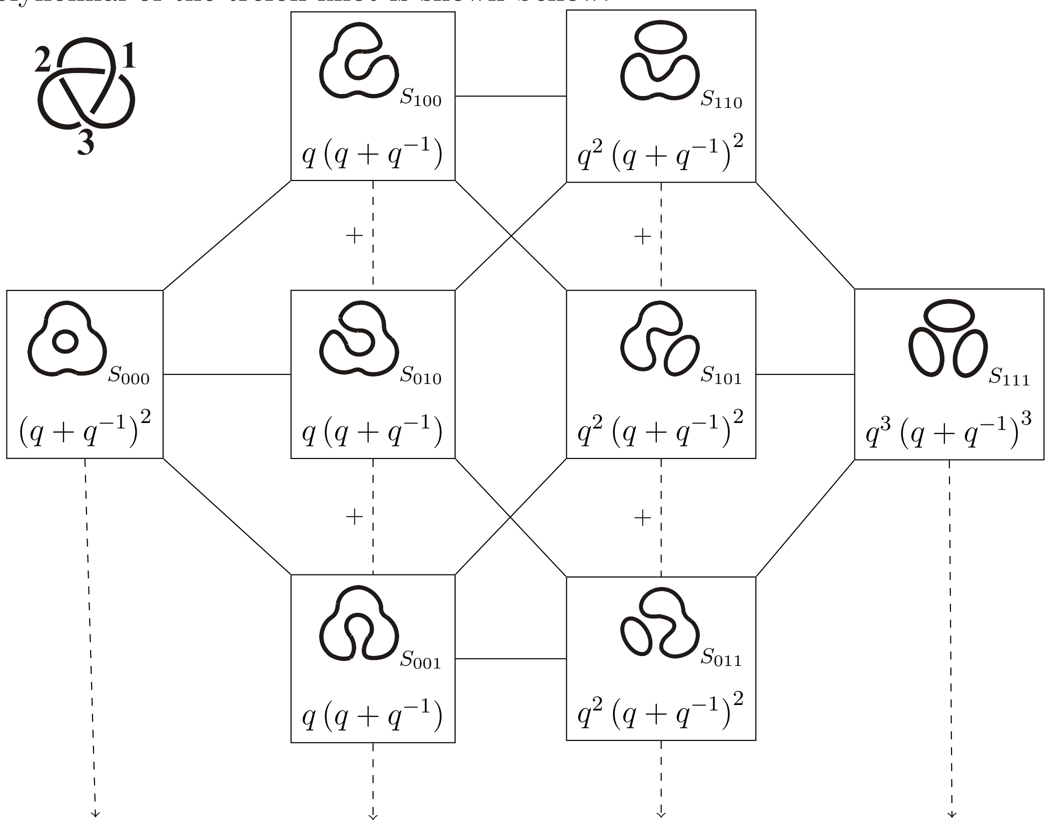
$$J\,(L) := \frac{\widehat{J}\,(L)}{q + q^{-1}}.$$

Each vertex α of the $n-$dimensional cube $\{0,1\}^{\mathcal{X}}$, corresponds to a complete smoothing S_α of L, where all the crossings are smoothed and according to α the result is a union of planar cycles. This complete smoothing is called a state of L. This cube is formed by the 2^n smoothings and each of the smoothings can be indexed by a word of n zeros and ones.

Each union S_α of k cycles is replaced by a term of the form

$$(-1)^r\, q^r\, \left(q + q^{-1}\right)^k,$$

where r is the height of the smoothing, that is, the number of 1 smoothings used. The Jones polynomial is the sum of the resulting terms over all $\alpha \in \{0,1\}^{\mathcal{X}}$ multiplied by the normalization term $(-1)^{n_-} q^{n_+ - 2n_-}$. An example of the computation of the Jones polynomial of the trefoil knot is shown bellow.



$$\left(q+q^{-1}\right)^2 \qquad -3q\left(q+q^{-1}\right) \qquad +3q^2\left(q+q^{-1}\right)^2 \qquad -q^3\left(q+q^{-1}\right)^3$$

$$= q^{-2} + 1 + q^2 - q^6 \xrightarrow[n_+ = 3,\, n_- = 0]{\cdot (-1)^{n_-} q^{n_+ - 2n_-}} q + q^3 + q^5 - q^9 \xrightarrow{\cdot \left(q+q^{-1}\right)^{-1}} J\left(\circledast\right) = q^2 + q^6 - q^8$$

2.6 Khovanov homology

M. Khovanov developed a categorification of the Jones polynomial, which associates to each knot a chain complex of graded vector spaces whose graded Euler characteristic is the Jones polynomial.

Definition 2.10. *A graded vector space is a vector space W together with a decomposition $W = \bigoplus_{m \in \mathbb{Z}} W_m$, with homogeneous components $\{W_m\}$.*

Definition 2.11. *Let W be a graded vector space, the graded dimension of W is the power series $dimW := \sum_m q^m \dim W_m$.*

Definition 2.12. *A chain complex $\mathcal{C}$ is a sequence of homomorphisms of abelian groups with differential maps d^r, ... $\to \overline{\mathcal{C}}^r \xrightarrow{d^r} \overline{\mathcal{C}}^{r+1}$..., such that $d^{r+1} \circ d^r = 0$ for each $r \in \mathbb{Z}$.*

Note that $\mathrm{Im}(d^i) \subseteq ker\,(d^{i+1})$.

Definition 2.13. *Let $\cdot\{l\}$ be the "degree shift" operation on graded vector spaces. That is, if $W = \bigoplus_{m\in\mathbb{Z}} W_m$ is a graded vector space, we set $W\{l\}_m := W_{m-l}$, so that $q\dim W\{l\} = q^l q \dim W$.*

Definition 2.14. *Likewise, let $\cdot[s]$ be the "height shift" operation on chain complexes. That is, if $\overline{\mathcal{C}}$ is a chain complex ... $\to \overline{\mathcal{C}}^r \xrightarrow{d^r} \overline{\mathcal{C}}^{r+1}$...of vector spaces (we call r the "height" of a piece $\overline{\mathcal{C}}^r$ of that complex)and if $\mathcal{C} =\overline{\mathcal{C}}\,[s]$, then $\mathcal{C}^r = \overline{\mathcal{C}}^{r-s}$(with all differentials shifted accordingly).*

Definition 2.15. *The homology of a chain complex is the set of modules, $\mathcal{H}^r(\mathcal{C})$ given by $\mathcal{H}^r(\mathcal{C}) = \ker d^r / Im\,(d^{r-1})$.*

Definition 2.16. *The graded Euler characteristic $\mathcal{X}_q(\mathcal{C})$ of a chain complex $\mathcal{C}$ is the alternating sum of the graded dimensions of its homology groups.*

2.6.1 Computing the Khovanov homology

Let L be an oriented knot projection and let $\mathcal{X}$, n, n_+ and n_-be as in the previous section.

Let V be a graded vector space and let v_+, v_- be its basis elements, whose degrees are $+1$ and -1 respectively, then $q\dim V = q + q^{-1}$. Each vertex $\alpha \in \{0,1\}^{\mathcal{X}}$ will be associated to the graded vector space $V_\alpha(L) := V^{\otimes k}\{r\}$, where k is the number of cycles in S_α and r is the height $|\alpha| = \sum_i \alpha_i$ of α.

Then, the r^{th} chain group $[\![L]\!]^r$ where $0 \leq r \leq n$, is the direct sum of all vector spaces at height r: $[\![L]\!]^r := \oplus_{\alpha : r = |\alpha|} V_\alpha(L)$.

Now, with the differential that will be defined bellow, the sequence of spaces $\mathcal{C}(L) := [\![L]\!]\,[-n_-]\,\{n_+ - 2n_-\}$will be a chain complex.

2.6.2 Maps on the cube

The edges of the $n-$dimensional cube $\{0,1\}^{\mathcal{X}}$ will be identified by sequences in $\{0,1,*\}^{\mathcal{X}}$ with just one $*$, where the tail of such an edge is found by setting $* \longrightarrow 0$ and the head by setting $* \longrightarrow 1$. The height $|\xi|$ of an edge ξ is the height of its tail, and hence if the maps on the edges are called d_ξ, then the maps with the same height are collapsed in $d^r := \sum_{|\xi|=r} (-1)^{\xi} d_\xi$.

For any edge d_ξ, the smoothing at the tail of ξ differs from the smoothing at the head of ξ by just a little: either two of the circles merge into one or one of the cycles splits in two. So for any ξ, let d_ξ be the identity on the tensor factors corresponding to the cycles that do not participate. The definition of ξ is completed by using two linear maps: $m : V \otimes V \longrightarrow V$ and $\Delta : V \longrightarrow V \otimes V$

$$m : \begin{cases} v_+ \otimes v_- \longmapsto v_- \quad v_+ \otimes v_+ \longmapsto v_+ \\ v_- \otimes v_+ \longmapsto v_- \quad v_- \otimes v_- \longmapsto 0 \end{cases}$$

$$\Delta : \begin{cases} v_+ \longmapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \longmapsto v_- \otimes v_- \end{cases}$$

Now, the sequence of spaces $\mathcal{C}(L) := [\![L]\!] [-n_-] \{n_+ - 2n_-\}$ with the differential explained above is a chain complex, the graded Euler characteristic of $\mathcal{C}$, $\mathcal{X}_q(\mathcal{C})$ by definition, is the alternating sum of the graded dimensions of its homology groups, and, if the degree of the differential d is 0 and all chain groups are finite dimensional, it is also equal to the alternating sum of the graded dimensions of the chain groups. Because of the explanation above, the following can be stated:

Theorem 3. *The graded Euler characteristic of $\mathcal{C}(L)$ is the unnormalized Jones polynomial of L:*

$$\mathcal{X}_q(\mathcal{C}(L)) = \widehat{J}(L).$$

2.6.3 The homology

Theorem 4. *The sequences $[\![L]\!]$ and $\mathcal{C}(L)$ are chain complexes.*

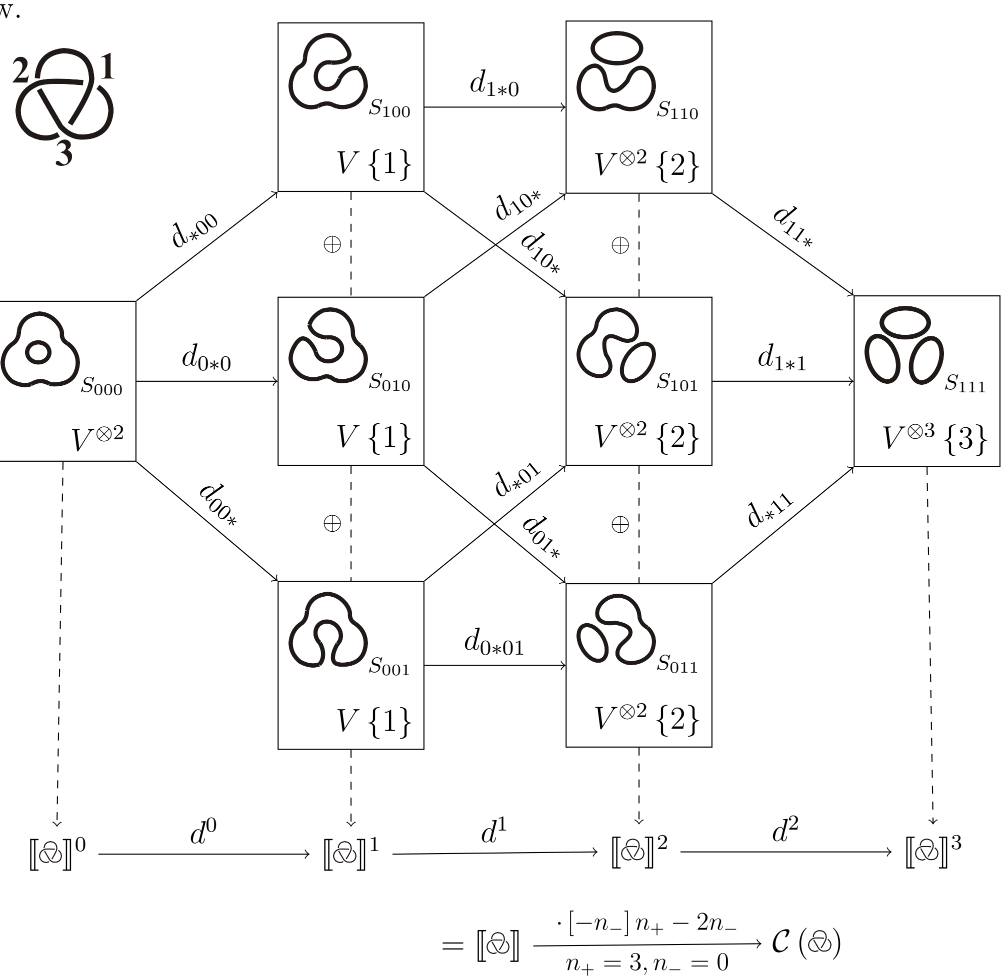
Let $\mathcal{H}^r(L)$ be the r-th homology of the complex $\mathcal{C}(L)$. It is a graded vector space depending on the link projection L. Let $Kh(L)$ be the graded Poincaré polynomial of the complex $\mathcal{C}(L)$in the variable t:

$$Kh(L) := \sum_r t^r q \dim \mathcal{H}^r(L).$$

Theorem 5. *The graded dimensions of the homology groups $\mathcal{H}^r(L)$ are knot invariants, and hence $Kh(L)$, a polynomial in the variables t and q, is a knot invariant that specializes to the unnormalized Jones polynomial at $t = -1$.*

The proofs of these theorems can be read in [5].

To demonstrate the calculation of Khovanov homology, consider the trefoil knot bellow.



CHAPTER 3
KHOVANOV BRACKET FOR $(3, k)$-TORUS KNOTS

To expose the calculation of Khovanov bracket for a $(3, k)$-torus knot, $k > 3$, consider a projection with the least number of crossings.

Because of Murasugi [7], the least number of crossings for a $(3, k)$-torus knot is the minimum of $3\,(k-1)$ or $2k$, that is, $2k$, since $k > 3$.

According to the definition of a torus knot and [6], a $(3, k)$-torus knot is a $(k, 3)$-torus knot, it goes k times meridionally around the torus and 3 times longitudinally. If a projection is drawn, a piece of the strand which passes under is going to cross 2 pieces of the strand that go above it. Since there are k pieces of strand going under the number of crossings in the projection will be $2k$. See an example in figure (3–1).

Definition 3.1. *In a projection with number of crossings $2k$ of a $(3, k)$-torus knot, the k crossings in the center of the knot will be called inner crossings and the k crossings in the outside of the knot will be called outer crossings.*

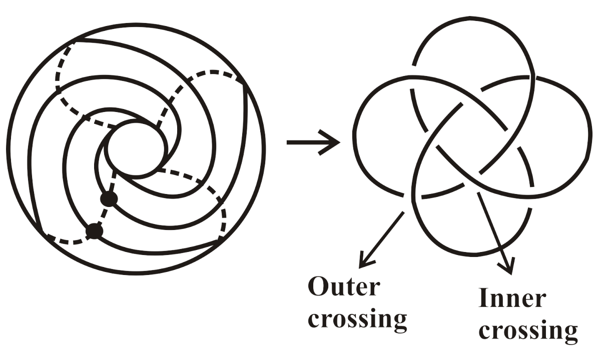


Figure 3–1: Projection of a $(3, 4)$-torus knot

First of all, the projection will be oriented arbitrarily by choosing a direction that remains constant throughout the projection; based on this, the $2k$ crossings of the knot will be labeled right handed (possitive) or left handed(negative) to find n_+ and n_-. Hence, $n_+ = 2k$ and $n_- = 0$.

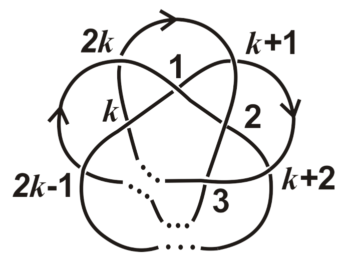
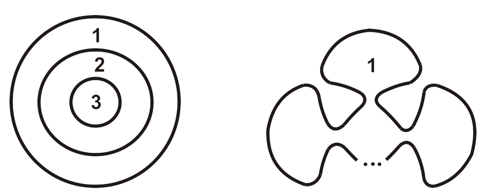


Figure 3–2: Oriented $(3, k)$-Torus knot

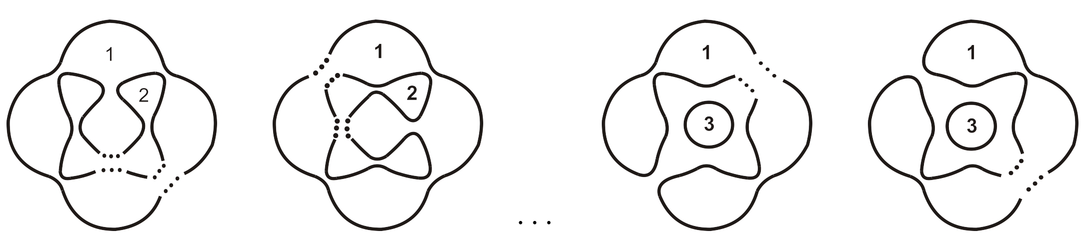
Since $T\,(3, k)$ has $2k$ crossings, it has 2^{2k} states. The states with height 0 and $2k$ will be shown, that is, the states with all $2k$ crossings $0-$smoothed or $1-$smoothed, respectively. Cycles that appear in each state are enumerated to later organize each piece of the differential maps according to the cycles it interacts with.



$S_{00\ldots00}$ $S_{11\ldots11}$

Figure 3–3: States of height 0 and $2k$

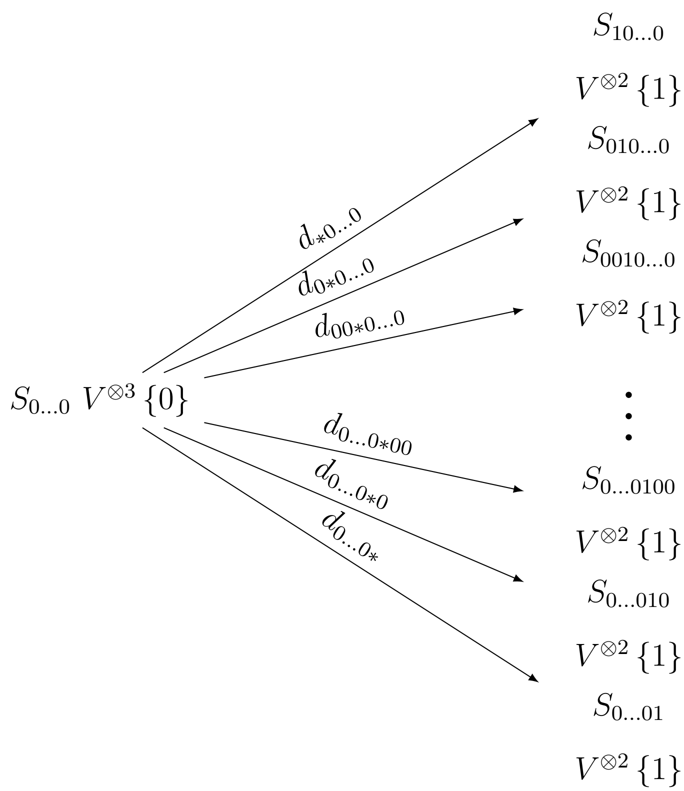
There are $2k$ states with height 1.



$S_{10\ldots0}$ $S_{010\ldots0}$ $\cdots$ $S_{0\ldots010}$ $S_{0\ldots01}$

Figure 3–4: States of height 1

The portion of the cube with states with height 0 and 1 is in the following figure.



The vertex in the cube with state $S_{00...0}$ has 3 cycles and its height is 0, hence it is associated to the graded vector space $V^{\otimes 3}\{0\}$. Therefore the 0-chain group is $[\![T(3,k)]\!]^0 = V^{\otimes 3}\{0\}$.

Each of the $2k$ vertices of the states $S_{10...0}$, $S_{010...0}$, $S_{0010...0}$, $S_{00010...0}$, $S_{0...010...0}$,... have 2 cycles and the height of each one is 1, then, the graded vector spaces associated are $V^{\otimes 2}\{1\}$. Therefore the 1^{st}- chain group $[\![T(3,k)]\!]^1$ is

$$V^{\otimes 2}\{1\} \oplus V^{\otimes 2}\{1\} \oplus V^{\otimes 2}\{1\} \oplus ... \oplus V^{\otimes 2}\{1\}.$$

Therefore, the edges identified with sequences of height 0 are m maps and they are collapsed in d^0. If the state reached by the edge has an inner crossing $1-$smoothed, the edge will be a m_{23} map, but if the state reached by the edge has an outer crossing $1-$smoothed, the edge will be a m_{12}. These are for example:

$$d_{*0\ldots 0} : S_{0\ldots 0} \to S_{10\ldots 0} \qquad d_{0\ldots 0*} : S_{0\ldots 0} \to S_{0\ldots 01}$$

$$m_{23} \otimes Id_1 : V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes V_2 \qquad m_{12} \otimes Id_3 : V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes V_2$$

Since cycles and tensor factors associated in states are labeled, then subscripts in m and Δ maps to denote labeled tensor factor which map is acting on are used. Therefore, the m_{12} map means the extension of m on the tensor factors V_1 and V_2 and it results in $V_{\min\{1,2\}} = V_1$.

A basis for $V^{\otimes 3}$ is

$$\{v_+ \otimes v_+ \otimes v_+, v_+ \otimes v_+ \otimes v_-, v_+ \otimes v_- \otimes v_+, v_- \otimes v_+ \otimes v_+,$$
$$v_+ \otimes v_- \otimes v_-, v_- \otimes v_- \otimes v_+, v_- \otimes v_+ \otimes v_-, v_- \otimes v_- \otimes v_-\}$$

Notice that $v_- \otimes v_- \otimes v_- \in \ker d^0$, furthermore , $m_{12}\left(v_- \otimes v_- \otimes v_+\right) = 0$ but $m_{23}\left(v_- \otimes v_- \otimes v_+\right) = v_- \otimes v_-$ and $m_{23}\left(v_+ \otimes v_- \otimes v_-\right) = 0$ but $m_{12}\left(v_+ \otimes v_- \otimes v_-\right) = v_- \otimes v_-$.

Then, $v_- \otimes v_- \otimes v_+ \oplus v_+ \otimes v_- \otimes v_- - v_- \otimes v_+ \otimes v_- \in \ker d^0$, furthermore,

$$\mathcal{H}^0 = \ker d^0 = \langle v_- \otimes v_- \otimes v_-, v_- \otimes v_- \otimes v_+ \oplus v_+ \otimes v_- \otimes v_- - v_- \otimes v_+ \otimes v_- \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$$

Since $v_- \otimes v_- \otimes v_-$ has degree -3, $v_- \otimes v_- \otimes v_+ \oplus v_+ \otimes v_- \otimes v_- - v_- \otimes v_+ \otimes v_-$ has degree -1 and the shifting $n_+ - 2n_- = 2k$, the grade for the first $\mathbb{Z}$ is $2k-3$ and the grade for the second $\mathbb{Z}$ is $2k-1$, the first term of Khovanov bracket is $q^{2k-3}t^0 + q^{2k-1}t^0 = q^{2k-3} + q^{2k-1}$.

Now, d^0 evaluated in the elements of the basis is, respectively:

$(v_+ \otimes v_+, \ldots 2k \ \times \ldots, v_+ \otimes v_+)$

$(v_+ \otimes v_-, \ldots 2k \ \times \ldots, v_+ \otimes v_-)$

$(v_+ \otimes v_-, \ldots k \ \times \ldots, v_- \otimes v_+, \ldots k \ \times \ldots, v_- \otimes v_+)$

$(v_- \otimes v_+, \ldots 2k \ \times \ldots, v_- \otimes v_+)$

$(0, \ldots k \ \times \ldots, 0, v_- \otimes v_-, \ldots k \ \times \ldots, v_- \otimes v_-)$

$(v_- \otimes v_-, ...k \ \times ..., v_- \otimes v_-, 0, ...k \ \times ..., 0)$

$(v_- \otimes v_-, ...2k \ \times ..., v_- \otimes v_-)$

Where $2k\ \times$ and $k\ \times$ mean how many times the element written before them is repeated. For example $(0, ...k \ \times ..., 0, v_- \otimes v_-, ...k \ \times ..., v_- \otimes v_-)$ means the first k components of this element are 0 and the last k components are $v_- \otimes v_-$.

Then,

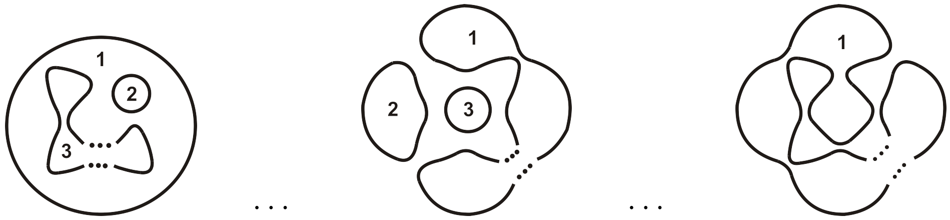
$$\begin{aligned} \text{Im } d^0 \ = \ & \langle (v_+ \otimes v_+, ...2k \ \times ..., v_+ \otimes v_+) \, , \\ & (v_- \otimes v_-, ...2k \ \times ..., v_- \otimes v_-) \, , \\ & (v_+ \otimes v_-, ...2k \ \times ..., v_+ \otimes v_-) \, , \\ & (v_- \otimes v_+, ...2k \ \times ..., v_- \otimes v_+) \, , \\ & (v_- \otimes v_+, ...k \ \times ..., v_- \otimes v_+, v_+ \otimes v_-, ...k \ \times ..., v_+ \otimes v_-) \, , \\ & (v_- \otimes v_-, ...k \ \times ..., v_- \otimes v_-, 0, ...k \ \times ..., 0) \rangle \end{aligned}$$

There are $\dfrac{(2k)!}{2!\,(n-2)!} = k\,(2k-1)$ states or vertices of height 2.

The states in which the $1-$smoothed crossings are two inner crossings are $\dfrac{k!}{2!\,(k-2)!} = \dfrac{k\,(k-1)}{2}$ and they have 3 cycles, then, the graded vector space associated to each one is $V^{\otimes 3}\,\{2\}$.

The states in which the $1-$smoothed crossings are one of the inner crossings and one of the outer crossings are k^2 and they have 1 cycle, then, the graded vector space associated to each one is $V\,\{2\}$.

The states in which the $1-$smoothed crossings are two outer crossings are $\dfrac{k!}{2!\,(k-2)!} = \dfrac{k\,(k-1)}{2}$ and they have 3 cycles, then, the graded vector space associated to each one is $V^{\otimes 3}\,\{2\}$.

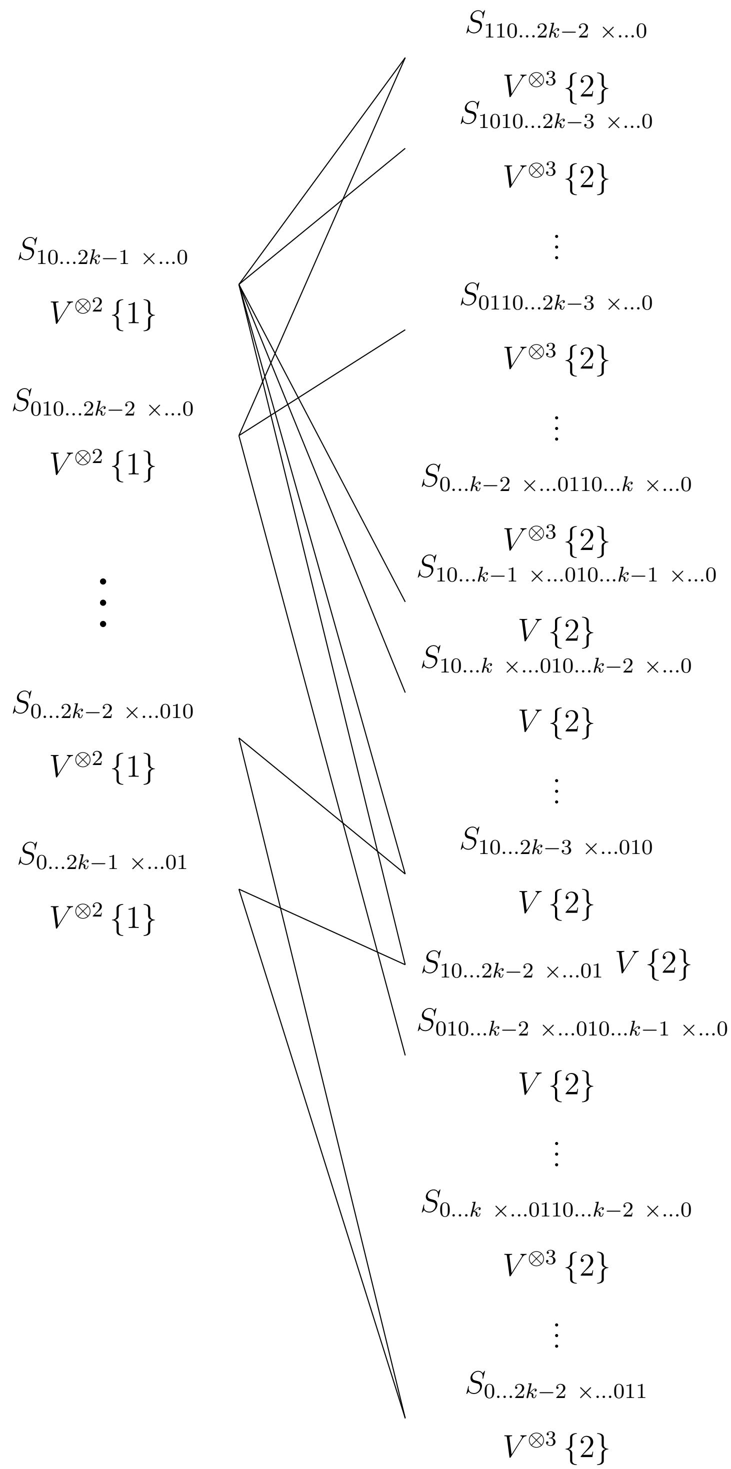


$S_{110\ldots 2k-2\ \times\ldots 0}$ $\cdots$ $S_{0\ldots 2k-2\ \times\ldots 011}$ $\cdots$ $S_{10\ldots k-1\ \times\ldots 010\ldots k-1\ \times\ldots 0}$

Figure 3–5: States of height 2

Without loss of generality, states of height 2 will be organized such that the first states will be the states in which the $1-$smoothed crossings are two inner crossings, then the states in which the $1-$smoothed crossings are one of the inner crossings and one of the outer crossings and last the states in which the $1-$smoothed crossings are two outer crossings.

The portion of the cube with the states with height 1 and 2 will be shown. The first $\dfrac{k(k-1)}{2}$ states of height 2 have 3 cycles, the following k^2 states have 1 cycle and the last $\dfrac{k(k-1)}{2}$ states have 3 cycles.



Therefore, the maps in the edges of height 1 collapsed in d^1 are Δ maps in cases of either two inner crossings or two outer crossings are 1-smoothed and m maps in the other case. These are:

$d_{1*0\dots 2k-2\ \times\dots 0} : S_{10\dots 2k-1\ \times\dots 0} \to S_{110\dots 2k-2\ \times\dots 0}$

$-\Delta_{23} \otimes Id_1 : V_1 \otimes V_2 \to V_1 \otimes V_2 \otimes V_3$

$d_{*10\dots 2k-2\ \times\dots 0} : S_{010\dots 2k-2\ \times\dots 0} \to S_{110\dots 2k-2\ \times\dots 0}$

$\Delta_{23} \otimes Id_1 : V_1 \otimes V_2 \to V_1 \otimes V_2 \otimes V_3$

$d_{10*0\dots 2k-3\ \times\dots 0} : S_{10\dots 2k-1\ \times\dots 0} \to S_{1010\dots 2k-3\ \times\dots 0}$

$-\Delta_{23} \otimes Id_1 : V_1 \otimes V_2 \to V_1 \otimes V_2 \otimes V_3$

$d_{*010\dots 2k-3\ \times\dots 0} : S_{0010\dots 2k-3\ \times\dots 0} \to S_{1010\dots 2k-3\ \times\dots 0}$

$\Delta_{23} \otimes Id_1 : V_1 \otimes V_2 \to V_1 \otimes V_2 \otimes V_3$

$$\vdots$$

$d_{10\dots k-1\ \times\dots 0*0\dots k-1\ \times\dots 0} : S_{10\dots 2k-1\ \times\dots 0} \to S_{10\dots k-1\ \times\dots 010\dots k-1\ \times\dots 0}$

$-m_{12} : V_1 \otimes V_2 \to V_1$

$d_{*0\dots k-1\ \times\dots 010\dots k-1\ \times\dots 0} : S_{0\dots k\ \times\dots 010\dots k-1\ \times\dots 0} \to S_{10\dots k-1\ \times\dots 010\dots k-1\ \times\dots 0}$

$m_{13} : V_1 \otimes V_3 \to V_1$

$$\vdots$$

$d_{10\dots 2k-2\ \times\dots 0*} : S_{10\dots 2k-1\ \times\dots 0} \to S_{10\dots 2k-2\ \times\dots 01}$

$-m_{12} : V_1 \otimes V_2 \to V_1$

$d_{*0\dots 2k-2\ \times\dots 01} : S_{0\dots 2k-1\ \times\dots 01} \to S_{10\dots 2k-2\ \times\dots 01}$

$m_{13} : V_1 \otimes V_2 \to V_1$

$$\vdots$$

$d_{0\dots k\ \times\dots 010\dots k-2\ \times\dots 0*} : S_{0\dots k\ \times\dots 010\dots k-1\ \dots 0} \to S_{0\dots k\ \times\dots 010\dots k-2\ \times\dots 01}$

$-\Delta_{13} \otimes Id_2 : V_1 \otimes V_2 \to V_1 \otimes V_2 \otimes V_3$

$d_{0\dots k\ \times\dots 0*0\dots k-2\ \times\dots 01} : S_{0\dots 2k-1\ \times\dots 01} \to S_{0\dots k\ \times\dots 010\dots k-2\ \times\dots 01}$

$\Delta_{13} \otimes Id_2 : V_1 \otimes V_2 \to V_1 \otimes V_2 \otimes V_3$

$$\vdots$$

$d_{0...2k-2\ \times ...01*} : S_{0...2k-2\ \times ...010} \to S_{0...2k-2\ \times ...011}$

$-\Delta_{13} \otimes Id_2 : V_1 \otimes V_2 \to V_1 \otimes V_2 \otimes V_3$

$d_{0...2k-2\ \times ...0*1} : S_{0...2k-1\ \times ...01} \to S_{0...2k-2\ \times ...011}$

$\Delta_{13} \otimes Id_2 : V_1 \otimes V_2 \to V_1 \otimes V_2 \otimes V_3$

Notice that if $\Delta\left((x, 0, ...2k-1\ \times ..., 0)\right) = \Delta\left((y, 0, ...2k-1\ \times ..., 0)\right)$, where $x, y \in \{v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-\}$, then $x = y$, but if $m\left((x, 0, ...2k-1\ \times ..., 0)\right) = m\left((y, 0, ...2k-1\ \times ..., 0)\right)$, then $x = y$ or $x = v_+ \otimes v_-$ and $y = v_- \otimes v_+$ or conversely, or $x = v_- \otimes v_-$ and $y = 0$ or conversely. Also, each state of height 2 is reached by 2 maps of height 1, one of these is positive and the other is negative.

To obtain the kernel of d^1 the following are considered:

The elements in which each state is the same element basis element are in the kernel of d^1, these are,

$$\left(v_+ \otimes v_+, ...2k\ \times ..., v_+ \otimes v_+\right),$$

$$\left(v_+ \otimes v_-, ...2k\ \times ..., v_+ \otimes v_-\right),$$

$$\left(v_- \otimes v_+, ...2k\ \times ..., v_- \otimes v_+\right),$$

$$\left(v_- \otimes v_-, ...2k\ \times ..., v_- \otimes v_-\right).$$

Since the first $\dfrac{k(k-1)}{2}$ states of height 2 are just reach by two Δ edges which tails are two of the k first states of height 1 and these states are combined between them, then the k first elements of $[\![T(3,k)]\!]^1$ have to be evaluated in the same basis element to be in the kernel of $d^{1\cdot}$

As the states of 1 cycle of height 2 are just reached by one m edge which tail is one of the k first elements of height 1 and one m edge which tail is one of the k last elements of height 1, then they can be evaluated in the same basis element or

$v_+ \otimes v_-$ and $v_- \otimes v_+$ or conversely or $v_- \otimes v_-$ and 0 or conversely to be in the kernel of d^1.

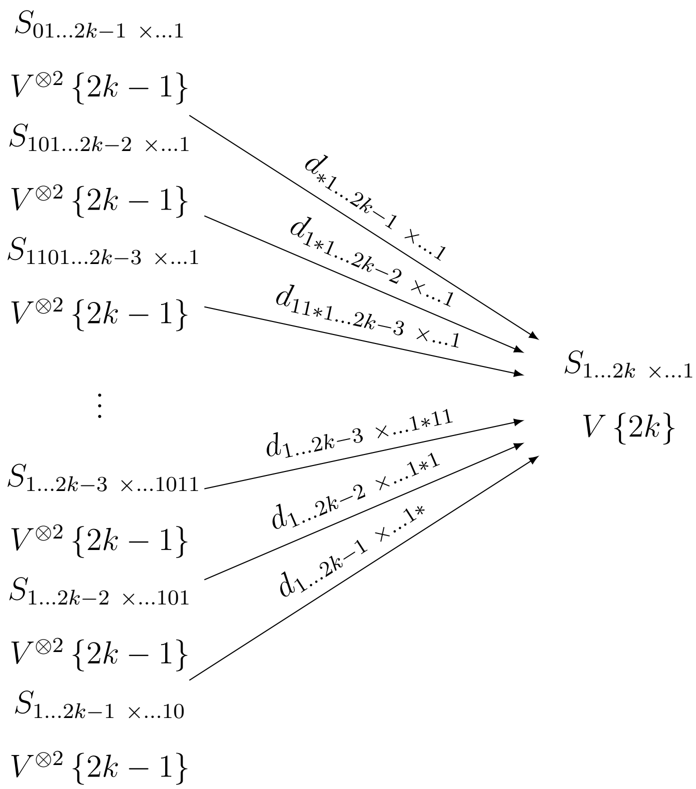
Due to the last $\dfrac{k(k-1)}{2}$ states of height 2 are just reach by two Δ edges which tails are two of the k last states of height 1 and these states are combined between them, then the k last elements of $[\![T(3,k)]\!]^1$ have to be evaluated in the same basis element to be in the kernel of $d^{1\cdot}$

Because of these considerations it is held that

$$
\begin{aligned}
\ker d^1 \quad = \quad & \langle (v_+ \otimes v_+, ...2k \ \times ..., v_+ \otimes v_+) \ , \\
& (v_- \otimes v_-, ...2k \ \times ..., v_- \otimes v_-) \, , \\
& (v_+ \otimes v_-, ...2k \ \times ..., v_+ \otimes v_-) \, , \\
& (v_- \otimes v_+, ...2k \ \times ..., v_- \otimes v_+) \, , \\
& (v_- \otimes v_+, ...k \ \times ..., v_- \otimes v_+, v_+ \otimes v_-, ...k \ \times ..., v_+ \otimes v_-) \, , \\
& (v_- \otimes v_-, ...k \ \times ..., v_- \otimes v_-, 0, ...k \ \times ..., 0)\rangle
\end{aligned}
$$

Hence $\mathcal{H}^1 = \ker d^1 / \mathrm{Im}\ d^0 = \{0\{$.

Now the portion of the cube with the states with height $2k$ and $2k-1$ will be shown.



The vertex in the cube with the state $S_{1\ldots 1}$ has 1 cycle and its height is $2k$, hence it is associated to the graded vector space $V\{2k\}$. Therefore the $2k$- chain group is $[\![T(3,k)]\!]^{2k} = V\{2k\}$.

Each of the vertices of the states with height $2k-1$ have 2 cycles, then, the graded vector space associated are $V^{\otimes 2}\{2k-1\}$. Therefore,

$$[\![T(3,k)]\!]^{2k-1} = V^{\otimes 2}\{2k-1\} \oplus V^{\otimes 2}\{2k-1\} \oplus \ldots \oplus V^{\otimes 2}\{2k-1\}.$$

$S_{01\ldots 2k-1\ \times\ldots 1}$ $\qquad$ $S_{1\ldots 2k-1\ \times\ldots 10}$

Figure 3–6: States of height $2k-1$

The edges identified with sequences of height $2k-1$ are m maps and they are collapsed in d^{2k-1}, these are

$$d_{*1...2k-1\ \times...1} : S_{01...2k-1\ \times...1} \rightarrow S_{1...2k\ \times...1}$$
$$m_{12} : V_1 \otimes V_2 \rightarrow V_1$$

$$d_{1*1...2k-2\ \times...1} : S_{101...2k-2\ \times...1} \rightarrow S_{1...2k\ \times...1}$$
$$-m_{12} : V_1 \otimes V_2 \rightarrow V_1$$

$$\vdots$$

$$d_{1...2k-2\ \times...1*1} : S_{1...2k-2\ \times...101} \rightarrow S_{1...2k\ \times...1}$$
$$m_{13} : V_1 \otimes V_3 \rightarrow V_1$$

$$d_{1...2k-1\ \times...1*} : S_{1...2k-1\ \times...10} \rightarrow S_{1...2k\ \times...1}$$
$$-m_{13} : V_1 \otimes V_3 \rightarrow V_1$$

Notice that $d^{2k-1}(v_+ \otimes v_+, 0, ..., 0) = v_+$ and $d^{2k-1}(v_+ \otimes v_-, 0, ..., 0) = v_-$, then $\mathrm{Im} d^{2k-1} = V$. Hence $\mathcal{H}^{2k} = \ker d^{2k} / \mathrm{Im} d^{2k-1} = V/V = \{0\{$.

Also,

$$\ker d^{2k-1} = \langle (v_- \otimes v_-, 0, ...2k-1\ \times ..., 0), (0, v_- \otimes v_-, 0, ...2k-2\ \times ..., 0), ,$$
$$(0, 0, v_- \otimes v_-, 0, ...2k-3\ \times ..., 0), ..., (0, ...2k-1\ \times ..., 0, v_- \otimes v_-),$$
$$(v_+ \otimes v_- + v_- \otimes v_+, v_+ \otimes v_- + v_- \otimes v_+, 0, ...2k-2\ \times ..., 0),$$
$$(v_+ \otimes v_- + v_- \otimes v_+, 0, 0, v_+ \otimes v_- + v_- \otimes v_+, 0, ...2k-4\ \times ..., 0),$$
$$(v_+ \otimes v_- + v_- \otimes v_+, 0, 0, 0, 0, v_+ \otimes v_- + v_- \otimes v_+, 0, ...2k-6\ \times ..., 0),$$
$$..., (v_+ \otimes v_- + v_- \otimes v_+, 0, ...2k-2\ \times ..., 0, v_+ \otimes v_- + v_- \otimes v_+),$$
$$(0, v_+ \otimes v_- + v_- \otimes v_+, v_+ \otimes v_- + v_- \otimes v_+, 0, ...2k-3\ \times ..., 0),$$
$$(0, v_+ \otimes v_- + v_- \otimes v_+, 0, 0, v_+ \otimes v_- + v_- \otimes v_+, 0, ...2k-5\ \times ..., 0),$$

$$... \left(0, v_+ \otimes v_- + v_- \otimes v_+, 0, ...2k-4\ \times ..., 0, v_+ \otimes v_- + v_- \otimes v_+, 0\right),$$

$$\left(v_+ \otimes v_+, v_+ \otimes v_+, 0, ...2k-2\ \times ..., 0\right),$$

$$\left(v_+ \otimes v_+, 0, 0, v_+ \otimes v_+, 0, ...2k-4\ \times ..., 0\right),$$

$$... \left(v_+ \otimes v_+, 0, ...2k-2\ \times ..., 0, v_+ \otimes v_+\right),$$

$$\left(0, v_+ \otimes v_+, v_+ \otimes v_+, 0, ...2k-3\ \times ..., 0\right),$$

$$\left(0, v_+ \otimes v_+, 0, 0, v_+ \otimes v_+, 0, ...2k-5\ \times ..., 0\right),$$

$$... \left(0, v_+ \otimes v_+, 0, ...2k-4\ \times ..., 0, v_+ \otimes v_+, 0\right),$$

$$\left(v_+ \otimes v_-, v_+ \otimes v_-, 0, ...2k-2\ \times ..., 0\right),$$

$$\left(v_+ \otimes v_-, 0, 0, v_+ \otimes v_-, 0, ...2k-4\ \times ..., 0\right),$$

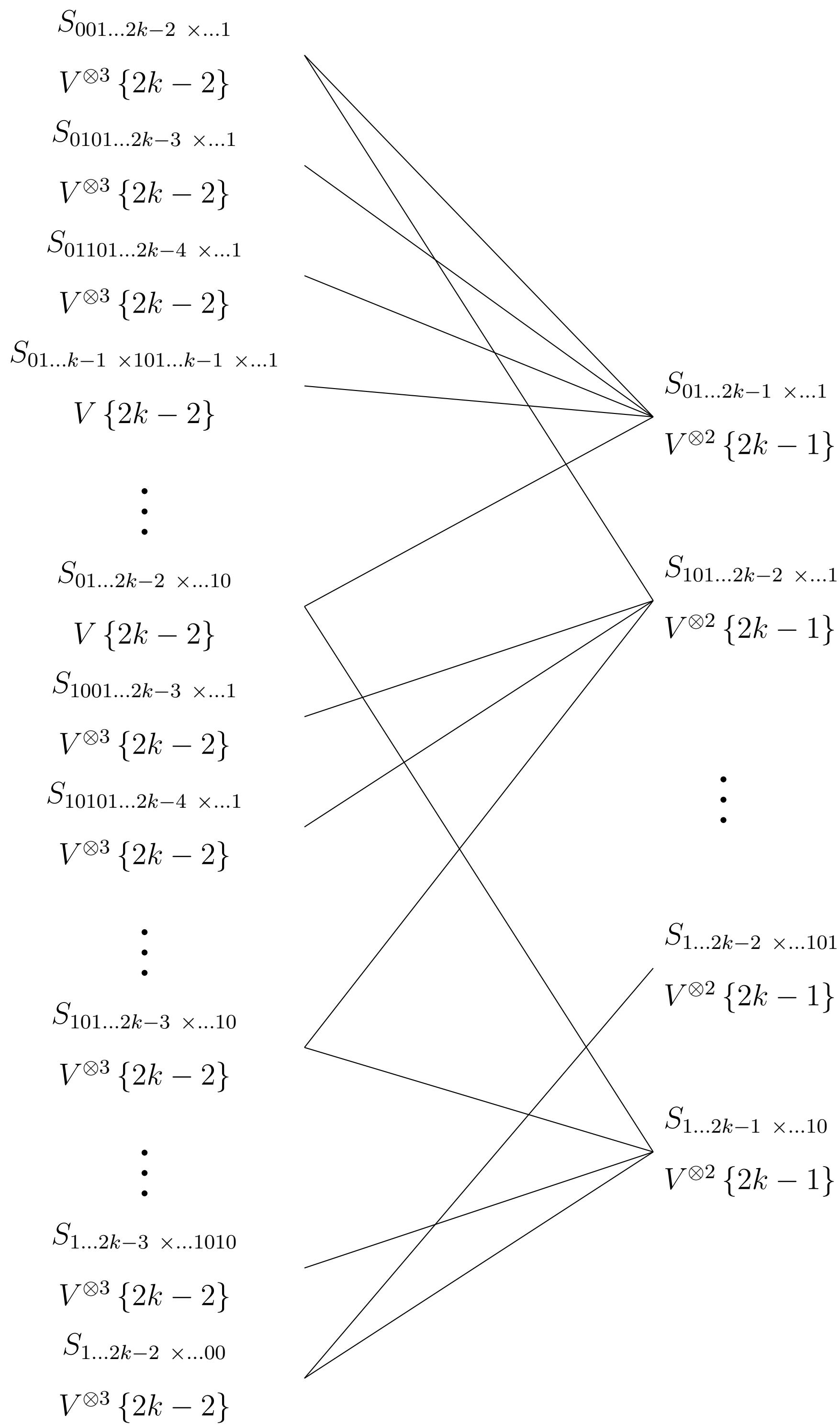
$$... \left(v_+ \otimes v_-, 0, ...2k-2\ \times ..., 0, v_+ \otimes v_-\right),$$

$$\left(0, v_+ \otimes v_-, v_+ \otimes v_-, 0, ...2k-3\ \times ..., 0\right),$$

$$\left(0, v_+ \otimes v_-, 0, 0, v_+ \otimes v_-, 0, ...2k-5\ \times ..., 0\right),$$

$$... \left(0, v_+ \otimes v_-, 0, ...2k-4\ \times ..., 0, v_+ \otimes v_-, 0\right)\rangle$$

Now the portion of the cube with the states with height $2k-1$ and $2k-2$ will be shown.



The edges identified with sequences of height $2k-2$ are collapsed in d^{2k-2}, these are

$$d_{0*1\ldots 2k-2\ \times\ldots 1} : S_{001\ldots 2k-2\ \times\ldots 1} \to S_{01\ldots 2k-1\ \times\ldots 1}$$

$m_{23} \otimes Id_1 : V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes V_2$

$d_{01*1\ldots 2k-3\ \times\ldots 1} : S_{0101\ldots 2k-3\ \times\ldots 1} \to S_{01\ldots 2k-1\ \times\ldots 1}$

$-m_{23} \otimes Id_1 : V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes V_2$

$d_{011*1\ldots 2k-4\ \times\ldots 1} : S_{01101\ldots 2k-4\ \times\ldots 1} \to S_{01\ldots 2k-1\ \times\ldots 1}$

$m_{23} \otimes Id_1 : V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes V_2$

$\vdots$

$d_{01\ldots k-1\ \times\ldots 1*1\ldots k-1\ \times\ldots 1} : S_{01\ldots k-1\ \times\ldots 101\ldots k-1\ \times\ldots 1} \to S_{01\ldots 2k-1\ \times\ldots 1}$

$-\Delta_{12} : V_1 \to V_1 \otimes V_2$

$d_{01\ldots k\ \times\ldots 1*1\ldots k-2\ \times\ldots 1} : S_{01\ldots k\ \times\ldots 101\ldots k-2\ \times\ldots 1} \to S_{01\ldots 2k-1\ \times\ldots 1}$

$m_{23} \otimes Id_1 : V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes V_2$

$\vdots$

$d_{01\ldots 2k-2\ \times\ldots 1*} : S_{01\ldots 2k-2\ \times\ldots 10} \to S_{01\ldots 2k-1\ \times\ldots 1}$

$\Delta_{12} : V_1 \to V_1 \otimes V_2$

$d_{*01\ldots 2k-2\ \times\ldots 1} : S_{001\ldots 2k-2\ \times\ldots 1} \to S_{101\ldots 2k-2\ \times\ldots 1}$

$m_{13} \otimes Id_2 : V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes V_2$

$d_{10*1\ldots 2k-3\ \times\ldots 1} : S_{1001\ldots 2k-3\ \times\ldots 1} \to S_{101\ldots 2k-3\ \times\ldots 1}$

$-m_{13} \otimes Id_2 : V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes V_2$

$d_{101*1\ldots 2k-4\ \times\ldots 1} : S_{10101\ldots 2k-4\ \times\ldots 1} \to S_{101\ldots 2k-2\ \times\ldots 1}$

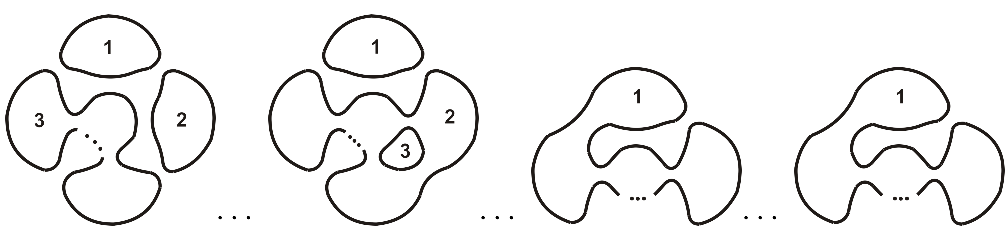
$m_{13} \otimes Id_2 : V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes V_2$

$\vdots$

These are $\dfrac{(2k)!}{2!\,(2k-2)!} = k\,(2k-1)$ states of height $2k-2$, the states in which the $0-$smoothed crossings are two of the inner crossings or two of the outer crossings have 3 cycles. Then the graded vector spaces associated to each one is $V^{\otimes 3}\{2k-2\}$. The edges which tail is on one of these states are m maps.

The states in which the $0-$smoothed crossings are adjacent crossings (one of these is inner and the other one is outer) have 1 cycle. Then the graded vector spaces associated to each one is $V\{2k-2\}$. The edges which tail is on one of these states are Δ maps.

The states in which the $0-$smoothed crossings are not adjacent crossings (one of these is inner and the other one is outer) have 3 cycles. Then the graded vector spaces associated to each one is $V^{\otimes 3}\{2k-2\}$. The edges which tail is on one of these states are m maps.



$S_{001\ldots 2k-2\ \times \ldots 1}$ ... $S_{01\ldots 2k-4\ \times \ldots 1011}$ ... $S_{01\ldots 2k-2\ \times \ldots 10}$ ... $S_{1\ldots 2k-2\ \times \ldots 100}$

Figure 3–7: States of height $2k-2$

Each state of height $2k-1$ is reached by $2k-1$ edges of which $2k-3$ are m maps and 2 are Δ maps, like was explained above, hence, it is enough to study the behavior of edges that reach the first state of height $2k-1$.

Now, d^{2k-2} evaluated in some elements of the basis is:

$d^{2k-2}\,(v_+ \otimes v_+ \otimes v_+, 0, 0, \ldots, 0) = (v_+ \otimes v_+, -v_+ \otimes v_+, 0, \ldots 2k-2\ \times \ldots, 0)$

$d^{2k-2}\,(v_+ \otimes v_+ \otimes v_-, 0, 0, \ldots, 0) = (v_+ \otimes v_-, -v_- \otimes v_+, 0, \ldots 2k-2\ \times \ldots, 0)$

$d^{2k-2}\,(v_+ \otimes v_- \otimes v_+, 0, 0, \ldots, 0) = (v_+ \otimes v_-, -v_+ \otimes v_-, 0, \ldots 2k-2\ \times \ldots, 0)$

$d^{2k-2}\,(v_- \otimes v_+ \otimes v_+, 0, 0, \ldots, 0) = (v_- \otimes v_+, -v_- \otimes v_+, 0, \ldots 2k-2\ \times \ldots, 0)$

$d^{2k-2}(v_+\otimes v_-\otimes v_-,0,0,...,0)=(0,-v_-\otimes v_-,0,...2k-2\ \times...,0)$

$d^{2k-2}(v_-\otimes v_+\otimes v_-,0,0,...,0)=(v_-\otimes v_-,0,...2k-1\ \times...,0)$

$d^{2k-2}(v_-\otimes v_-\otimes v_+,0,0,...,0)=(v_-\otimes v_-,-v_-\otimes v_-,0,...2k-2\ \times...,0)$

$d^{2k-2}(v_-\otimes v_-\otimes v_-,0,0,...,0)=(0,...2k\ \times...,0)$

$d^{2k-2}(0,v_+\otimes v_+\otimes v_+,0,...,0)=(-v_+\otimes v_+,0,v_+\otimes v_+,0,...2k-3\ \times...,0)$

$d^{2k-2}(0,v_+\otimes v_+\otimes v_-,0,...,0)=(-v_+\otimes v_-,0,v_+\otimes v_-,0,...2k-3\ \times...,0)$

$d^{2k-2}(0,v_+\otimes v_-\otimes v_+,0,...,0)=(-v_+\otimes v_-,0,v_-\otimes v_+,0,...2k-3\ \times...,0)$

$d^{2k-2}(0,v_-\otimes v_+\otimes v_+,0,...,0)=(-v_-\otimes v_+,0,v_-\otimes v_+,0,...2k-3\ \times...,0)$

$d^{2k-2}(0,v_+\otimes v_-\otimes v_-,0,...,0)=(0,0,v_-\otimes v_-,0,...2k-3\ \times...,0)$

$d^{2k-2}(0,v_-\otimes v_+\otimes v_-,0,...,0)=(-v_-\otimes v_-,0,v_-\otimes v_-,0,...2k-3\ \times...,0)$

$d^{2k-2}(0,v_-\otimes v_-\otimes v_+,0,...,0)=(-v_-\otimes v_-,0,...2k-3\ \times...,0)$

$d^{2k-2}(0,v_-\otimes v_-\otimes v_-,0,...,0)=(0,...2k\ \times...,0)$

$d^{2k-2}(0,0,v_+\otimes v_+\otimes v_+,0,...,0)=(v_+\otimes v_+,0,0,v_+\otimes v_+,0,...2k-4\ \times...,0)$

$d^{2k-2}(0,0,v_+\otimes v_+\otimes v_-,0,...,0)=(v_+\otimes v_-,0,0,v_+\otimes v_-,0,...2k-4\ \times...,0)$

$d^{2k-2}(0,0,v_+\otimes v_-\otimes v_+,0,...,0)=(v_+\otimes v_-,0,0,v_-\otimes v_+,0,...2k-4\ \times...,0)$

$d^{2k-2}(0,0,v_-\otimes v_+\otimes v_+,0,...,0)=(v_-\otimes v_+,0,0,v_-\otimes v_+,0,...2k-4\ \times...,0)$

$d^{2k-2}(0,0,v_+\otimes v_-\otimes v_-,0,...,0)=(0,0,0,v_-\otimes v_-,0,...2k-4\ \times...,0)$

$d^{2k-2}(0,0,v_-\otimes v_+\otimes v_-,0,...,0)=(v_-\otimes v_-,0,0,v_-\otimes v_-,0,...2k-4\ \times...,0)$

$d^{2k-2}(0,0,v_-\otimes v_-\otimes v_+,0,...,0)=(v_-\otimes v_-,0,...2k-1\ \times...,0)$

$d^{2k-2}(0,0,v_-\otimes v_-\otimes v_-,0,...,0)=(0,...2k\ \times...,0)$

Then,

$\mathrm{Im}d^{2k-2}=\langle(v_-\otimes v_-,0,...2k-1\ \times...,0),(0,v_-\otimes v_-,0,...2k-2\ \times...,0),\ ,$

$(0,0,v_-\otimes v_-,0,...2k-3\ \times...,0),...,(0,...2k-1\ \times...,0,v_-\otimes v_-),$

$(v_+\otimes v_-+v_-\otimes v_+,v_+\otimes v_-+v_-\otimes v_+,0,...2k-2\ \times...,0),$

$(v_+\otimes v_-+v_-\otimes v_+,0,0,v_+\otimes v_-+v_-\otimes v_+,0,...2k-4\ \times...,0),$

$(v_+\otimes v_-+v_-\otimes v_+,0,0,0,0,v_+\otimes v_-+v_-\otimes v_+,0,...2k-6\ \times...,0),$

$...,(v_+\otimes v_-+v_-\otimes v_+,0,...2k-2\ \times...,0,v_+\otimes v_-+v_-\otimes v_+),$

$$\left(0, v_+ \otimes v_- + v_- \otimes v_+, v_+ \otimes v_- + v_- \otimes v_+, 0, ...2k-3 \ \times ..., 0\right),$$
$$\left(0, v_+ \otimes v_- + v_- \otimes v_+, 0, 0, v_+ \otimes v_- + v_- \otimes v_+, 0, ...2k-5 \ \times ..., 0\right),$$
$$...\left(0, v_+ \otimes v_- + v_- \otimes v_+, 0, ...2k-4 \ \times ..., 0, v_+ \otimes v_- + v_- \otimes v_+, 0\right),$$
$$\left(v_+ \otimes v_+, v_+ \otimes v_+, 0, ...2k-2 \ \times ..., 0\right),$$
$$\left(v_+ \otimes v_+, 0, 0, v_+ \otimes v_+, 0, ...2k-4 \ \times ..., 0\right),$$
$$...\left(v_+ \otimes v_+, 0, ...2k-2 \ \times ..., 0, v_+ \otimes v_+\right),$$
$$\left(0, v_+ \otimes v_+, v_+ \otimes v_+, 0, ...2k-3 \ \times ..., 0\right),$$
$$\left(0, v_+ \otimes v_+, 0, 0, v_+ \otimes v_+, 0, ...2k-5 \ \times ..., 0\right),$$
$$...\left(0, v_+ \otimes v_+, 0, ...2k-4 \ \times ..., 0, v_+ \otimes v_+, 0\right),$$
$$\left(v_+ \otimes v_-, v_+ \otimes v_-, 0, ...2k-2 \ \times ..., 0\right),$$
$$\left(v_+ \otimes v_-, 0, 0, v_+ \otimes v_-, 0, ...2k-4 \ \times ..., 0\right),$$
$$...\left(v_+ \otimes v_-, 0, ...2k-2 \ \times ..., 0, v_+ \otimes v_-\right),$$
$$\left(0, v_+ \otimes v_-, v_+ \otimes v_-, 0, ...2k-3 \ \times ..., 0\right),$$
$$\left(0, v_+ \otimes v_-, 0, 0, v_+ \otimes v_-, 0, ...2k-5 \ \times ..., 0\right),$$
$$...\ \left(0, v_+ \otimes v_-, 0, ...2k-4 \ \times ..., 0, v_+ \otimes v_-, 0\right)\rangle$$
$$= \ker d^{2k-1}$$

Hence, $\mathcal{H}^{2k-1} = \ker d^{2k-1} / \mathrm{Im} d^{2k-2} = \{0\{$

CHAPTER 4
CONCLUSION AND FUTURE WORK

4.1 Conclusion

In this work, part of Khovanov homology for $(3, k)$-torus knot was calculated for $k > 3$, by analizing the states of the cube of smoothings and the differentials on the edges to build the chain complex.

A minimal projection was used with all positive crossings, to study the different states obtained from $0-$smooth or $1-$smooth specific crossings of the knot. Part of the cube of smoothing is shown explicitly for height 0, 1, 2, $2k-2$, $2k-1$, $2k$ and the differentials of height 0, 1, $2k-2$, $2k-1$. This makes it possible to compute $\mathcal{H}^0$, $\mathcal{H}^1$, $\mathcal{H}^{2k}$, $\mathcal{H}^{2k-1}$ and the exponents of the variable q in the Khovanov bracket.

The result was the trivial *r-th* homology where $r = 1$, $2k$, $2k-1$, and $\mathcal{H}^0 \cong \mathbb{Z} \oplus \mathbb{Z}$. The degree of each $\mathbb{Z}$ depends on the minimal number of crossings of the knot and the shifting that come from the grade of the generators of the kernel of d^0, which are, -3 and -1. Therefore the Khovanov bracket for a $T\,(3, k)$ starts with $q^{2k-3} + q^{2k-1}$.

4.2 Future work

For future work, it is proposed:

- To find formulas for the *r-th* homology where $r = 2, 3, 4, ..., 2k-3$.
- To analyze the repeated block of the matrix of the Khovanov bracket shown in Appendix A for each $(3, k)$-torus knot.
- To find a formula for the Khovanov bracket mod 2 for $(3, k)$-torus knots using the construction of the cube of smoothings. In the tables of Appendix A it can be

observed the common characteristics of the terms mod 2 in t^2, t^3, and the periodic appearance.

- To extend the formulas of the Khovanov bracket of $(3, k)$-torus knots to (p, q)-torus knots, where p, q are relatively prime.

APPENDIX A
TABLES OF KHOVANOV BRACKET FOR $(3, k)$-TORUS KNOTS

The following tables show information about the Khovanov bracket and Khovanov bracket mod 2. The data organized was obtained from Wolfram Mathematica by loading the KnotTheory´ package which can be obtained from [9].

Let L be a torus knot, $Kh(L)$ is the Khovanov bracket of L and $Kh\,(L)\,(\mathrm{mod}2)$ is the Khovanov bracket mod 2. Let $T\,(3, k)$ be the $(3, k)$-torus knot. -1 means there is torsion 2.

$T\,(3, 4)$

$Kh\,(T\,(3, 4)) = q^5 + q^7 + q^9t^2 + q^{13}t^3 + q^{11}t^4 + q^{13}t^4 + q^{15}t^5 + q^{17}t^5$

$Kh\,(T\,(3, 4))\,(\mathrm{mod}2) - Kh\,(T\,(3, 4)) = q^{11}t^2 + q^{11}t^3$

	t^0	t^1	t^2	t^3	t^4	t^5
q^5	1					
q^7	1					
q^9			1			
q^{11}			-1	-1	1	
q^{13}				1	1	
q^{15}						1
q^{17}						1

Table A–1: Khovanov bracket for $T\,(3, 4)$

$T(3,5)$

$$Kh\,(T(3,5)) = q^7 + q^9 + q^{11}t^2 + q^{15}t^3 + q^{13}t^4 + q^{15}t^4 + q^{17}t^5 + q^{19}t^5 + q^{17}t^6 + q^{21}t^7$$

$$Kh\,(T(3,5))\,(\mathrm{mod}2) - Kh\,(T(3,5)) = q^{13}t^2 + q^{13}t^3 + q^{19}t^6 + q^{19}t^7$$

	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7
q^7	1							
q^9	1							
q^{11}			1					
q^{13}			-1	-1	1			
q^{15}				1	1			
q^{17}						1	1	
q^{19}						1	-1	-1
q^{21}								1

Table A–2: Khovanov bracket for $T\,(3,5)$

$T(3,7)$

$Kh\,(T(3,7)) = q^{11} + q^{13} + q^{15}t^2 + q^{19}t^3 + q^{17}t^4 + q^{19}t^4 + q^{21}t^5 + q^{23}t^5 + q^{21}t^6 + q^{25}t^7 + q^{23}t^8 + q^{25}t^8 + q^{27}t^9 + q^{29}t^9$

$Kh\,(T(3,7))\,(\mathrm{mod}2) - Kh\,(T(3,7)) = q^{17}t^2 + q^{17}t^3 + q^{23}t^6 + q^{23}t^7$

	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9
q^{11}	1									
q^{13}	1									
q^{15}			1							
q^{17}			-1	-1	1					
q^{19}				1	1					
q^{21}						1	1			
q^{23}						1	-1	-1	1	
q^{25}								1	1	
q^{27}										1
q^{29}										1

Table A–3: Khovanov bracket for $T\,(3,7)$

$T(3,8)$

$Kh\,(T(3,8)) = q^{13} + q^{15} + q^{17}t^2 + q^{21}t^3 + q^{19}t^4 + q^{21}t^4 + q^{23}t^5 + q^{25}t^5 + q^{23}t^6 + q^{27}t^7 + q^{25}t^8 + q^{27}t^8 + q^{29}t^9 + q^{31}t^9 + q^{29}t^{10} + q^{33}t^{11}$

$Kh\,(T(3,8))\,(\mathrm{mod}2) - Kh\,(T(3,8)) = q^{19}t^2 + q^{19}t^3 + q^{25}t^6 + q^{25}t^7 + q^{31}t^{10} + q^{31}t^{11}$

	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9	t^{10}	t^{11}
q^{13}	1											
q^{15}	1											
q^{17}			1									
q^{19}			-1	-1	1							
q^{21}				1	1							
q^{23}						1	1					
q^{25}						1	-1	-1	1			
q^{27}								1	1			
q^{29}										1	1	
q^{31}										1	-1	-1
q^{33}												1

Table A–4: Khovanov bracket for $T\,(3,8)$

$$T(3,10)$$

$$Kh\left(T\left(3,10\right)\right)=q^{17}+q^{19}+q^{21}t^{2}+q^{25}t^{3}+q^{23}t^{4}+q^{25}t^{4}+q^{27}t^{5}+q^{29}t^{5}+q^{27}t^{6}+ q^{31}t^{7}+q^{29}t^{8}+q^{31}t^{8}+q^{33}t^{9}+q^{35}t^{9}+q^{33}t^{10}+q^{37}t^{11}+q^{35}t^{12}+q^{37}t^{12}+q^{39}t^{13}+q^{41}t^{13}$$

$$Kh\left(T\left(3,10\right)\right)(\mathrm{mod}2)-Kh\left(T\left(3,10\right)\right)=q^{23}t^{2}+q^{23}t^{3}+q^{29}t^{6}+q^{29}t^{7}+q^{35}t^{10}+q^{35}t^{11}$$

	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9	t^{10}	t^{11}	t^{12}	t^{13}
q^{17}	1													
q^{19}	1													
q^{21}			1											
q^{23}			-1	-1	1									
q^{25}				1	1									
q^{27}						1	1							
q^{29}						1	-1	-1	1					
q^{31}								1	1					
q^{33}										1	1			
q^{35}										1	-1	-1	1	
q^{37}												1	1	
q^{39}														1
q^{41}														1

Table A–5: Khovanov bracket for $T\left(3,10\right)$

$T(3,11)$

$Kh\,(T\,(3,11)) = q^{19} + q^{21} + q^{23}t^2 + q^{27}t^3 + q^{25}t^4 + q^{27}t^4 + q^{29}t^5 + q^{31}t^5 + q^{29}t^6 + q^{33}t^7 + q^{31}t^8 + q^{33}t^8 + q^{35}t^9 + q^{37}t^9 + q^{35}t^{10} + q^{39}t^{11} + q^{37}t^{12} + q^{39}t^{12} + q^{41}t^{13} + q^{43}t^{13} + q^{41}t^{14} + q^{45}t^{15}$

$Kh\,(T\,(3,11))\,(\mathrm{mod} 2) - Kh\,(T\,(3,11)) = q^{25}t^2 + q^{25}t^3 + q^{31}t^6 + q^{31}t^7 + q^{37}t^{10} + q^{37}t^{11} + q^{43}t^{14} + q^{43}t^{15}$

	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9	t^{10}	t^{11}	t^{12}	t^{13}	t^{14}	t^{15}
q^{19}	1															
q^{21}	1															
q^{23}			1													
q^{25}			-1	-1	1											
q^{27}				1	1											
q^{29}						1	1									
q^{31}						1	-1	-1	1							
q^{33}								1	1							
q^{35}										1	1					
q^{37}										1	-1	-1	1			
q^{39}												1	1			
q^{41}														1	1	
q^{43}														1	-1	-1
q^{45}																1

Table A–6: Khovanov bracket for $T\,(3,11)$

$T(3,13)$

$Kh\,(T\,(3,13)) = q^{23} + q^{25} + q^{27}t^2 + q^{31}t^3 + q^{29}t^4 + q^{31}t^4 + q^{33}t^5 + q^{35}t^5 + q^{33}t^6 + q^{37}t^7 + q^{35}t^8 + q^{37}t^8 + q^{39}t^9 + q^{41}t^9 + q^{39}t^{10} + q^{43}t^{11} + q^{41}t^{12} + q^{43}t^{12} + q^{45}t^{13} + q^{47}t^{13} + q^{45}t^{14} + q^{49}t^{15} + q^{47}t^{16} + q^{49}t^{16} + q^{51}t^{17} + q^{53}t^{17}$

$Kh\,(T\,(3,13))\,(\mathrm{mod}2) - Kh\,(T\,(3,13)) = q^{29}t^2 + q^{29}t^3 + q^{35}t^6 + q^{35}t^7 + q^{41}t^{10} + q^{41}t^{11} + q^{47}t^{14} + q^{47}t^{15}$

	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9	t^{10}	t^{11}	t^{12}	t^{13}	t^{14}	t^{15}	t^{16}	t^{17}
q^{23}	1																	
q^{25}	1																	
q^{27}			1															
q^{29}			-1	-1	1													
q^{31}				1	1													
q^{33}						1	1											
q^{35}						1	-1	-1	1									
q^{37}								1	1									
q^{39}										1	1							
q^{41}										1	-1	-1	1					
q^{43}												1	1					
q^{45}														1	1			
q^{47}														1	-1	-1	1	
q^{49}																1	1	
q^{51}																		1
q^{53}																		1

Table A–7: Khovanov bracket for $T\,(3,13)$

$T(3,14)$

$Kh\left(T\left(3,14\right)\right) = q^{25}+q^{27}+q^{29}t^{2}+q^{33}t^{3}+q^{31}t^{4}+q^{33}t^{4}+q^{35}t^{5}+q^{37}t^{5}+q^{35}t^{6}+q^{39}t^{7}+q^{37}t^{8}+q^{39}t^{8}+q^{41}t^{9}+q^{43}t^{9}+q^{41}t^{10}+q^{45}t^{11}+q^{43}t^{12}+q^{45}t^{12}+q^{47}t^{13}+q^{49}t^{13}+q^{47}t^{14}+q^{51}t^{15}+q^{49}t^{16}+q^{51}t^{16}+q^{53}t^{17}+q^{55}t^{17}+q^{53}t^{18}+q^{57}t^{19}$

$Kh\left(T\left(3,14\right)\right)(\mathrm{mod}2) - Kh\left(T\left(3,14\right)\right) = q^{31}t^{2}+q^{31}t^{3}+q^{37}t^{6}+q^{37}t^{7}+q^{43}t^{10}+q^{43}t^{11}+q^{49}t^{14}+q^{49}t^{15}+q^{55}t^{18}+q^{55}t^{19}$

	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9	t^{10}	t^{11}	t^{12}	t^{13}	t^{14}	t^{15}	t^{16}	t^{17}	t^{18}	t^{19}
q^{25}	1																			
q^{27}	1																			
q^{29}			1																	
q^{31}			−1	−1	1															
q^{33}				1	1															
q^{35}						1	1													
q^{37}						1	−1	−1	1											
q^{39}								1	1											
q^{41}										1	1									
q^{43}										1	−1	−1	1							
q^{45}												1	1							
q^{47}														1						
q^{49}														1	−1	−1	1			
q^{51}																1	1			
q^{53}																		1	1	
q^{55}																		1	−1	−1
q^{57}																				1

Table A–8: Khovanov bracket for $T\left(3,14\right)$

$$T(3,16)$$

$$Kh\left(T\left(3,16\right)\right) = q^{29}+q^{31}+q^{33}t^{2}+q^{37}t^{3}+q^{35}t^{4}+q^{37}t^{4}+q^{39}t^{5}+q^{41}t^{5}+q^{39}t^{6}+q^{43}t^{7}+q^{41}t^{8}+q^{43}t^{8}+q^{45}t^{9}+q^{47}t^{9}+q^{45}t^{10}+q^{49}t^{11}+q^{47}t^{12}+q^{49}t^{12}+q^{51}t^{13}+q^{53}t^{13}+q^{51}t^{14}+q^{55}t^{15}+q^{53}t^{16}+q^{55}t^{16}+q^{57}t^{17}+q^{59}t^{17}+q^{57}t^{18}+q^{61}t^{19}+q^{59}t^{20}+q^{61}t^{20}+q^{63}t^{21}+q^{65}t^{21}$$

$$Kh\left(T\left(3,16\right)\right)(\mathrm{mod}2) - Kh\left(T\left(3,16\right)\right) = q^{35}t^{2}+q^{35}t^{3}+q^{41}t^{6}+q^{41}t^{7}+q^{47}t^{10}+q^{47}t^{11}+q^{53}t^{14}+q^{53}t^{15}+q^{59}t^{18}+q^{59}t^{19}$$

	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9	t^{10}	t^{11}	t^{12}	t^{13}	t^{14}	t^{15}	t^{16}	t^{17}	t^{18}	t^{19}	t^{20}	t^{21}
q^{29}	1																					
q^{31}	1																					
q^{33}			1																			
q^{35}			−1	−1	1																	
q^{37}				1	1																	
q^{39}						1	1															
q^{41}						1	−1	−1	1													
q^{43}								1	1													
q^{45}										1	1											
q^{47}										1	−1	−1	1									
q^{49}												1	1									
q^{51}														1	1							
q^{53}														1	−1	−1	1					
q^{55}																1	1					
q^{57}																		1	1			
q^{59}																		1	−1	−1	1	
q^{61}																				1	1	
q^{63}																						1
q^{65}																						1

Table A–9: Khovanov bracket for $T\left(3,16\right)$

Some common characteristics in the tables of Khovanov bracket for $T(3,k)$-torus knots can be observed. First, the ones proved in this thesis:

- The first two numbers in the t^0 column are 1's and they appear in q^{2k-3} and q^{2k-1} rows.
- The t^1 column do not have any number because $\mathcal{H}^1 = 0$.
- The t^{2k}, t^{2k-1}, t^{2k-2} columns do not appear in the tables because $\mathcal{H}^r = 0$ with $r = 2k$, $2k-1$, $2k-2$.

And a few others for future work:

- The repeated block in table (A–10), which suggests that a formula can be obtained for $\mathcal{H}^r$, with $r = 3$, 4, ..., $2k-3$.

1	1		
1	-1	-1	1
		1	1

Table A–10: Repeated block

- The first two numbers in the t^2 column are 1, -1 and suggest that can be a formula for $\mathcal{H}^2$.
- The terms mod 2 seem to have a periodic appearance in t^2, t^3, t^6, t^7, t^{10}, t^{11}, t^{14}, t^{15}, ...

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KHOVANOV HOMOLOGY FOR $(3, k)$-TORUS KNOTS

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